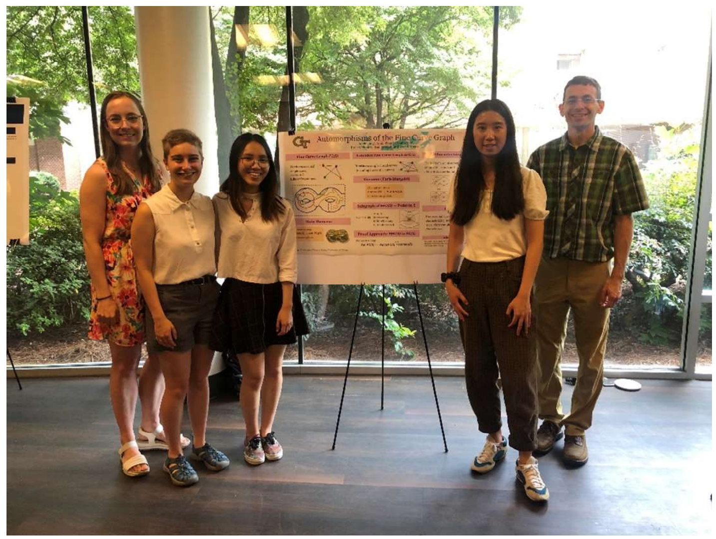
Automorphisms of the fine curve graph



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Joint with Adele Long, Dan Margalit, Anna Pham, and Claudia Yao

Overview

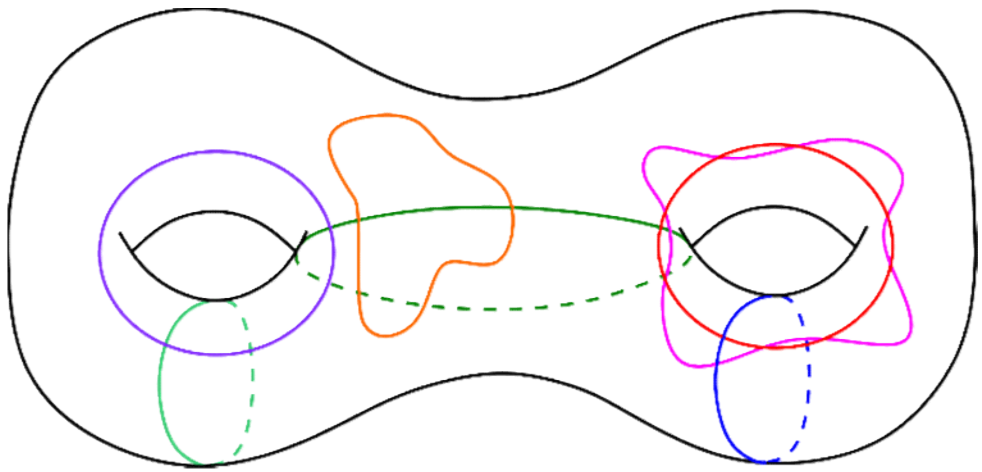
The group of automorphisms of the fine curve graph for a surface is isomorphic to the group of homeomorphisms of the surface.

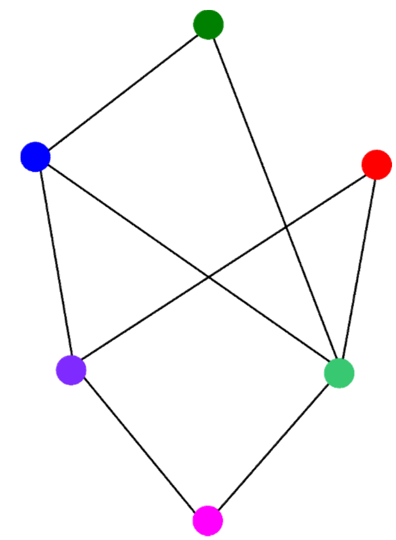
Curve Graphs

Fine Curve Graph (Bowden–Hensel–Webb)

Vertices: Essential simple closed curves

Edges: Disjointness





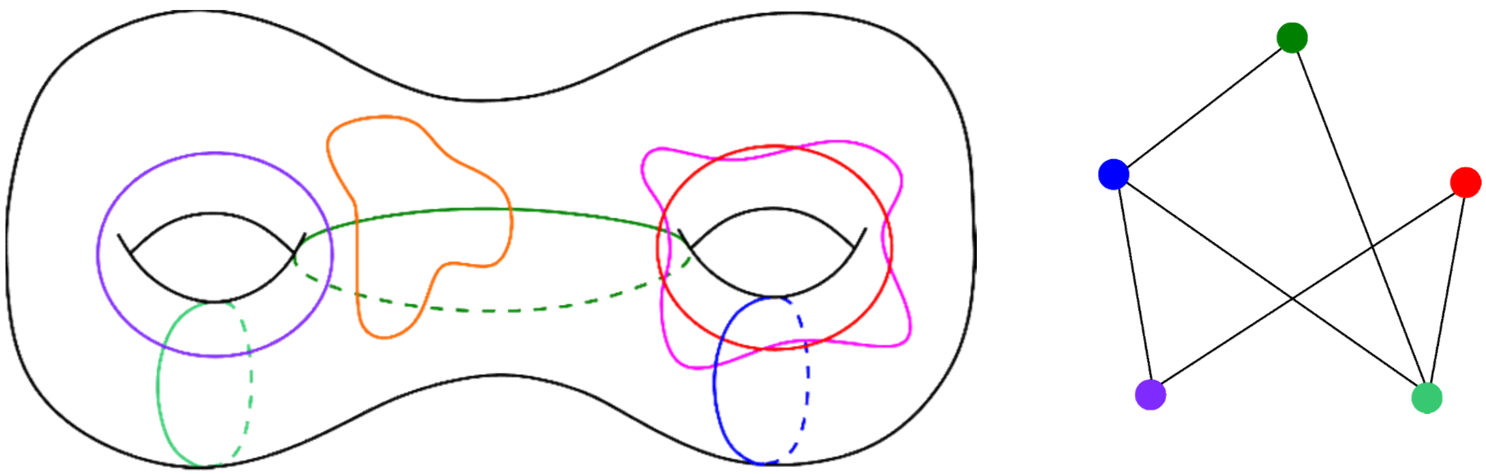
Bowden–Hensel–Webb use $FC(S)$ to show $\mathrm{Diff}_0(S)$ admits many unbounded quasi-morphisms

$\implies \mathrm{Diff}_0(S)$ is not uniformly perfect

Curve Graph (Harvey)

Vertices: Isotopy classes of essential simple closed curves

Edges: Disjointness



Extended Mapping Class Group

Group of symmetries of a surface

$$\mathrm{MCG}^{\pm}(S) = \text{Homeomorphisms/isotopy}$$

What is Known for the Mapping Class Group

Ivanov:

$$\mathrm{MCG}^{\pm}(S) \cong \mathrm{AutMCG}(S) \cong \mathrm{Aut}(\mathcal{C}(S))$$

Natural map: $\mathrm{MCG}^{\pm}(S) \to \mathrm{Aut}(\mathcal{C}(S))$

$f \in \mathrm{MCG}^{\pm}(S)$ maps disjoint curves to disjoint curves.

Ivanov: For $g \geq 3$, the natural map

$$\mathrm{MCG}^{\pm}(S_g) \to \mathrm{Aut}(\mathcal{C}(S_g))$$

is an isomorphism.

Ivanov:

$$\mathrm{MCG}^{\pm}(S) \cong \mathrm{AutMCG}(S) \cong \mathrm{Aut}(\mathcal{C}(S))$$

$$\mathrm{MCG}^{\pm}(S) \to \mathrm{Aut}(\mathrm{MCG}(S))$$

$$f \mapsto \text{conjugation by } f$$

Automorphisms of $\mathrm{MCG}(S)$ preserve powers of Dehn twists.

Reduce to problem using curve graph.

$\rightsquigarrow \mathcal{C}(S)$ a combinatorial tool to study $\mathrm{MCG}^{\pm}(S)$

Ivanov (1997):

$$\mathrm{MCG}^{\pm}(S) \cong \mathrm{AutMCG}(S) \cong \mathrm{Aut}(\mathcal{C}(S))$$

Inspired theorems of the following form:

1. Automorphism group of a simplicial complex associated to S is isomorphic to $\mathrm{MCG}^{\pm}(S)$

2. Automorphism group of some normal subgroup of $\mathrm{MCG}(S)$ is isomorphic to $\mathrm{MCG}^{\pm}(S)$

Ivanov Metaconjecture:

Every object naturally associated to a surface S and having a sufficiently rich structure has $\mathrm{MCG}^{\pm}(S)$ as its group of automorphisms.

Evidence for Ivanov Metaconjecture:

Irmak: Complex of non-separating curves

Brendle–Margalit: Complex of separating curves

McCarthy–Papadopoulos: Truncated complex of domains

Irmak–McCarthy and Disarlo: Arc complex

Korkmaz–Papadopoulos: Arc and curve complex

Bowditch: Complex of strongly separating curves

Evidence for Ivanov Metaconjecture:

Farb–Ivanov: The Torelli group

Brendle–Margalit: The Johnson kernel

Bridson–Pettet–Souto: Every term of the Johnson filtration

Evidence for Ivanov Metaconjecture:

Bridson–Pettet–Souto: Every term of the Johnson filtration

Many examples of normal subgroups satisfying metaconjecture

Q: Do all nontrivial normal subgroups of $\mathrm{MCG}(S)$ satisfy the metaconjecture?

Dahmani–Guirardel–Osin: Construct examples of infinitely generated, free, normal subgroups of $\mathrm{MCG}(S)$ $\implies$ automorphism group is not isomorphic to $\mathrm{MCG}(S)$

Evidence for Ivanov Metaconjecture:

Dahmani–Guirardel–Osin: Construct examples of infinitely generated, free, normal subgroups of $\mathrm{MCG}(S)$
$\implies$ automorphism group is not isomorphic to $\mathrm{MCG}(S)$

All elements are pseudo-Anosov

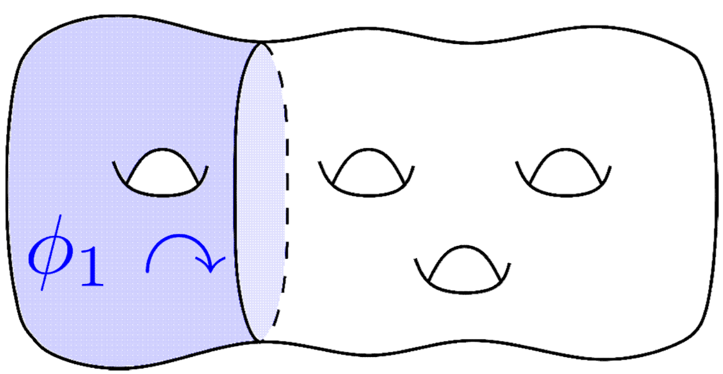
Support of every nontrivial element is the whole surface

Brendle–Margalit: consider elements with *small* support

Evidence for Ivanov Metaconjecture:

A subsurface is small if it is contained in a subsurface of genus k with connected boundary where $k < g/3$.

Brendle–Margalit: If N is a normal subgroup of $\mathrm{MCG}(S)$ which contains at least one element of small support, then $\mathrm{Aut}\, N \cong \mathrm{MCG}^{\pm}(S)$.



Eg. $N = \langle\!\langle \phi_1 \rangle\!\rangle$

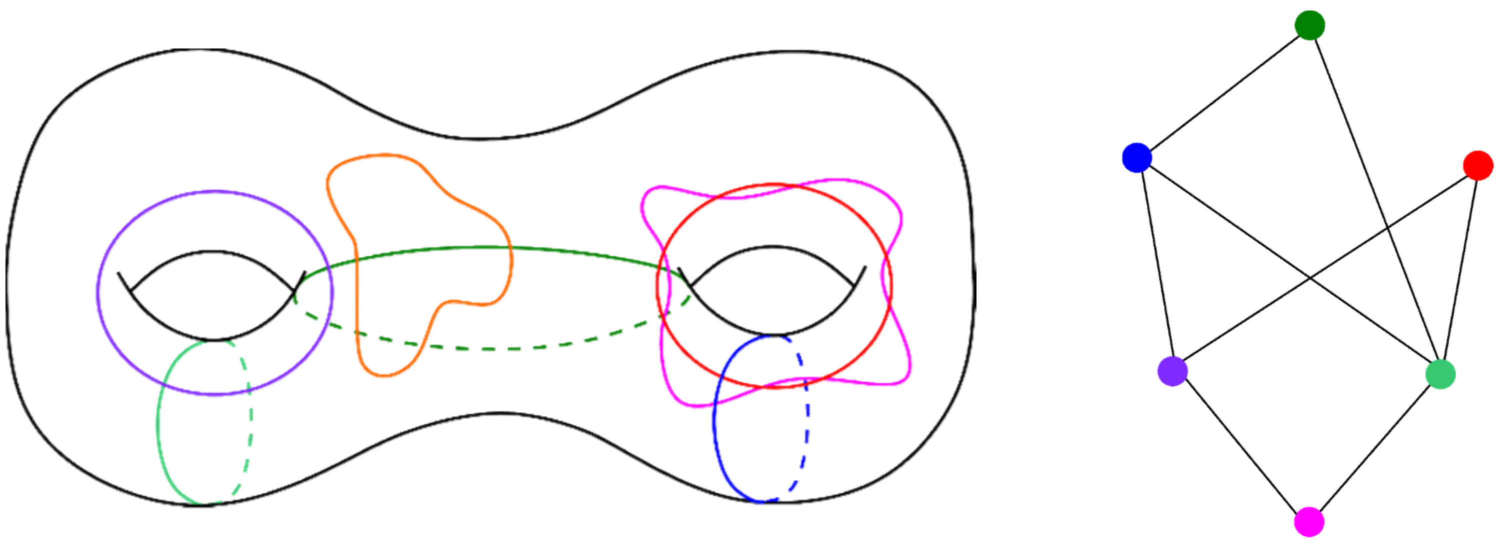
$\implies$ Torelli subgroup, Johnson kernel, and Johnson filtration

Main Theorem

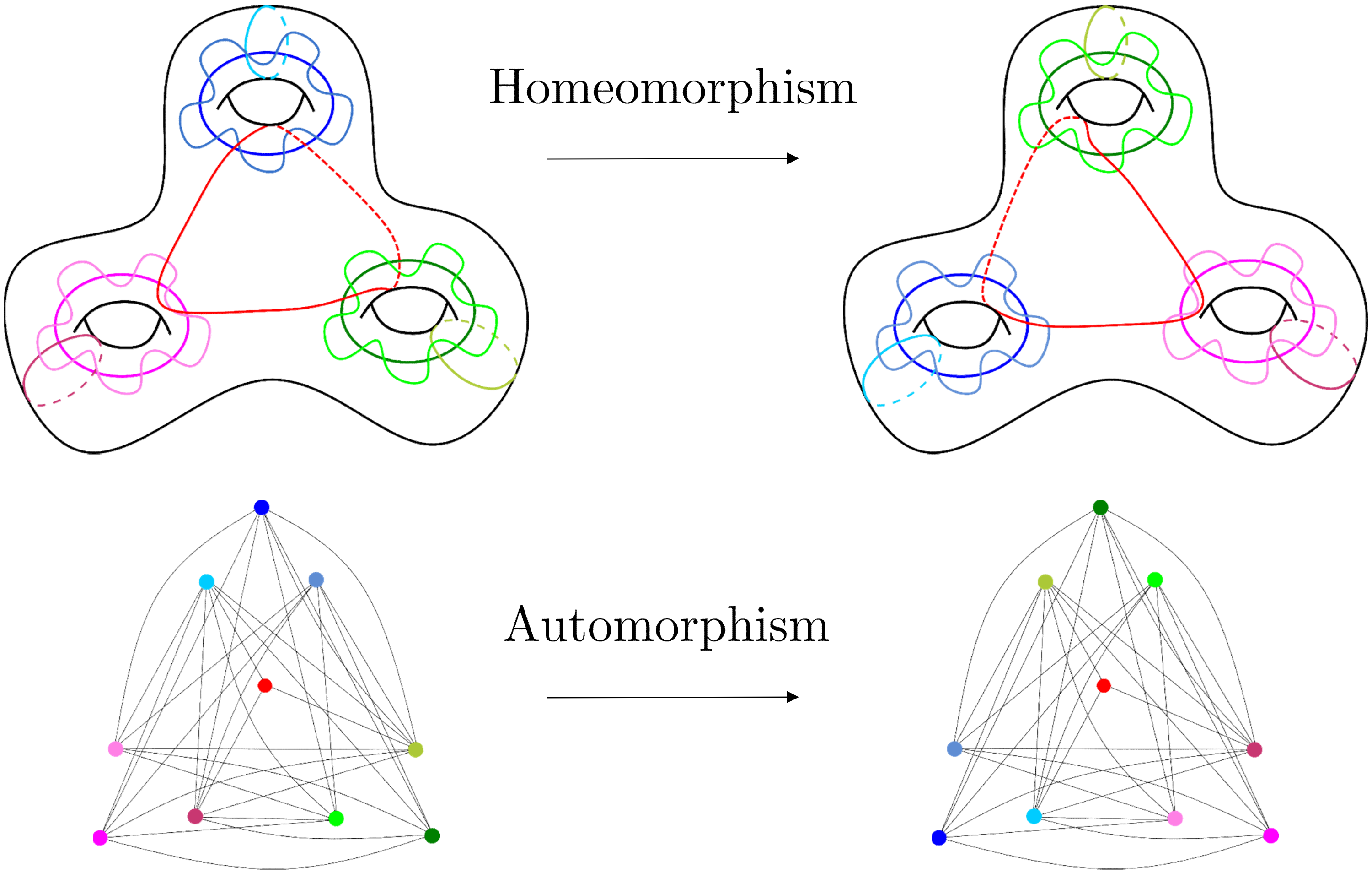
Recall: Fine Curve Graph

Vertices: Essential simple closed curves

Edges: Disjointness



The Natural Map



Long–Margalit–Pham–V.–Yao:

For $g \geq 2$, the natural map

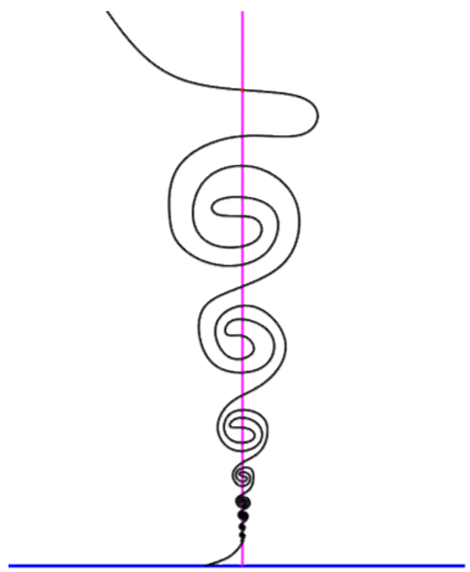
$$\eta : \mathrm{Homeo}(S_g) \to \mathrm{Aut}(FC(S_g))$$

is an isomorphism.

Main Difficulty

Two vertices of $FC(S)$ can

- bound countably many bigons
- intersect uncountably many times



Main Goal

Using collections of vertices to encode points in the surface.

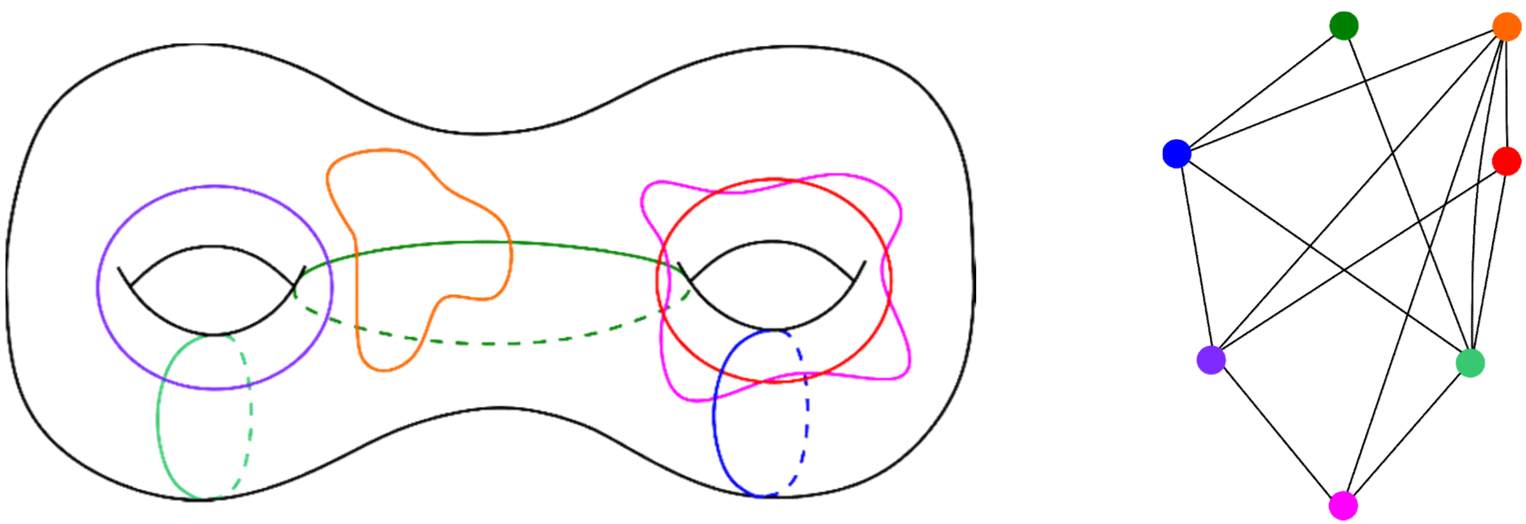
Main Tool

Use the work of Farb–Margalit on the extended fine curve graph.

Extended Fine Curve Graph

Vertices: Simple closed curves

Edges: Disjointness



Farb–Margalit:

For any surface without boundary, the natural map

$$\nu : \mathrm{Homeo}(S_g) \to \mathrm{Aut}(EFC(S_g))$$

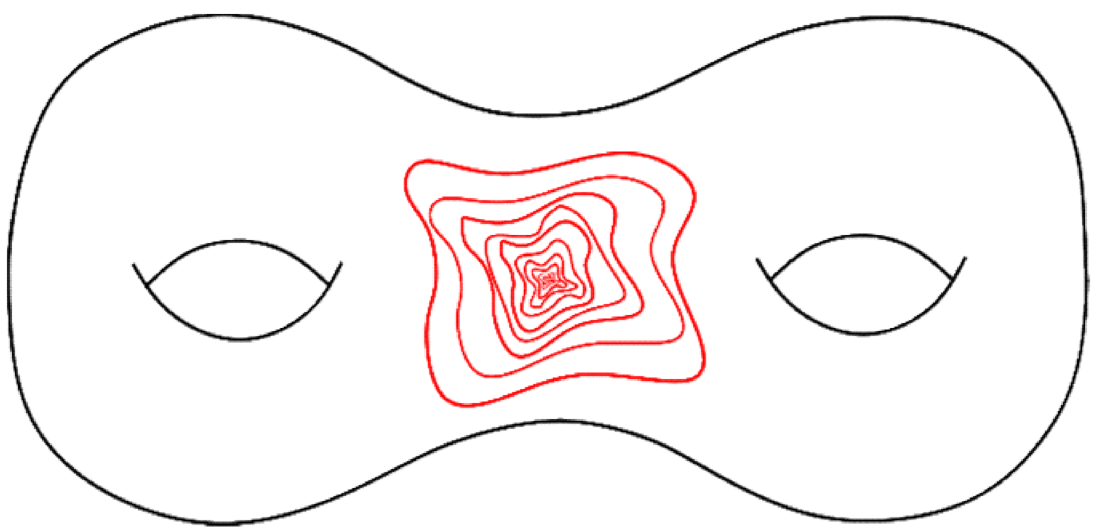
is an isomorphism.

Main tool:

Convergent sequences.

Convergent Sequence

A sequence of vertices (c_i) of $EFC(S)$ converges to a point $x \in S$ if every neighborhood of x contains all but finitely many of the corresponding curves c_i.



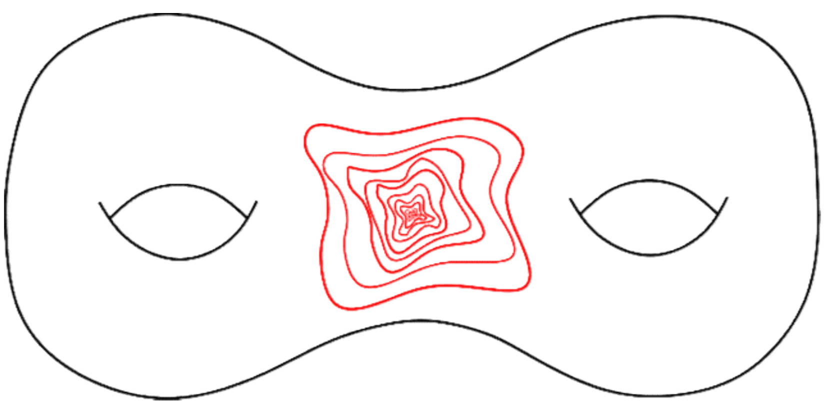
Farb–Margalit:

Automorphisms of $EFC(S)$ preserve convergent sequences.

Topology $\longleftrightarrow$ Graph Theory

(c_i) convergent sequence $\iff$

1. $\exists$ vertex a that intersects the tail of (c_i)
2. If a and b are distinct vertices that intersect the tail of (c_i), then $a \cap b \neq \emptyset$



Farb–Margalit:

Automorphisms of $EFC(S)$ preserve convergent sequences.

Use this lemma to build an inverse map of

$$\nu : \text{Homeo}(S_g) \to \text{Aut}(EFC(S_g)).$$

Extending to the Fine Curve Graph

Construct a homomorphism

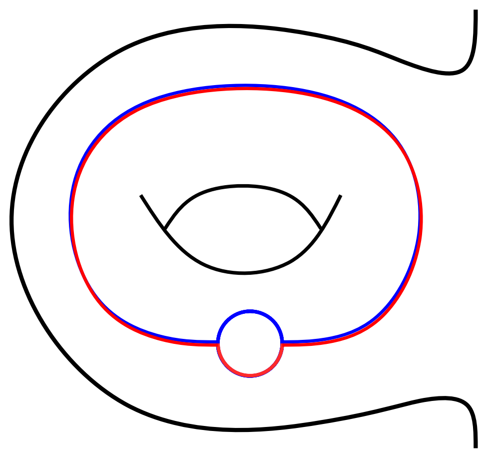
$$\epsilon : \mathrm{Aut}FC(S_g) \to \mathrm{Aut}(EFC(S_g)) \cong \mathrm{Homeo}(S).$$

Main tool:

Bigon pairs.

Bigon Pair

The vertices $c, d \in FC(S_g)$ is a bigon pair if $c \cap d$ is a nontrivial closed interval and c and d are homotopic.



If the two curves in the bigon pair are nonseparating, we call the pair a nonseparating bigon pair.

Note: Nonseparating bigon pairs are separating!

Long–Margalit–Pham–V.–Yao:

Automorphisms of $FC(S)$ preserves the set of nonseparating bigon pairs.

Use this lemma to define a homomorphism

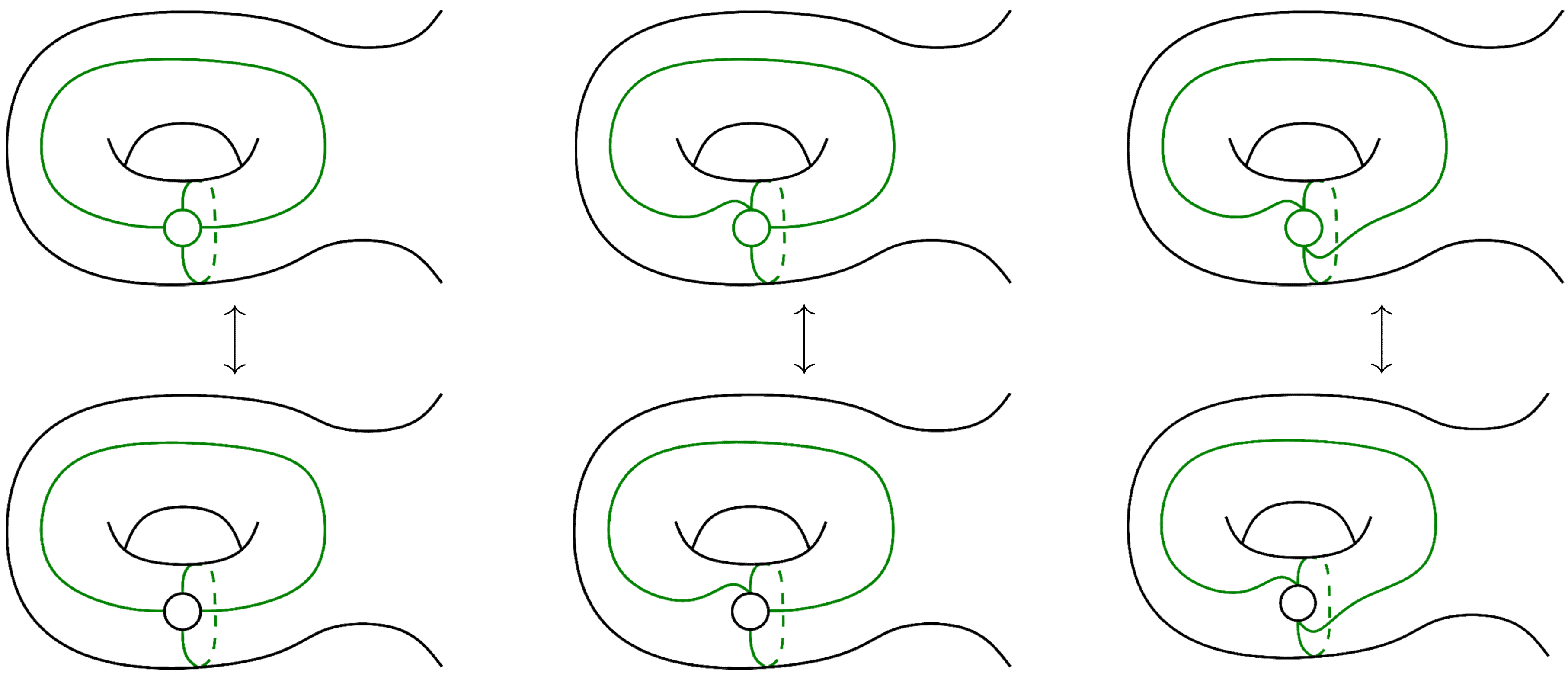
$$\epsilon : \mathrm{Aut}FC(S_g) \to \mathrm{Aut}(EFC(S_g)).$$

Showing this homomorphism is well-defined is difficult.

Solution: Use sharing pairs.

Sharing Pair

A pair of bigon pairs $\{\{a, b\}, \{a', b'\}\}$ is a sharing pair if the corresponding arcs in S_g^1 have disjoint interiors.



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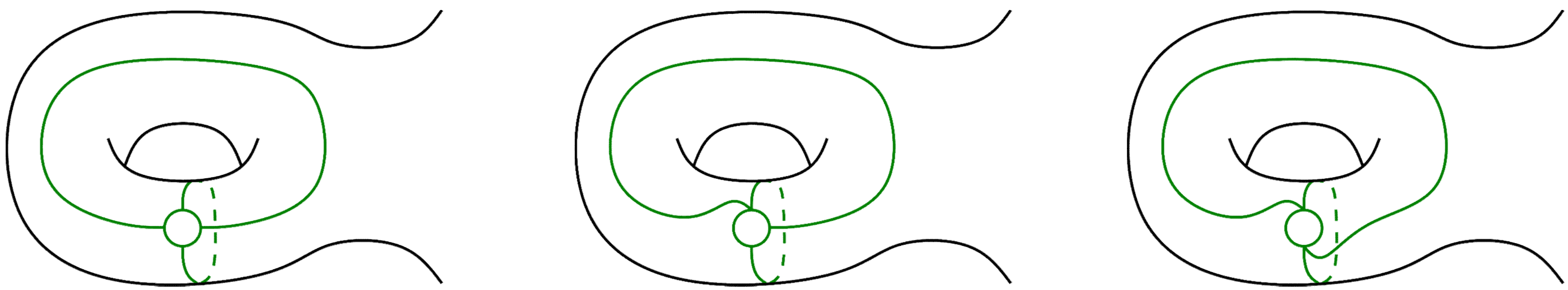
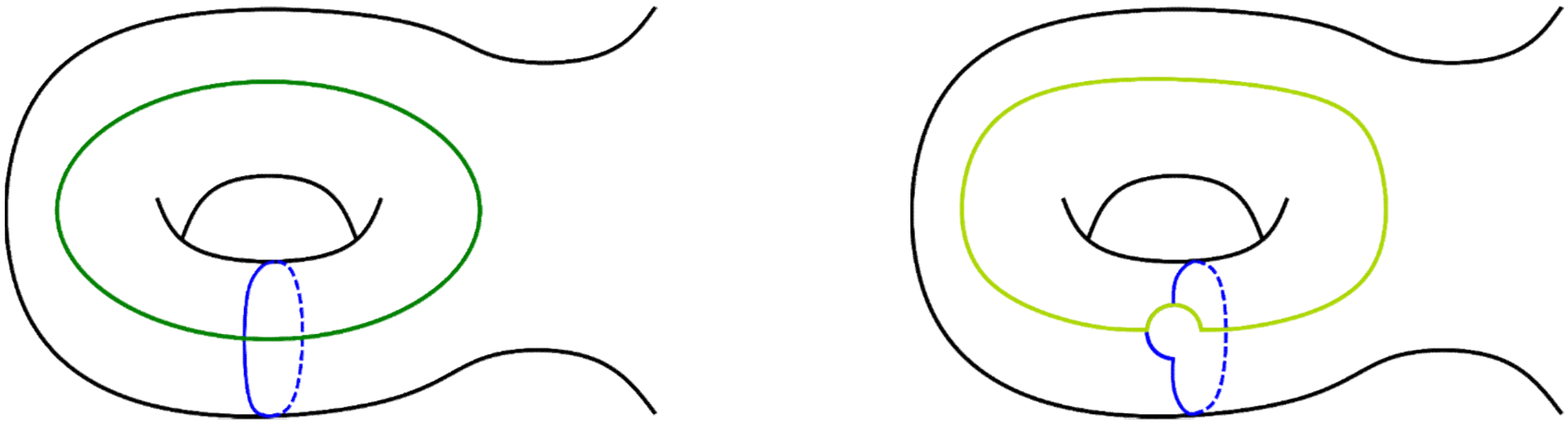


A sharing pair is linked if every boundary parallel curve in S_g^1 sufficiently close to the boundary intersects the two arcs alternately.

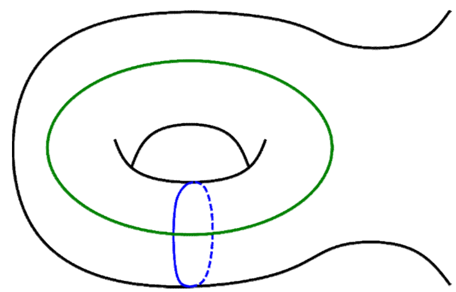
Torus Pair

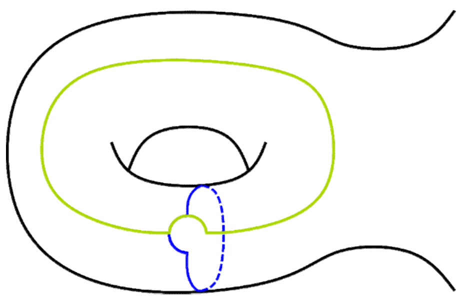
A pair of vertices $\{c, d\}$ in $FC(S_g)$ is a torus pair if $c \cap d$ is a single interval and c and d cross at that interval.



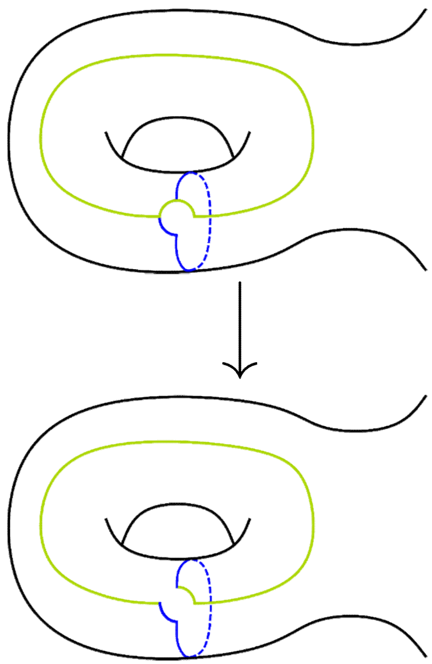
Torus Triple

A torus pair $\{c, d\}$ is degenerate if $c \cap d$ is a single point, and nondegenerate otherwise

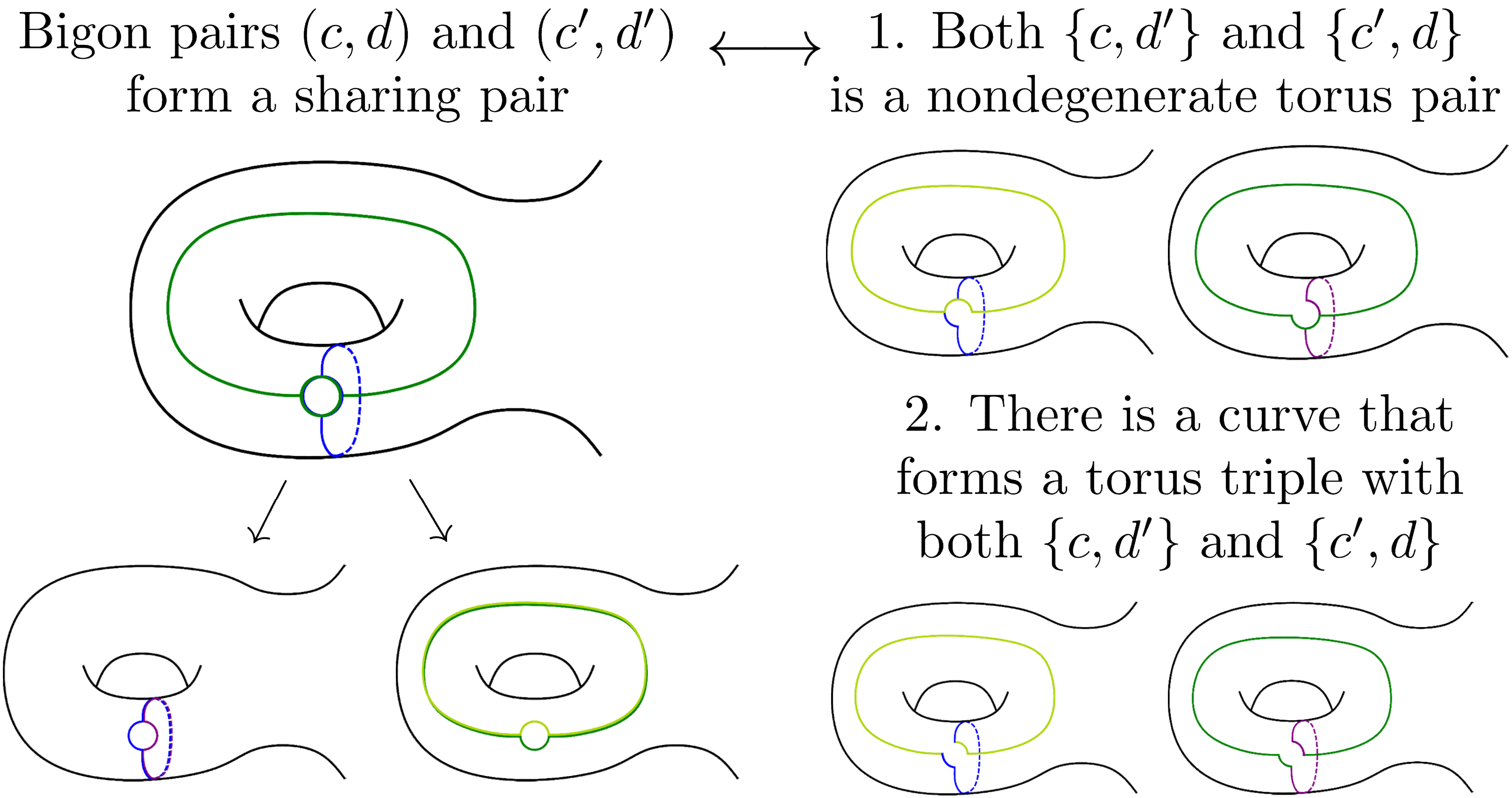




If $\{c, d\}$ nondegenerate torus pair, $\exists$ third curve e in $c \Delta d$



$\{c, d, e\}$ is a torus triple


Bigon pairs (c, d) and (c', d')
form a sharing pair
1. Both $\{c, d'\}$ and $\{c', d\}$
is a nondegenerate torus pair
2. There is a curve that
forms a torus triple with
both $\{c, d'\}$ and $\{c', d\}$

Long–Margalit–Pham–V.–Yao:

Automorphisms of $FC(S)$ preserves the set of linked sharing pairs.

Use this lemma to show the homomorphism

$$\epsilon : \mathrm{Aut}FC(S_g) \to \mathrm{Aut}(EFC(S_g))$$

is well-defined.

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Proof Sketch:

For $\alpha \in \mathrm{Aut}FC(S_g)$ define $\hat{\alpha} \in \mathrm{Aut}EFC(S_g)$ to be

- $\alpha(c) = \hat{\alpha}(c)$ for c an essential simple closed curve
- For $e \in S_g$ inessential, we take any bigon pair $\{c, d\}$ determining e and define $\hat{\alpha}(e)$ to be the inessential curve determined by $\{\alpha(c), \alpha(d)\}$.

Proof Sketch:

For $\alpha \in \mathrm{Aut}FC(S_g)$ define $\hat{\alpha} \in \mathrm{Aut}EFC(S_g)$ to be

- For $e \in S_g$ inessential, we take any bigon pair $\{c, d\}$ determining e and define $\hat{\alpha}(e)$ to be the inessential curve determined by $\{\alpha(c), \alpha(d)\}$.

If $\{c', d'\}$ is another bigon pair determining e, there exists a sequence of bigon pairs
$\{c, d\} = \{c_0, d_0\}, \ldots, \{c_n, d_n\} = \{c', d'\}$ where each pair $\{\{c_i, d_i\}, \{c_{i+1}, d_{i+1}\}\}$ is a linked sharing pair.

This path is found via a path in fine arc graph!

Automorphisms preserve the set of linked sharing pairs
$\rightsquigarrow$ well-defined.

Final Step

Compose natural map

$$\eta : \text{Homeo}(S_g) \to \text{Aut}(FC(S_g))$$

with the well defined homomorphism

$$\epsilon : \text{Aut}FC(S_g) \to \text{Aut}(EFC(S_g))$$

with the map

$$\nu^{-1} : \text{Aut}(EFC(S_g)) \to \text{Homeo}(\mathrm{S_g}).$$

Long–Margalit–Pham–V.–Yao:

For $g \geq 2$, the natural map

$$\eta : \text{Homeo}(S_g) \to \text{Aut}(FC(S_g))$$

is an isomorphism.

Fine Arc Graph

Vertices: Essential simple proper arcs

Edges: Disjointness

Fact:

For any $S = S_g^b$ with $b > 0$, the graph $FA(S)$ is connected.

Fact:

For any $S = S_g^b$ with $b > 0$, the graph $FA(S)$ is connected.

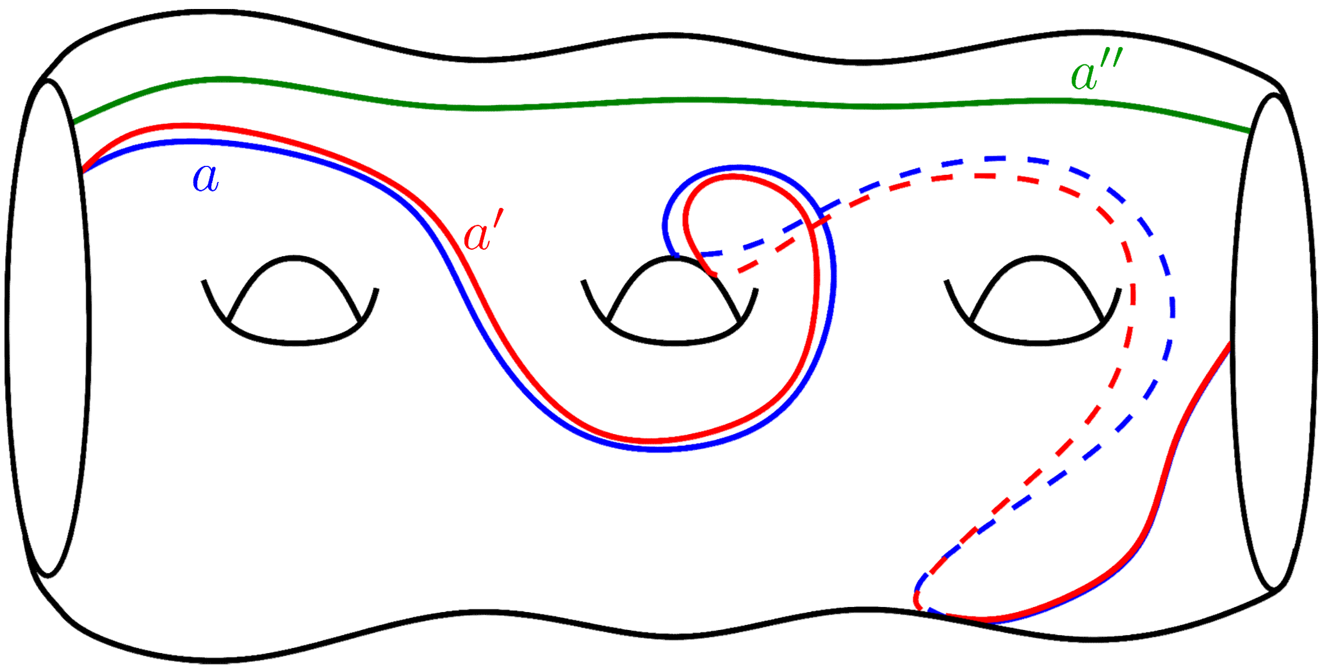
We know: $A(S)$ is connected.

Notice: Simplicial map $FA(S) \to A(S)$ given by taking isotopy classes.

Need to Show: Between any two isotopic essential simple proper arcs in S there is a path in $FA(S)$ connecting the two.

Proof:

Let $a, b \in FA(S)$ be isotopic, let $H : I \times [0,1] \to S$ be an isotopy from a to b.



Path of length 2 from a to a'

Compactness of $[0,1] \implies FA(S)$ connected

New Directions

Le Roux–Wolff:

Vertices: Nonseparating simple closed curves

Edges: Disjoint, or exactly one transverse point of intersection

Theorem: Let S be a connected, nonspherical surface without boundary. Then $\mathrm{Aut}(NC^1_{\pitchfork}(S)) \cong \mathrm{Homeo}(S)$

Booth–Minahan–Shapiro:

Vertices: Essential simple closed curves

Edges: Intersecting at most once

Theorem: For $g \geq 1$, $\mathrm{Aut}(FC^1(S_g)) \cong \mathrm{Homeo}(S_g)$

Notice: This provides evidence for an Ivanov-like metaconjecture for $\mathrm{Homeo}(S)$

Open Problems

Is $\mathrm{Aut} FC(S) \cong \mathrm{Homeo}(S)$ when...

- when S has punctures?
- when S is non-orientable?
- when S is of infinite-type?

Open Problems

Which simplicial complexes, $SC(S)$,
associated to a surface, S, have
$\mathrm{Aut}SC(S) \cong \mathrm{Homeo}(S)$?