Contact Geometry and Low-Dimensional Topology

PCMI Lecture 1

John Etnyre

Notes by: Shea Vick

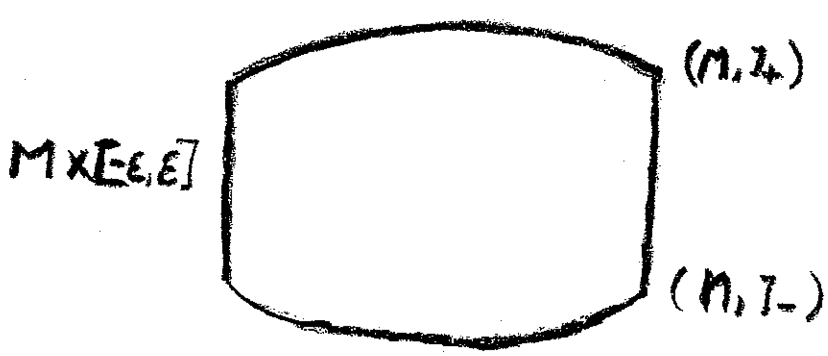
Contact Geometry has been a key tool in the following recent results in low-dimensional topology.

(1) **Kronheimer and Mrovka's** proof that non-trivial knots satisfy property p. (i.e. non-trivial surgery on non-trivial knots yields non-simply connected manifolds.)
(2) **Ozváth and Szabó's** proof that the unknot, trefoil, and figure eight knot are determined by surgeries on them. (i.e. Let K =unknot, trefoil, or figure eight knot. Let K' be a knot. If $\exists\ r \in \mathbb{Q}$ such that $S^3_r(K) \cong S^3_r(K')$ (orientation preserving), then $K = K'$.)
(3) **Ozváth and Szabó's** proof that Heegaard-Floer invariants detect the Thurston norm of a manifold and the seifert genus of a knot.

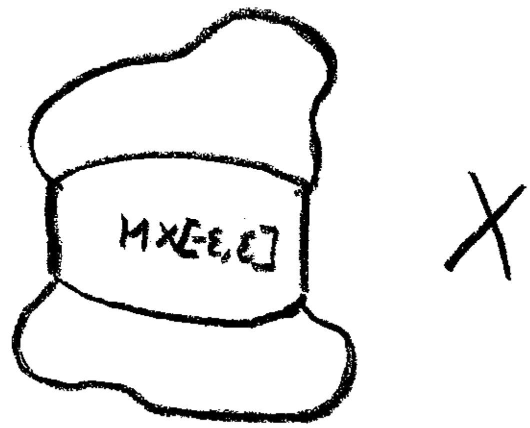
The following are some of the key ideas that went into these results. (All definitions will be given later.)

Start with a closed 3-manifold M and a surface $\Sigma \subset M$ such that Σ is minimal genus among surfaces homologous to it (assume the genus of Σ is positive).

(1) **Gabai** gives a taught foliation $\mathcal{F}$ that contains Σ as a leaf.
(2) **Eliashberg - Thurston** give a positive and negative contact structure $\xi_\pm$ on M that is C^0 close to $\mathcal{F}$.
(3) They also give a symplectic structure on $M \times [-\epsilon, \epsilon]$ that "dominates" (or "fills") $(M, \xi_+) \coprod (M, \xi_-)$.



(4) **Eliashberg, E.** find a closed symplectic manifold X that $M \times [-\epsilon, \epsilon]$ embeds into



these "caps" are constructed using

(a) **Giroux's** correspondence between open books and contact structures.

(b) **Eliashberg, Weinstein's** ideas on contact surgery and symplectic handle attachment.

(5) Uses Seiberg-Witten or Heegaard-Floer to conclude something about M and Σ based on the existence of X. (e.g. HF invariant of $X \neq 0 \Rightarrow HF^+(M, \mathfrak{s}_{\mathcal{F}}) \neq 0 \Rightarrow |\langle c_1(\mathfrak{s}_{\mathcal{F}}), [\Sigma] \rangle| \leq 2g - 2$; but since Σ is a leaf of $\mathcal{F}$ we have $|\langle c_1(\mathfrak{s}_{\mathcal{F}}), [\Sigma] \rangle| = 2g - 2$; thus the Heegaard-Floer basic classes detect the Thurston norm).

Then contact geometry part of the above is steps (2) $\rightarrow$ (4). The goal of these lecture is to

- Introduce the basic ideas in contact geometry.
- Explain the ideas behind (2), (3), and (4b), and explain what (4a) means.

Part 1: Basic Ideas and Definitions

Let M be an oriented 3-manifold.

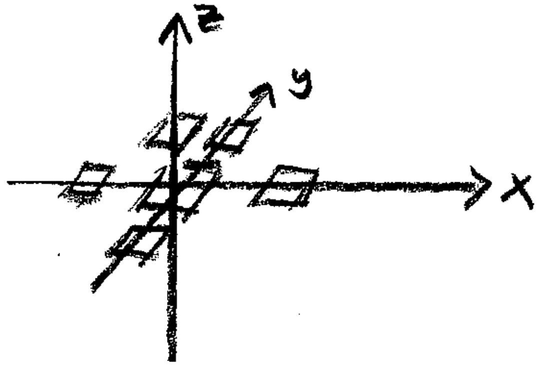
A **plane field** ξ on M can (locally) be given as the kernel of a 1-form α.

$$\xi_x = \ker(\alpha_x), \quad x \in M$$

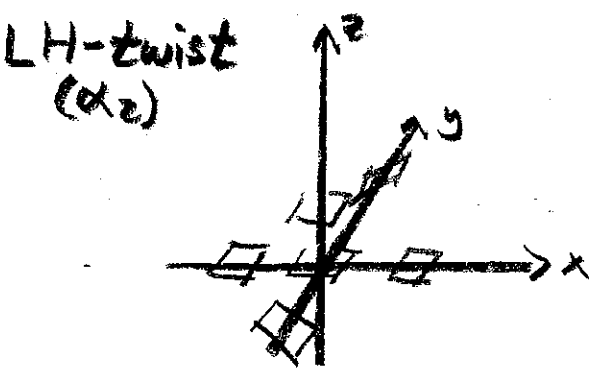
Exercise 1: Show that ξ is orientable if and only if α can be chosen as a global 1-form.

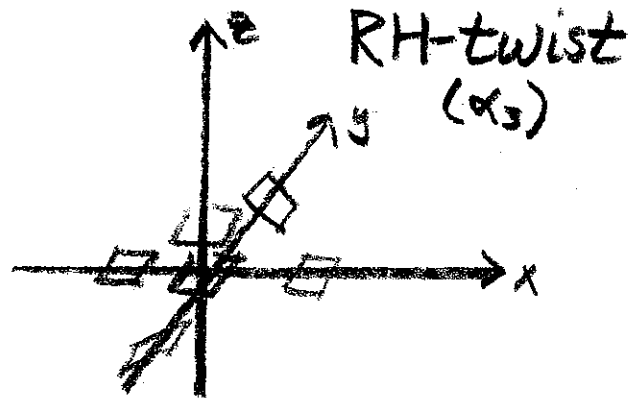
Examples:

(1) $\mathbb{R}^3$, $\xi_1 = \ker(\alpha_1)$, $\alpha_1 = dz$



(2) $\mathbb{R}^3$, $\xi_2 = \ker(\alpha_2)$, $\alpha_2 = dz - y\,dx$, or
$\mathbb{R}^3$, $\xi_3 = \ker(\alpha_3)$, $\alpha_3 = dz + y\,dx$





Definition

(1) ξ is a **foliation** if $\alpha \wedge d\alpha \equiv 0$.
(2) ξ is a positive (resp. negative) **contact structure** if $\alpha \wedge d\alpha$ is never zero and induces the given (resp. opposite) orientation on M (i.e. $\alpha \wedge d\alpha > 0$ (resp. $\alpha \wedge d\alpha < 0$)).
(3) ξ is a positive (resp. negative) **confoliation** if $\alpha \wedge d\alpha \geq 0$ (resp. $\alpha \wedge d\alpha \leq 0$).

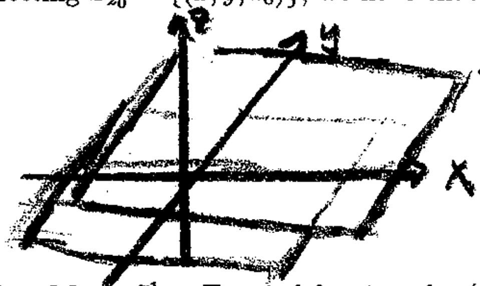
Exercise 2: Show that these definitions don't depend on the choice of α.

Frobenius Theorem "If a plane filed ξ is closed under Lie bracket, then $M = \coprod S_\lambda$ (S_λ a surface with $\xi_x = T_x S_\lambda$)."

Exercise 3: Show that $\alpha \wedge d\alpha \equiv 0$ if and only if ξ is closed under Lie bracket. (Here closed under Lie bracket means if v and w are sections of ξ then $[v, w]$ is also a section of ξ.)

Examples:

(1) $\mathbb{R}^3$, $\xi_1 = \ker(\alpha_1)$, $\alpha_1 = dz$. In this case, $d(dz) = 0$, so α_1 is a foliation. Letting $S_{z_0} = \{(x, y, z_0)\}$, we have that $T_{(}x, y, z_0)S_{z_o} = \xi_{(x,y,z_0)}$.



(planes parallel to the xy-plane.)

(2) Let $M = S^1 \times \Sigma_g$, and let $\xi = \ker(d\theta)$. Then $d(d\theta) = 0$, and $\xi_{(\theta,p)} = T_p(\{\theta\} \times \Sigma_g)$.
(3) ξ_2 is a positive contact structure on $\mathbb{R}^3$ since $\alpha_2 \wedge d\alpha_2 = dx \wedge dy \wedge dz$. Similarly, ξ_3 is a negative contact structure on $\mathbb{R}^3$ since $\alpha_3 \wedge d\alpha_3 = -dx \wedge dy \wedge dz$.
(4) $S^3 \subset \mathbb{C}^2$ with $\xi_4 = \ker(\alpha_4)$, where $\alpha_4 = r_1^2\, d\theta_1 + r_2^2\, d\theta_2$.

Exercise 4: Check that $\alpha_4 \wedge d\alpha_4 > 0$ on S^3.

One natural question to ask at this point is the following. "How prevalent are contact structure and foliations?"

Answer: All oriented 3-manifolds have foliations and positive (negative) contact structures.

Lemma Given a plane field ξ one can find local coordinates (x, y, z) such that α can be written

$$\alpha = dz - a(x, y, z)\, dx$$

Lemma

(1) ξ is a positive (resp. negative) contact structure if and only if $\partial a/\partial y > 0$ (resp. $\partial a/\partial y < 0$).
(2) ξ is a foliation if and only if $\partial a/\partial y \equiv 0$

Exercise 5: prove the previous two lemmas. (Hint: For the first lemma, let $\phi : [-1, 1] \times [-1, 1] \to M$ be an embedded disk such that $d\phi(\partial_z) \pitchfork \xi$. Now use ϕ and the flow of v to build the desired coordinate neighborhood.)

Theorem (Darboux, Pfaff)

(1) If ξ is a foliation, then we can take $\alpha = dz$.
(2) If ξ is a positive (resp. negative) contact structure, then we can take $\alpha = dz - y\, dx$ (resp. $\alpha = dz + y\, dx$).

This tells us that, locally all foliations and positive/negative contact structures "look the same". So unlike in Riemannian geometry, nothing interesting happens locally. Instead, contact structures and foliations give us global information about our manifold.

The above theorem is one example of the many similarities between contact structures and foliations. There is, however, one very big difference. Foliations have non-trivial deformations, whereas contact structure do not.

Theorem (Gray's Theorem) Let ξ_t, $t \in [0,1]$ be a family of contact structure on M^3, then there exists a family of diffeomorphisms $\psi_t : M \to M$ such that $(\psi_t)_*(\xi_t) = \xi_0$.

In the above theorem, we call ψ_t a **contactomorphism** (for a fixed t), and we say that ξ_0, ξ_1 are **isotopic**.

This tells us that we can't "deform" a contact structure. On the other hand, foliations DO have non-trivial deformations, as can be seen in the following example.

Example:
Let $\mathcal{F}_s$ = the foliation of T^2 by lines of slope s, and let $\xi_s = \mathcal{F}_s \times S^1$ be a foliation of T^3.

Exercise 6: Show that there does not exist a family of diffeomorphisms $\psi_s : T^3 \to T^3$ such that $(\psi_s)_*(\xi_s) = \xi_0$.

Contact Geometry and Low-Dimensional Topology

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John Etnyre
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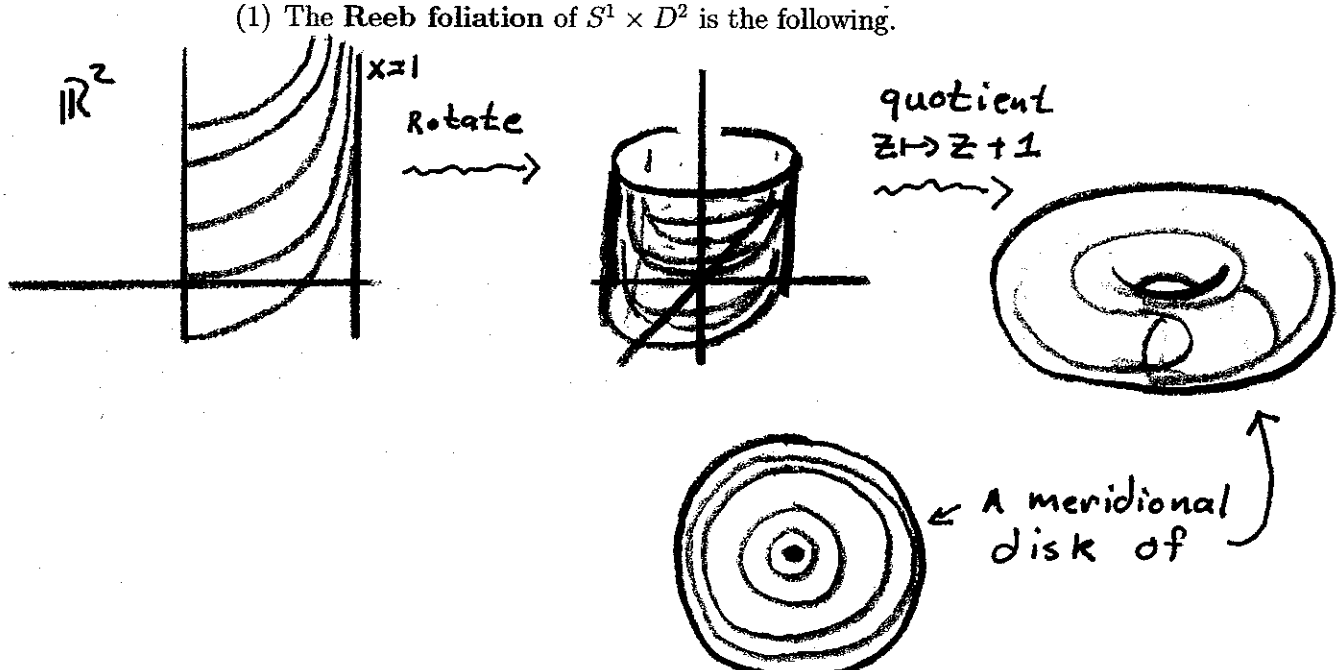
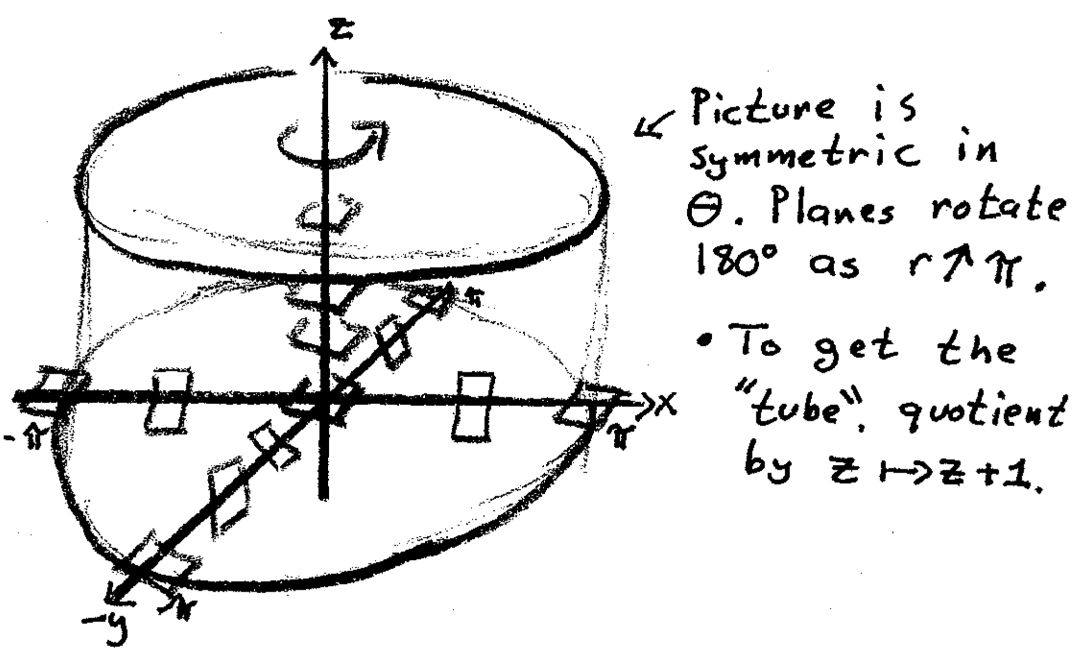
To see more similarities between contact structures and foliations, we need a few more definitions and examples.

Examples:

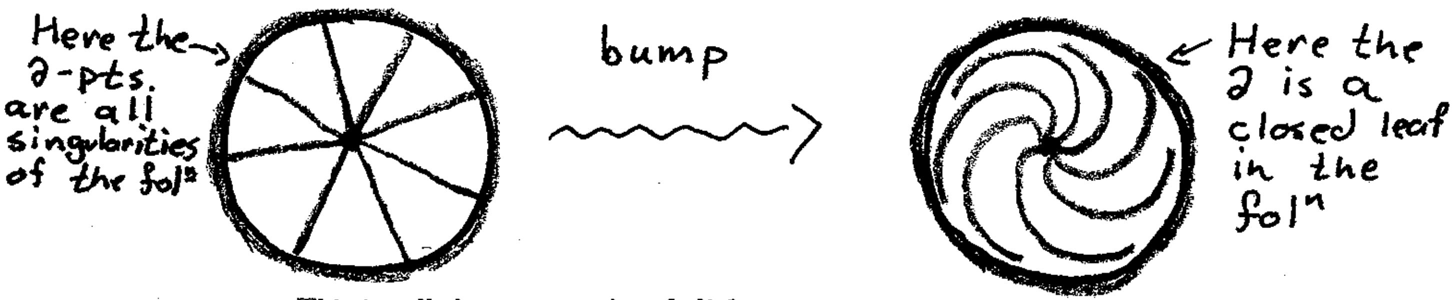
(1) The **Reeb foliation** of $S^1 \times D^2$ is the following.



(2) A **Lutz tube** is a contact structure on $S^1 \times D^2$ obtained as follows. Let $S^1 \times D^2 = \{(r, \theta, z) \mid \ r \leq \pi\} \subset \mathbb{R}^3$, and let $\xi_{ot} = \ker(\cos(r)\, dz - r\sin(r)\, d\theta)$.



If D is a miridional disk, then the singular foliation induced on D is.



This is called an **overtwisted disk**.

Exercise 1: Show that if $\widetilde{D}$ is given by "bumping the interior of D slightly in the positive z-direction, while fixing ∂D", then the above picture of the singular line field on $\widetilde{D}$ is accurate.

It is easy to construct foliations with Reeb components and contact structures with Lutz tubes, but it is harder to construct foliations and contact structures without them.

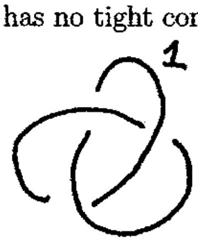
Exercise 2: Try to prove the above statement. (This is a difficult exercise if you don't know how to get started, but is still probably worth pondering.)

A contact structure without overtwisted disks (or equivalently without Lutz tubes) is called **tight**. Otherwise it is called **overtwisted**.

Eliashberg classified overtwisted contact structures on 3-manifolds. In short, there are lots of them (infinitely many), and, up to isotopy, they are in 1-1 correspondence with homotopy classes of plane fields.

Exercise 3: Show that all 3-manifolds have infinitely many homotopy classes of plane fields.

Not all 3-manifolds have tight contact structures! For instance the Poincaré homology sphere has no tight contact structure (E-Honda).



Exercise 4: Find a hyperbolic manifold without a tight contact structure. (This might be hard.)

Recall an oriented 2-dimensional bundle (like ξ) has an Euler class $e \in \mathrm{H}^2(M;\mathbb{Z})$.

Theorem

(1) (Thurston) If ξ is a Reebless foliation, then for any surface Σ embedded in M,

$$\begin{aligned} |\langle e(\xi), [\Sigma]\rangle| &\leq -\chi(\Sigma) \quad \text{if } \Sigma \neq S^2 \\ &= 0 \qquad\quad \text{otherwise} \end{aligned}$$

(2) (Eliashberg) If ξ is a tight contact structure, then the same inequality/equality holds.

Exercise 5: Show that this theorem implies that only finitely many $c \in \mathrm{H}^2(M;\mathbb{Z})$ can be the Euler class of a Reebless foliation or a tight contact structure.

This theorem and a result of Gabai (see below) imply that the Euler classes of Reebless foliations characterize the Thurston norm. Then Eliashberg-Thurston tell us that the same is true for tight contact structures.

How do we find tight contact structure?

Recall that a 4-manifold X is **symplectic** if there exists a 2-form ω such that $d\omega = 0$, and $\omega \wedge \omega$ is never zero.

If $M = \partial X$, and ξ is a contact structure on M, then we say ω **dominates** ξ if $\omega|_\xi > 0$.

If (M,ξ) is one component of a contact manifold (Y,ξ'), and (X,ω) is a compact symplectic 4-manifold such that ω dominates ξ', then we say (X,ω) is a **week semi-filling** of (M,ξ) (if Y is connected, then we call this a **weak-filling**).

Theorem (Eliashberg-Gromov) If (M,ξ) is weakly (semi-)fillable, then ξ is tight.

Example:
Consider $S^3 \subset \mathbb{C}^4$, with $\xi = \ker(\alpha)$, where $\alpha = r_1^2\, d\theta_1 + r_2^2\, d\theta_2$. Then we have that $\omega = d\alpha = 2r_1\, dr_1 \wedge d\theta_1 + 2r_2\, dr_2 \wedge d\theta_2$ is a symplectic form on $\mathbb{C}^2$ (and therefore also on B^4).

Now $S^3 = \partial B^4$, and $\omega|_\xi = d\alpha|_\xi > 0$ (since $\alpha \wedge d\alpha > 0$). This tells us that (S^3,ξ) is tight (i.e. has no Lutz tubes).

The above is in stark contrast with a theorem of Novikov, showing that any foliation of S^3 has Reeb components.

We will find many other fillable contact structures later, but it is worth noting that there are tight non-fillable contact structures (see E-Honda).

Part 2: (Con-)Foliations Into Contact Structures

Consider the interesting foliation of $S^2 \times S^1$.

$$\zeta_{(p,\theta)} = T_p(S^2 \times \{\theta\})$$

Theorem (Eliashberg-Thurston) Any oriented C^2-foliation ξ on an oriented 3-manifold M, other than the above foliation of $S^2 \times S^1$, may be C^0-approximated by positive and negative contact structures.

We say ξ can be **C^k-deformed** into a contact structure if there is a C^k-family ξ_t such that ξ_t is contact for $t > 0$ and $\xi_0 = \xi$.

ξ is **C^k-approximated** by a contact structure if in any C^k-neighborhood of ξ, there is a contact structure (need not be a deformation).

Example: On T^3 consider $\alpha_n^t = dz + t\,(\cos(2\pi nz)\ dx + \sin(2\pi nz)\ dy)$. At $t = 0$, we get a foliation of T^3 by T^2's. For $t > 0$, we get contact structure ξ_n^t.

Note that Gray's theorem tells us that ξ_n^t is independent of t, so we denote it ξ_n.

Kanda, Giroux tell us that ξ_n are all distinct and give all the tight contact structures on T^3.

Remarks:

(1) A foliation can be approximated by (deformed into) infinitely many different contact structures!
(2) The above theorem implies only that a foliation can be approximated by contact structure, not deformed (though it still may be true for deformations).
(3) We loose smoothness on C^0 (C^2 is still maybe true).

Contact Geometry and Low-Dimensional Topology

PCMI Lecture 3

John Etnyre
Notes by: Shea Vick

Theorem (Eliashberg-Thurston) Any oriented C^2-foliation ξ on an oriented 3-manifold M, other than the above foliation of $S^2 \times S^1$, may be C^0-approximated by positive and negative contact structures.

Why is ξ on $S^2 \times S^1$ special?

Reeb Stability (for confoliations) Suppose a confoliation ξ on M admits an embedded integral 2-sphere S (i.e. $\forall x \in S,\ T_xS = \xi_x$). Then (M, ξ) is diffeomorphic to $(S^2 \times S^1, \zeta)$.

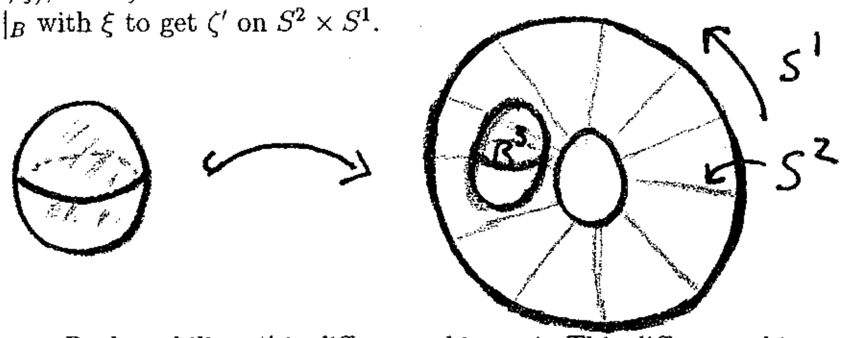
Exercise 5: Try to show this if ξ is a foliation. (Hint: Try to show that the subset of M foliated by S^2's is closed and open.)

Similarly,

Theorem Any confoliation of $S^2 \times S^1$, C^0-close to the foliation ζ is a foliation diffeomorphic to ζ. (Here we just need that the S^1 factor is transverse to the plane field.)

Theorem Let ξ be a confoliation on the 3-ball B which is standard near ∂B (i.e. near ∂B, ξ is given by $\ker(dz)$). Then ξ is a foliation and diffeomorphic to the standard foliation on all of B.

Proof
Given (B, ξ), take ζ on $S^2 \times S^1$. Now embed B into $S^2 \times S^1$ as shown below. Now replace $\zeta|_B$ with ξ to get ζ' on $S^2 \times S^1$.



According to Reeb stability, ζ' is diffeomorphic to ζ. This diffeomorphism sends ξ to $\zeta|_B$.

♣

Not only do these theorems explain the unique nature of ζ on $S^2 \times S^1$, they also allow us to see you can't "locally" perturb a foliation into a contact structure.

The proof of the Eliashberg-Thurston theorem involves two steps. Given a foliation on M.

(1) Perturb ξ into a confoliation ξ' such that ξ' is contact on a "sufficiently large" portion of M.
(2) Perturb ξ' into a contact structure.

Let's think about step 2 first, so we can figure out what "sufficiently large" in step 1 means.

Given a confoliation ξ' let

$$H(\xi') = \{x \in M \mid \xi_x \text{ is contact at } x \ \text{ (i.e. } (\alpha \wedge d\alpha)_x > 0)\}.$$

This is called the **hot region** (or **contact region**). Now let

$$G(\xi') = \{x \in M \mid \exists \text{ a path } \gamma \text{ from } x \text{ to } y \text{ with } y \in H(\xi') \text{ and } \gamma \text{ tangent to } \xi'\}.$$

Theorem If $G(\xi') = M$ then ξ' can be C^∞-deformed into a contact structure.

There are two ways to prove this.

- An **analytic way** due to Altschuler.
- A **topological way** due to Eliashberg and Thurston.

Analytic Way

Choose a Riemannian metric on M. Choose a 1-form α such that $\ker(\alpha) = \xi'$ and $|\alpha| = 1$ at all points of M.

Consider the equations

$$\begin{aligned} \frac{\partial}{\partial t}\beta &= *(\alpha \wedge df) \\ \beta(\cdot, 0) &= \alpha(\cdot) \end{aligned}$$

Where $f = *(\alpha \wedge d\beta + \beta \wedge d\alpha)$. Now try to solve for $\beta \in \Omega^1(M) \times \mathbb{R}^+$.

This is a weakly-parabolic system. Altschuler proved that there exists a unique smooth solution for $t \in [0, \infty]$. The function evolves by the equation

$$\frac{\partial f}{\partial t} = \Delta_\alpha f + \nabla_X f$$

Where X is some time-dependent vector field, and Δ_α is the "Laplacian on $\ker(\alpha)$".

A version of the maximum principal gives

If $f(p, 0) > 0$, then $f(p, t) > 0$ for all $t > 0$.

If q connected to a point p by a path tangent to ξ' and $f(p,0) > 0$, then $f(q,t) > 0$ for all $t > 0$.
"Heat flows infinitely fast to all points on M accessible to the hot region."

So if $G(\xi') = M$, then $f(p,t) > 0$ for all $p \in M$ and $t > 0$.

Now consider

$$\eta = \alpha + \epsilon\beta_1 \qquad \beta_1 = \beta(\cdot, 1)$$

Then

$$d\eta = d\alpha + \epsilon d\beta_1,$$

and hence

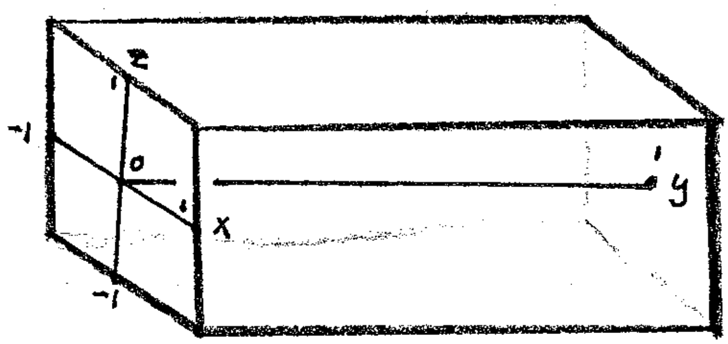
$$\eta \wedge d\eta = \alpha \wedge d\alpha + \epsilon(\alpha \wedge d\beta_1 + \beta_1 \wedge d\alpha) + \epsilon^2\beta_1 \wedge d\beta_1.$$

Thus, for ϵ small enough, η is a contact form.

♣

Topological Way

First observe we have a neighborhood V,

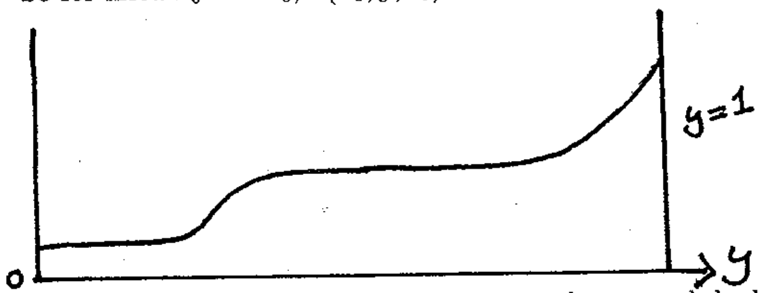


and in V $\xi' = \ker(\alpha)$, $\alpha = dz - a(x,y,z)dx$. Suppose ξ' is contact near $y = 1$. Then we will show there is a C^k-small deformation of ξ' in V to a contact structure that agrees with ξ' along ∂V.

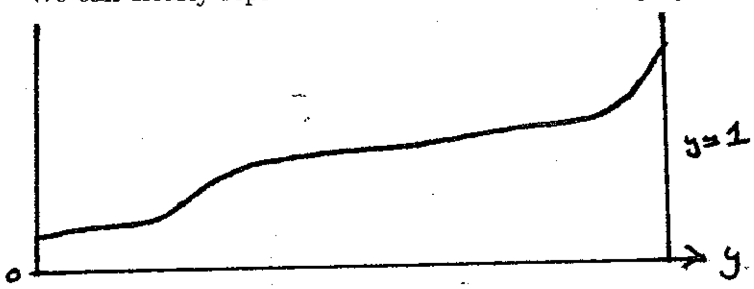
Since ξ' is a confoliation, we know that

$$\frac{\partial a}{\partial y} \geq 0 \quad \text{in } V \text{ and}$$
$$\frac{\partial a}{\partial y} > 0 \quad \text{near } y = 1, \text{ since } \xi' \text{ is contact here.}$$

So for fixed x_0 and z_0, $a(x_0, y, z_0)$ is of the form



We can clearly replace this with a new a whose graph looks like



In particular, $\partial a/\partial y(x_0, y, z_0) > 0$ for this new a.

> **Exercise d:** Show you can do this for all x, z simultaneously.

Now pick arcs $\gamma_1, \ldots, \gamma_n$ such that γ_i is tangent to ξ', each γ_i has a neighborhood V_i as above, and the V_i's cover $M - H(\xi')$.

We can fix ξ' on one V_i at a time. We needs to be cautious of the fact that as you change ξ' on V_i, we might mess up the modles on the other V_j's.

> **Exercise 6:** Convince yourself that if the perturbation on V_i is sufficiently small, then you can slightly modify the V_j's so that they still have the appropriate form.

♣

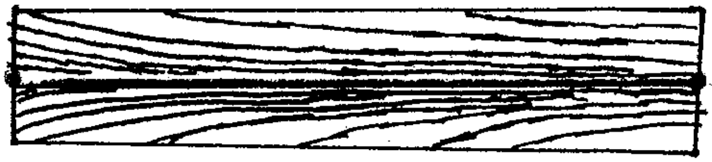
Now we know that in step 1, we need to perturb ξ to a confoliation ξ' such that $G(\xi') = M$ (i.e. every point in M is connected to a contact region by a path tangent to ξ'.)

For this we need to consider the **holonomy** of the foliation ξ.

Let γ be a closed curve in M tangent to ξ. Let $A = (-\epsilon, \epsilon) \times S^1$ be an embedded annulus in M such that

(1) $\{0\} \times S^1 = \gamma$
(2) A is transverse to ξ

ξ induces a **line field** on A.



Considering this foliation of A, pick $p \in S^1$ and set $I = (-\epsilon, \epsilon) \times \{p\}$.

When defining a return map

$$\phi_\gamma : I \to I,$$

we note that ϕ_γ might not be defined on all of I, but will be defined in a neighborhood of 0 (since $\phi_\gamma(0) = 0$).

ϕ_γ is called the **holonomy** of ξ at p along γ.

Note: ϕ_γ only depends on the homotopy class of γ (through curves tangent to ξ).

Definition

Holonomy is called

- **non-trivial** if $\phi_\gamma \neq \mathrm{id}_I$
- **non-trivial linear holonomy** if $\phi'_\gamma(0) \neq 1$
- **attracting** if $|\phi_\gamma(x)| < |x|$
- **repelling** if $|\phi_\gamma(x)| > |x|$
- **weakly attracting** if $|\phi_\gamma(x)| < |x|$ on intervals arbitrarily close to 0.
- **weakly repelling** if $|\phi_\gamma(x)| > |x|$ on intervals arbitrarily close to 0.

Contact Geometry and Low-Dimensional Topology

PCMI Lecture 4

John Etnyre
Notes by: Shea Vick

Recall that we're trying to prove the Eliashberg-Thurston theorem. To finish things off, we need to show that ξ can be perturbed into a confoliation ξ' such that every point in M can be connected to a point where ξ' is a contact structure. To prove this, we use holonomy?

Theorem (M, ξ) a C^k-foliation

(1) If Γ is a curve in a leaf with non-trivial linear holonomy, then ξ can be C^k-deformed into a positive (negative) contact structure in a neighborhood U of Γ, leaving ξ fixed outside a larger neighborhood.

(2) If Γ has weakly attracting holonomy, then ξ can be C^0-approximated by a positive (negative) contact structure in a neighborhood V of Γ, leaving ξ fixed outside a large neighborhood.

proof

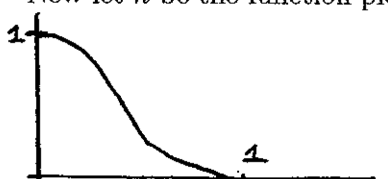
(1) Let $U = \underbrace{\Gamma}_{x} \times \underbrace{[-1,1]}_{y} \times \underbrace{[-1,1]}_{z}$ be a neiborhood such that $\xi = \ker(\alpha)$, where $\alpha = dz - a(x,z)dx$.

Claim: We can choose coordinates (x, y, z) such that

$$\frac{\partial a}{\partial z} \geq c \quad \text{for some} \quad c > 0.$$

Exercise 1: Prove the above claim. (Hint: Consider $\Gamma \times [-1-1]$, and find an a with the desired properties so that the induced return map is the same as the given one. Now find a diffeomorphism to the original $\Gamma \times [-1,1]$ so that the flow is preserved. Now construct the neighborhood as before.)

Now let h be the function pictured below.

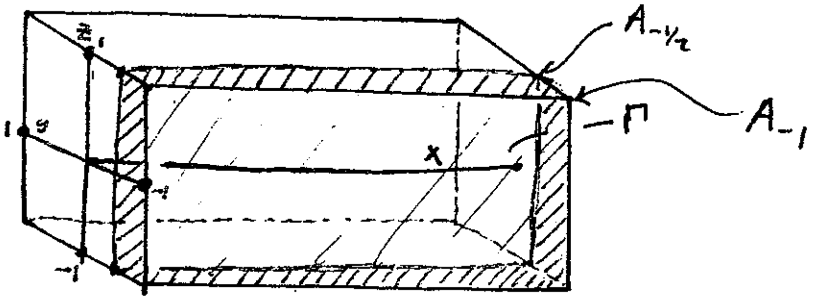


Let $\beta = h(y^2 + z^2)\, dy$, and note that

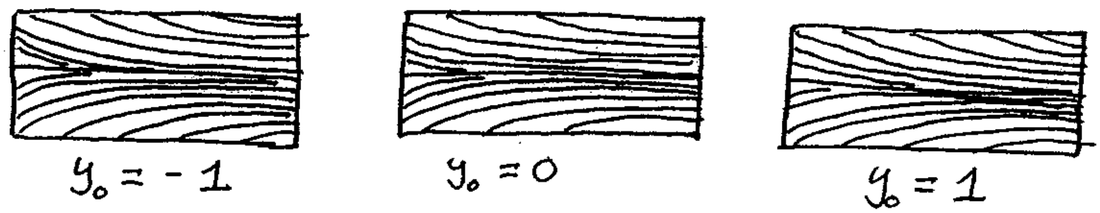
$$\alpha \wedge d\beta = \underbrace{-ah' 2z\, dx \wedge dy \wedge dz}_{\geq 0} + \underbrace{\frac{\partial a}{\partial z} h\, dx \wedge dy \wedge dz}_{>0}.$$

Exercise 2: Check that $\widetilde{\alpha} = \alpha + \epsilon\beta$ is a contact form for ϵ sufficiently small.

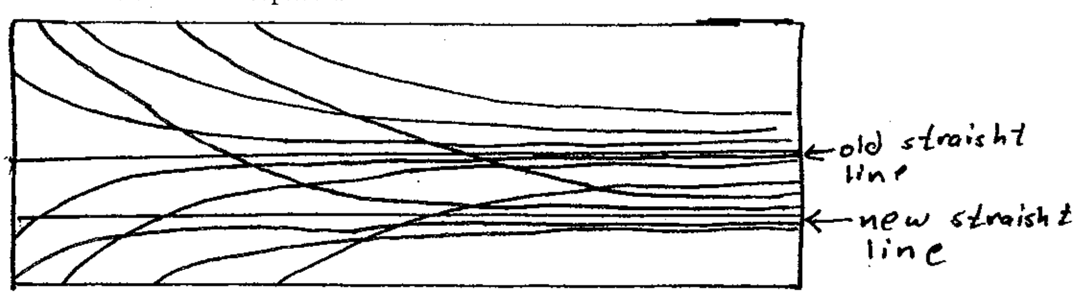
A More General Idea



Consider the vertical annuli $A_{y_0} = \{(x, y, z) \mid y = y_0\}$.



In the region R between A_{-1} and $A_{-1/2}$ apply a diffeomorphism that is the identity on $\partial U \cap R$ and so that on $A_{-1/2}$ all the tangents to the foliation are rotated clockwise from where they were. For example $A_{-1/2}$ should look something like the following after the diffeomorphism.



Now as y goes from $A_{-1/2}$ to A_1 rotate these tangents back to where they were. This turns the region between $A_{-1/2}$ and A_1 into a contact region.

> **Exercise 3:** Make the above argument more rigorous. (Hint: in the linear holonomy case, the following lemma will be helpful. In the non-linear holonomy case, a similar lemma is true.)

Lemma Let v_x be a family of smooth functions on $[-1, 1]$ such that $v_x(0) = 0$ and v_x is monotonically increasing in x. Then there exists a diffeomorphism $f : [-1, 1] \to [-1, 1]$, C^∞-close to the identity and C^∞-tangent to the identity at $\{-1, 1\}$ satisfying

$$f'(z)v_x(z) > v_x(f(z)) \quad \text{for all } z \in (-1, 1) \text{ and } x.$$

Similarly one can find such an f satisfying the opposite inequality.

Hint 2: take $v_x(z) = a(x, z)$, with a as above.

The moral of the story is that holonomy is good. If we can arrange that every leaf in a foliation is arbitrarily close to a leaf with holonomy, then we would be done.

With that in mind, let ξ be a foliation on M, and define a **minimal set** in M to be a non-empty closed union of leaves that contains no such smaller set. (i.e. a closed union of leaves that is the closure of any leaf in it.)

Exercise 4: Show that any leaf limits to a minimal set.

Thus we just need to see we can perturb ξ so that every minimal set has curves with holonomy.

Theorem In a C^2-foliation on a 3-manifold M, every minimal set is one of the following.

(1) All of M. (In this case the foliation is called **minimal.**)
(2) A closed, compact leaf.
(3) An **exceptional minimal set**. (There are only finitely many of these.)

Theorem(Stacksteder) An exceptional minimal set contains leaves with linear holonomy.

We now consider the two remaining possibilities. If ξ is minimal, then either it has has holonomy or it doesn't. If ξ has holonomy, then a theorem of Ghys tells us that it has linear holonomy, and we're done. If it has no holonomy, then a theorem of Tishler implies that ξ can be C^0-approximated by a fibration over a circle (this is one of the places "C^0" and "approximate" enter the picture).

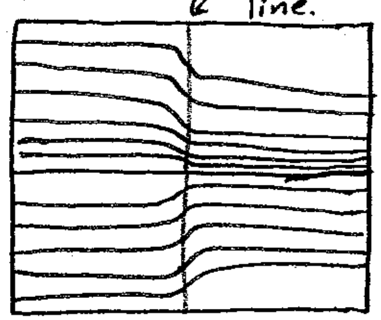
Now perturb the picture as shown below.

shifting things along this line.

Have $\Sigma \hookrightarrow M$
$\downarrow$
S^1

think of this as

$$M = \Sigma \times [0,1]/\sim$$
$$(0,x) \sim (1, f(x))$$
$$f : \Sigma \to \Sigma$$



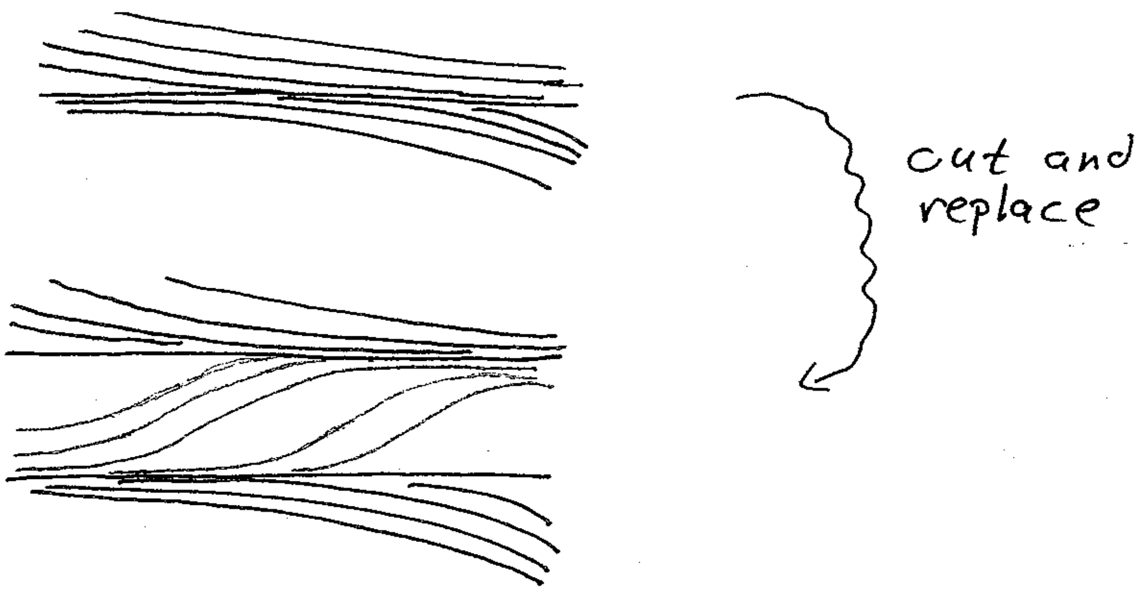
We have 2 closed leaves, but they both have holonomy.

This perturbation gives us linear holonomy, finishing the minimal case.

In the case where the minimal set is a closed leaf Σ, we first perturb our foliation ξ to get a new foliation with only finitely many closed leaves. Now we have three basic cases (this three case list is actually not exhaustive, but is "essentially" complete). In the first case, our leaf has linear or weakly attracting holonomy, and we're done.

In the second case, our closed leaf has only trivial holonomy. If this happens, we can apply a different form of Reeb stability to conclude that our foliation is $\Sigma \times (-1, 1)$ in a neighborhood of σ. This is a contradiction since ξ has only finitely man closed leafs.

The final case to consider is where ξ has holonomy, but it is neither weakly attracting, nor weakly repelling. This tells us that our holonomy is weakly attracting on one side, and weakly repelling on the other. If this is the case, we cut out Σ, and replace it as follows.



Contact Geometry and Low-Dimensional Topology

PCMI Lecture 5

John Etnyre
Notes by: Shea Vick

Part 3: Taut foliations and fillability

Given a foliation ξ on M we can perturb it into a contact structure ξ', but what can we say about ξ'? Is ξ' tight, fillable, overtwisted,...?

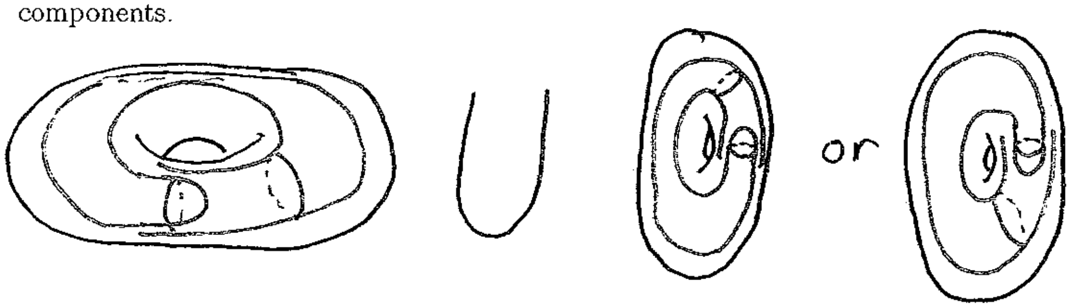
Recall that if ξ is Reebless, then for all $\Sigma \neq S^2$, we have that

$$|\langle e(\xi), [\Sigma]\rangle| \begin{cases} \leq \chi(\Sigma) & \text{if } \Sigma \neq S^2 \\ = 0 & \text{otherwise} \end{cases}$$

If ξ' is C^0-close to ξ, then we get the same inequality for ξ'. In particular, this indicates that ξ' might be tight. A slightly stronger form of this inequality (for surfaces with boundary) actually does show that ξ' is tight.

So if ξ is Reebless, we have that ξ' is tight. But what if ξ is not Reebless; can ξ' still be tight?

Example: Consider S^3 built from two solid tori, each of which is given the standard Reeb foliation. There are two ways to do this, both of which give S^3 with Reeb components.



Claim One of these perturbs to a tight contact structure on S^3, and the other to an overtwisted contact structure.

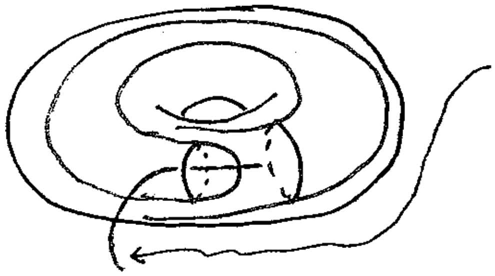
Exercise 1: Figure out which is which.

Definition A foliation is called **taut** if each leaf is intersected by a closed transverse curve.

An equivalent definition of taut is that there exists a vector field v transverse to ξ that preserves some volume for Ω on M.

Exercise 2: Show that the two definitions of taut are equivalent.

Note: If ξ has a Reeb component, then ξ is not taut.



now that this curve has crossed the boundary, it can't get back inside to close up.

This tells us that if a foliation is taut, then it is also Reebless.

Theorem (Eliashberg-Thurston) Suppose that ξ' is a contact structure that is C^0-close to ξ. Then if ξ is taut, ξ' is symplectically fillable.

Proof
Let $X = M \times [-1,1]$, and let $\xi = \ker(\alpha)$. Let ξ_+, ξ_- be positive, negative contact structures C^0 close to ξ. Set $\widetilde{\omega} = \iota_v \Omega$ (from above), and note that

(1) $\widetilde{\omega}|_\xi > 0$ (If $\widetilde{\omega}|_\xi < 0$, then reverse v.)
(2) $d\widetilde{\omega} = d\iota_v\Omega = d\iota_v\Omega + \iota_v d\Omega = \mathcal{L}_v\Omega = 0$

Set $\omega = \widetilde{\omega} + \epsilon\, d(t\alpha) = \widetilde{\omega} + \epsilon\, dt \wedge \alpha$. Then $d\omega = 0$, and $\omega \wedge \omega = 2\epsilon\, \widetilde{\omega} \wedge dt \wedge a$.

Exercise 3: Show that $\omega \wedge \omega > 0$.

This tells us that ω is a symplectic form for X, and that $\omega|_{\xi\times\{\pm1\}} > 0$. Thus, since ξ' is C^0-close to ξ, we have that $\omega|_{\xi'\times\{\pm1\}} > 0$. Similarly $\omega|_{\xi_\pm} > 0$. Therefore ω fills $(M,\xi') \coprod (-M,\xi_-)$, (note ξ_- is a positive contact structure on $-M$).

We can now construct lots of tight contact structures using the following theorem.

Theorem (Gabai) Let M be an irreducible 3-manifold, and let $F \subset M$ be an orientable surface representing a non-trivial homology class. Suppose F has minimal genus among all representatives of this class. Then there exists a taut foliation ξ on M with F as a leaf.

Corollary Let M and F be as above. Then there is a fillable contact structure ξ' on M such that $\langle e(\xi'), [\Sigma]\rangle = \pm(2 - 2g(\Sigma))$.

Actually we need to argue a little more to get this corollary. The ξ from Gabai's theorem is only C^2 if the genus of Σ is ≥ 2. If the genus of Σ is equal to one, then the only non-C^2-part of the foliation is along Σ, and this will have linear holonomy. So the above proof still works.

Part 4: Constructing Symplectic Fillings and Legendrian Surgery

Let (X, ω) be a symplectic manifold. A vector field v is **symplectically dialating** if

$$\mathcal{L}_v \omega = \omega.$$

Suppose v is transverse to $M = \partial X$, and that v points out of M. Set $\alpha = (\iota_v \omega)|_M$, then

$$d\alpha = d\iota_v\omega + \iota_v d\omega = \mathcal{L}_v\omega = \omega.$$

So

$$\alpha \wedge d\alpha = (\iota_v\omega) \wedge \omega = \frac{1}{2}\iota(\omega \wedge \omega)$$

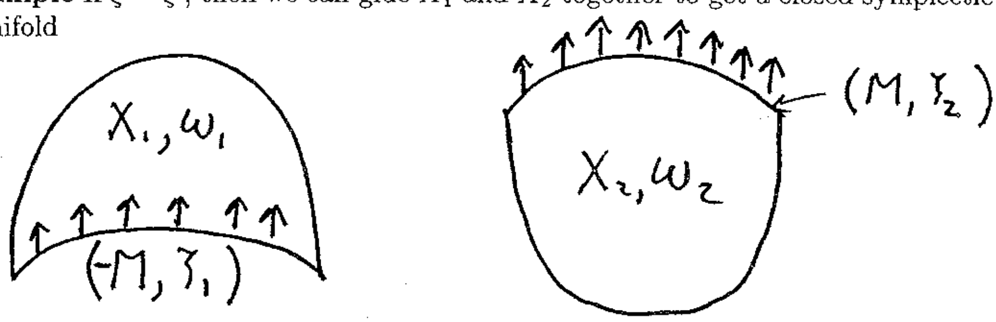
This tells us that α is a contact form on M.

A contact manifold (M, ξ) is said to be **strongly filled** by a compact symplectic manifold (X, ω) if there exists a dialating vector field v for ω that is transversely pointing out of $M = \partial X$ and $\iota_v\omega$ is a contact form for ξ. We also say (X, ω) is a **strong convex** filling of (M, ξ) (will be **concave** if v points inward).

Exercise 4: Show that if (X, ω) strongly fills (M, ξ), then (X, ω) weakly fills (M, ξ).

Why do we care so much about strong filings? They allow us to make gluing arguments work.

Example If $\xi = \xi'$, then we can glue X_1 and X_2 together to get a closed symplectic manifold

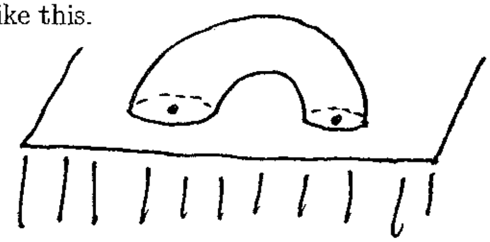


Exercise 5: Show that in the above situation, you can glue X_1 and X_2 together to get a closed symplectic manifold. (this might be a bit hard if you haven't ever studied symplectic geometry.)

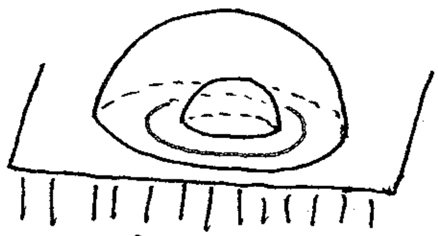
Exercise 6: If M is a homology sphere then a weak filling can be made strong.

Now for some four dimensional topology. We start with a 4-manifold X.

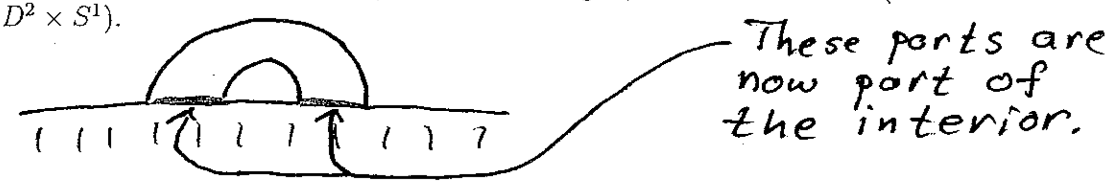
A **1-handle** is $h^1 = D^1 \times D^3$ glued to ∂X along $(\partial D^1) \times D^3 = S^0 \times D^3 = \{2-pts\} \times D^3$. To glue h^1 down, we just need to identify the two points. In three dimensions, the picture looks like this.



A **2-handle** is $h^2 = D^2 \times D^2$ glued to ∂X along $(\partial D^2) \times D^2 = S^1 \times D^2$. To glue h^2 to ∂X we must identify a knot K (that $S^1 \times pt$ will be glued to) and a framing of the normal bundle of K. In three dimensions, the picture looks as follows.



Note that if $X' = X \cup h^2$ glued along K with framing $\mathcal{F}$, then $\partial X' = \partial X - (S^1 \times D^2) \cup (D^2 \times S^1)$.



So $\partial X'$ is obtained form ∂X by Dehn surgery on K with framing $\mathcal{F}$.

Exercise 7: Check that this is the correct Dehn surgery. (Hint: think about where $\partial D^2 \times pt$ is mapped.)

Definition Let (M, ξ) be a contact 3-manifold. A knot K in M is **Legendrian** if $T_x K \subset \xi_x$ for all $x \in K$.

Note that if $v_x \in \xi_x$ is a vector transverse to K along K, then it defines a framing of K called the **contact framing**.

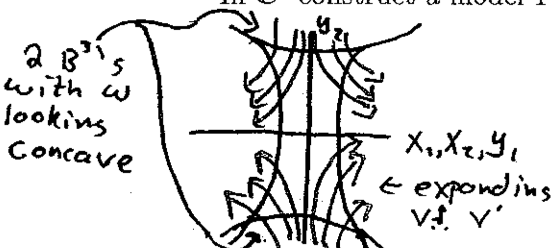
Theorem (Weinstein) If (X, ω) is a symplectic manifold with strongly/weakly convex boundary and X' is obtained from X by attaching a 1-handle or attaching a

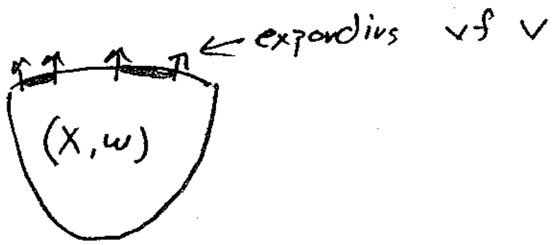
2-handle along a Legendrian knot in ∂X with framing one less than the contact framing, then ω extends to a symplectic form ω' on X' such that $\partial X'$ is strongly/weakly convex.

Note that if X' is obtained from X by a 2-handle attachment as in the theorem, then $\partial X' = (M', \xi')$. If $(M, \xi) = \partial(X, \omega)$, then we say that (M', ξ') is obtained from (M, ξ) by Legendrian surgery.

Idea of the proof for strong filling:

In $\mathbb{C}^2$ construct a model 1-handle.





Use v and v' to glue the handle to X.

> **Exercise 8:** Try to make this work. (Hint: you need a strong form of Darboux's theorem and flows of v and v'.)

> **Exercise 9:** Try to figure out the 2-handle attachment. In particular, why must you attach with contact framing -1?

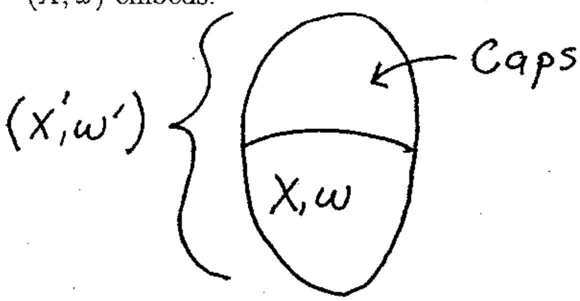
Contact Geometry and Low-Dimensional Topology

PCMI Lecture 6

John Etnyre
Notes by: Shea Vick

Part 5: Open Books and Symplectic Caps

Theorem (Eliashberg, E.) Let (X, ω) be a compact symplectic manifold which weakly fills (M, ξ), then there is a closed symplectic manifold (X', ω') into which (X, ω) embeds.



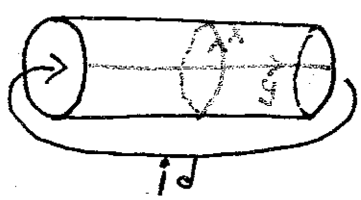
We need one more ingredient to prove this.

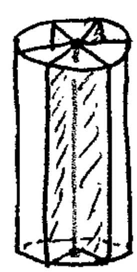
Definition Let Σ be a compact, oriented surface with boundary, and let $\phi : \Sigma \to \Sigma$ be an orientation preserving diffeomorphism that is the identity near the boundary of Σ. Then

$$T_\phi = \Sigma \times [0,1]/(x,0) \sim (\phi(x), 1)$$

is called the mapping torus of (Σ, ϕ).

For each component of $\partial\Sigma$, we see the following.





Let

$$M_{(\Sigma,\phi)} = T_\phi \cup \coprod (S^1 \times D^2)/\sim,$$

Where $\sim$ tells us to glue the solid tori to T_ϕ so that $\{pt\} \times \partial D^2$ goes to the y curve in the picture above and $S^1 \times \{pt\}$ goes to the x curve.

Exercise 1: If $L =$ cores of all $S^1 \times D^2$'s in $M_{(\Sigma,\phi)}$, show that $M - L$ fibers over the circle with fibers $\cong \Sigma$ (called **pages**). L is called the **binding**.

An **open book decomposition** of a 3-manifold M is an identification of M with $M_{(\Sigma,\phi)}$ for some (Σ, ϕ) as above.

Fact: All 3-manifolds have open book de decompositions.

> **Exercise 2:** Try to prove the above fact. (Hint: Think about branched covers over braids.)

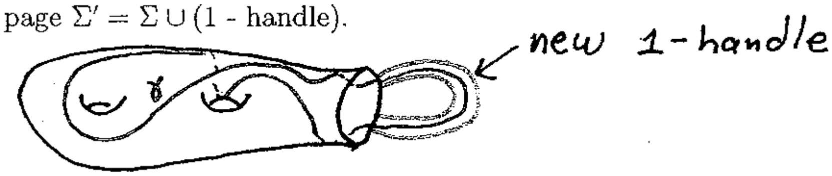
An open book decomposition (Σ, ϕ) for M is said to be **compatible** with (or **support** by) a contact structure ξ if there exists a contact form α for ξ such that,

(1) $\alpha(TL) > 0$
(2) $d\alpha|_{\text{pages}} \neq 0$ is a volume form for Σ.

A theorem of Thurston and Winkelnkemper tells us that all open books support contact structures.

Given an open book (Σ, ϕ) for (M, ξ) a **positive stabilization** is the open book with

(1) page $\Sigma' = \Sigma \cup (1 \text{ - handle})$.



(2) $\phi' = \phi \circ D_\gamma$, where D_γ is a positive Dehn twist along any curve γ such that γ runs over the 1-handle once.

> **Exercise 3:** Show that $M_{(\Sigma,\phi)} \cong M_{(\Sigma',\phi')}$, and that both open books support the same contact structure.

Theorem (Giroux) There is a 1-1 correspondence between oriented contact structure up to isotopy, and open book decompositions up to positive stabilization.

To see how to use this, we need to figure out how Legendrian surgery interacts with open books.

Given (M, ξ) and an open book (Σ, ϕ) supporting it, let γ be a simple closed curve a the page of the open book. Observe that the γ gets an induce framing, $\mathcal{F}$, from the page.

Let M' be obtained from M by $\mathcal{F} \pm 1$-Dehn surgery on γ.

> **Exercise 4:** Show that an open book for M' is $(\Sigma, \phi \circ D_\gamma^{\pm})$. (Hint: Think about the proof that all 3-manifolds are obtained by surgery on a link in S^3.)

Fact: If γ is non-separating on the page, then we can isotope the open book slightly so that γ is Legendrian and so that the contact framing agrees with the page framing.

Let (M', ξ') be obtained from (M, ξ) by Legendrian surgery along γ.

Fact: In this situation, (M', ξ') is supported by $(\Sigma, \phi \circ D_\gamma)$.

So given a symplectic manifold (X, ω) filling (M, ξ) with open book (Σ, ϕ) and Legendrian γ on a page of the open book, if we attach a symplectic 2-handle to (X, ω) to get (X', ω'), then $\partial(X', \omega') = (M', \xi')$ with supporting open book $(\Sigma, \phi \circ D_\gamma)$.

To go farther, we need a few facts about the mapping class group of surfaces.

Fact: If Σ is a surface with one boundary component, then any diffeomorphism of Σ (fixed near the boundary) can be written, up to isotopy, as

$$\phi = D_c^m \circ D_{\gamma_1}^{-1} \circ \cdots \circ D_{\gamma_n}^{-1}.$$

Where c is a curve parallel to the boundary and the γ_i are non-separating closed curves on Σ.

Now we start to construct the caps.

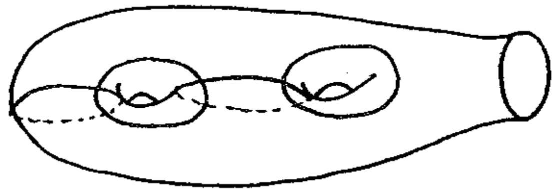
Given (X, ω) weakly filling (M, ξ), (Σ, ϕ) an open book supporting (M, ξ), assume Σ has one boundary component, and assume

$$\phi = D_c^m \circ D_{\gamma_1}^{\pm 1} \circ \cdots \circ D_{\gamma_n}^{\pm 1}.$$

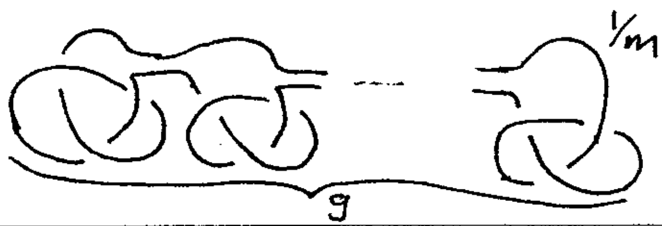
Now using the idea above, we can Legendrian realize all the γ_i, then construct (X', ω') by attaching 2-handles to (X, ω) along the $\gamma_1, \ldots, \gamma_n$.

Then we will have $\partial(X', \omega') = (M', \xi')$, with open book (Σ, ϕ'), where $\phi' = D_c^m$.

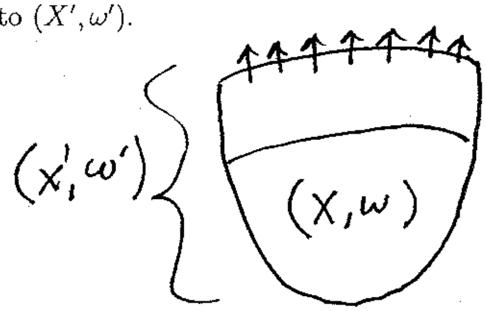
Now attach more handles so that $\phi' = D_c^m \circ D_{\delta_1} \circ \cdots \circ D_{\delta_{2g}}$.



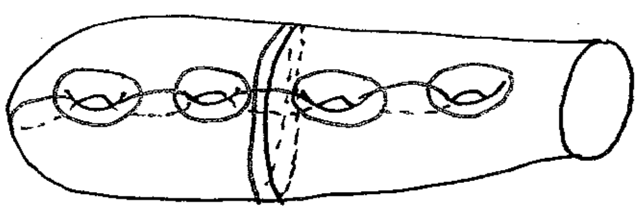
Exercise 4: If genus $\Sigma = g$, then M' is the following homology sphere.



Fact: If (X,ω) is a weak filling of (M,ξ) and M is a homology sphere, then we can slightly perturb ω so that (X',ω') is a strong filling of (M',ξ') and (X,ω) embeds into (X',ω').



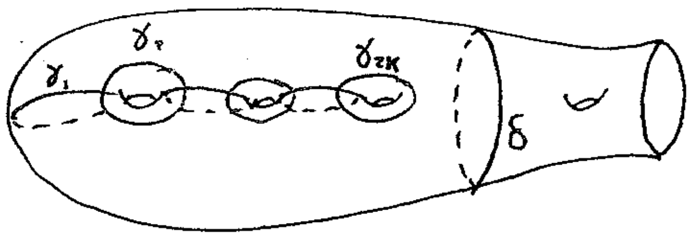
We can now stabilize the open book for (M',ξ') so that the page looks as follows.



If we stabilize (Σ',ϕ') enough, then after adding positive Dehn twists, we get $\phi'' = D_c$.

To do this we use the chain relation

$$(D_{\gamma_1} \circ \cdots \circ D_{\gamma_{2k}})^{4k+2} = D_\delta.$$



> **Exercise 5:** Use this relation to verify the above claim. In particular, show that (Σ, ϕ') can be stabilized and more positive Dehn twists can be added to get the claimed open book.

After symplectic 2-handles have been added corresponding to the above added Dehn twists, we have a symplectic 4-manifold (X'', ω'') with $\partial(X'', \omega) = (M'', \xi'')$ supported by (Σ', ϕ'''), $\phi''' = D_{c'}$, and so that (X, ω) embeds into (X'', ω'').

> **Exercise 6:** Show that M'' is an S^1-bundle over Σ'' with Euler number -1, where $\Sigma'' = \Sigma'$ with a disk glued to $\partial \Sigma'$.
>
> 
>

> **Exercise 7:** Show that if $Y = D^2$-bundle over Σ'' with Euler number 1, then Y admits a symplectic structure with concave boundary and $\partial Y = -M''$. Also show that the induced contact structure is contactomorphic to ξ''.

Therefore, using the fact about gluing form last time, we can glue Y and (X'', ω'') to get a closed symplectic manifold.