TIGHT CONTACT STRUCTURES WITH NO SYMPLECTIC FILLINGS

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ABSTRACT. We exhibit tight contact structures on 3-manifolds that do not admit any symplectic fillings.

1. INTRODUCTION

In the early 1980's, D. Bennequin [2] proved the existence of *exotic* contact structures on $\mathbf{R}^3$. These were obtained from the *standard* contact structure on $\mathbf{R}^3$ given by the 1-form $\alpha = dz - ydx$, by performing modifications called *Lutz twists.* The key distinguishing feature was that the exotic contact structures contained *overtwisted disks*, *i.e.*, disks D which are everywhere tangent to the 2-plane field distribution along ∂D. Using an ingenious argument which used braid foliations, Bennequin succeeded in proving that the standard contact structure, on the other hand, contained no overtwisted disks. Although initially believed to be very complex, the world of exotic contact structures eventually turned out not to be so exotic, when Eliashberg [7] gave a complete classification of contact structures which contain overtwisted disks (now called *overtwisted* contact structures) in terms of homotopy theory.

With the advent of Gromov's theory of holomorphic curves [21], it became easier to determine when a contact structure on a 3-manifold is *tight*, *i.e.*, contains no overtwisted disks [9]. Loosely speaking, a contact structure is *symplectically fillable* if it is the boundary of a symplectic 4-manifold. Gromov and Eliashberg showed that a symplectically fillable contact structure is necessarily tight. In fact, until the mid-1990's, almost all known tight contact structures were shown to be tight using symplectic fillings. The notable exceptions were the standard contact structure on $\mathbf{R}^3$ and a few of its quotients, originally shown to be tight by Bennequin using completely different techniques — however even these structures could be shown to be tight using symplectic fillings. Symplectic filling provided two rich sources of tight contact structures — perturbations of taut foliations as in [13] and

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Legendrian surgery as in [8] and [35]. This prompted Eliashberg and others to ask whether tight contact structures are the same as symplectically fillable contact structures. Subsequently, gluing techniques were developed by Colin [3], [4] and Makar-Limanov [32], and strengthened by the second author in [25]. Largely due to the improvements in gluing techniques, tight contact structures could now be constructed without resorting to symplectic filling techniques. The main result of this paper shows that the symplectically fillable contact structures form a proper subset of tight contact structures.

Theorem 1.1. *Let M_1 (resp. M_2) be the Seifert fibered space over S^2 with Seifert invariants $(-\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ (resp. $(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$). Then M_1 admits one tight contact structure and M_2 admits two nonisotopic tight contact structures that are not weakly symplectically semi-fillable.*

In this paper we will provide a complete proof for $M = M_1$; the proof for $M = M_2$ is similar, and we will briefly discuss the necessary modifications at the end of Section 3.

Remark on notation. A Seifert fibered space over a closed oriented surface Σ with n singular fibers is often denoted by $(g; (1, e), (\alpha_1, \beta_1), \cdots, (\alpha_n, \beta_n))$, or by $(g; e, \frac{\beta_1}{\alpha_1}, \cdots, \frac{\beta_n}{\alpha_n})$, where g is the genus of the base Σ, $e \in \mathbf{Z}$ is the Euler number, and $\alpha_i, \beta_i \in \mathbf{Z}^+$ are relatively prime. In this notation, $(-\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ would correspond to $(0; -1, \frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ and $(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$ to $(0; -1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

2. Background and preliminary notions

We briefly review the basic notions in Section 2.1 and proceed to a discussion of symplectic fillings in Section 2.2. There we introduce the various types of symplectic fillings and discuss the work of Lisca concerning the nonexistence of fillable structures on certain manifolds. Finally, in Section 2.3, we discuss the contact surgery technique which is usually called Legendrian surgery.

Convex surface theory will be our main tool throughout this paper. Originally developed by Giroux in [17], there have been many recent papers discussing convex surfaces. All the facts relevant to this paper concerning convex surfaces may be found in [23], [16] (see also [18], [27], [24]), and we assume the reader is familiar with the terminology in these papers.

2.1. Contact structures and Legendrian knots.

In this section we review a few basic notions of contact topology in dimension three. This is more to establish terminology than to introduce the readers to these ideas. Readers unfamiliar with these ideas should see [1] or [10].

An oriented 2-plane field distribution ξ on an oriented 3-manifold M is called a *positive contact structure* if $\xi = \ker \alpha$ for some global 1-form α satisfying $\alpha \wedge d\alpha > 0$. The 1-form α is called the *contact form* for ξ. In this paper we will always assume that our ambient manifold M is oriented and the contact structure ξ is positive and oriented. If Σ is a surface in a contact manifold (M, ξ), then Σ has a singular foliation Σ_ξ, called the *characteristic foliation*, given by integrating the singular line field $T_x\Sigma \cap \xi_x$. It is important to remember that the characteristic foliation on a surface determines the germ of the contact structure along the surface. A contact structure ξ is said to be *overtwisted* if there is an embedded disk D which is everywhere tangent to ξ along ∂D. A contact structure is *tight* if it is not overtwisted. For a complete classification of overtwisted contact structures see [7]. A contact structure ξ on M is *universally tight* if the pullback to the universal cover is tight, and is *virtually overtwisted* if ξ is tight but its pullback is overtwisted in some finite cover of M. It is an interesting problem to determine whether every tight contact structure is either universally tight or virtually overtwisted.

A knot K embedded in a contact manifold (M, ξ) is called *Legendrian* if it is everywhere tangent to ξ. A choice of nonzero section of ξ transverse to K gives a framing of the normal bundle of K, usually called the *contact framing*. If $\mathcal{F}$ is some preassigned framing of K, then we associate an integer $t(K, \mathcal{F})$ (or just $t(K)$ if the framing $\mathcal{F}$ is understood), called the *twisting number of ξ along K* relative to $\mathcal{F}$, which is the difference in twisting between the contact framing and $\mathcal{F}$. If K is a null-homologous knot and $\mathcal{F}$ is given by a Seifert surface for K, then $t(K)$ is called the *Thurston-Bennequin invariant* of K and is usually denoted $tb(K)$.

A closed surface or a properly embedded compact surface Σ with Legendrian boundary is called *convex* if there exists a contact vector field everywhere transverse to Σ. To a convex surface Σ we associate an isotopy class of multicurves called the *dividing set* Γ_Σ (or simply Γ). If Σ is closed, then components of Γ_Σ are closed curves, and if Σ has boundary, there may also be properly embedded arcs. The number of components of Γ_Σ is written as $\#\Gamma_\Sigma$. Informally, the Flexibility Theorem of Giroux [17] states that Γ_Σ, not the precise characteristic foliation, encodes all the contact-topological information in a small neighborhood of a closed surface Σ. (The Flexibility Theorem was extended to compact surfaces with Legendrian boundary in [28], and a complete proof appears in [23].) The complement of the dividing set is the union of two subsets $\Sigma \setminus \Gamma_\Sigma = \Sigma_+ - \Sigma_-$. Here Σ_+ is the subsurface where the orientation of Σ and the normal orientation of ξ coincide, and Σ_- is the subsurface where they are opposite. Therefore, we can refer to *positive* and *negative* components of $\Sigma \setminus \Gamma_\Sigma$.

2.2. **Symplectic fillings.**

The easiest way to prove a contact structure is tight is to show it "bounds" a symplectic 4-manifold. There are several notions of "symplectic filling", and we assemble the various notions here for the convenience of the reader. (For more details, see the survey paper [14].)

A symplectic manifold (X, ω) is said to have *ω-convex* boundary if there is a vector field v defined in the neighborhood of ∂X that points transversely out of X and for which $\mathcal{L}_v\omega = \omega$, where $\mathcal{L}$ denotes the Lie derivative. One may easily check that $\alpha = (\iota_v\omega)|_{\partial X}$ is a contact form on ∂X. A symplectic manifold (X, ω) is said to have *weakly convex* boundary if ∂X admits a contact structure ξ such that $\omega|_\xi > 0$ (and the orientations induced on ∂X by X and ξ agree). A contact structure ξ on a closed 3-manifold M is:

(1) *Holomorphically fillable* if (M, ξ) is the ω-convex boundary of some Kähler manifold (X, J, ω) and J preserves ξ.
(2) *Strongly symplectically fillable* if (M, ξ) is the ω-convex boundary of some symplectic manifold (X, ω).
(3) *Weakly symplectically fillable* if (M, ξ) is the weakly convex boundary of some symplectic manifold (X, ω).
(4) *Weakly symplectically semi-fillable* if (M, ξ) is one component of the weakly convex boundary of some symplectic manifold (X, ω).

Theorem 2.1 (Gromov-Eliashberg). *Let (M, ξ) be a contact 3-manifold which satisfies any of the above conditions for fillability. Then ξ is tight.*

The following diagram indicates the hierarchy of tight contact structures on closed 3-manifolds:

Tight		
$\cup \not\vert$		
Weakly symplectically semi-fillable	$\supsetneq$	Strongly symplectically semi-fillable
$\cup$		$\cup$
Weakly symplectically fillable	$\supsetneq$	Strongly symplectically fillable
		$\cup$
		Holomorphically fillable

The proper inclusion of the set of weakly symplectically semi-fillable contact structures into the set of tight contact structures is the content of Theorem 1.1. The proper inclusion of the set of strongly fillable structures into the set of weakly fillable structures is already seen on T^3 (due to Eliashberg [11]). This result was recently extended by Ding-Geiges [6] to T^2-bundles over S^1. For all other inclusions in the diagram it is not known whether the inclusions are strict.

We briefly discuss when the various notions of fillability become the same. If $H^2(M; \mathbf{Q}) = 0$, a weak symplectic filling can be modified into a strong symplectic filling [34]. Using work of Kronheimer and Mrowka [29] and Seiberg-Witten theory, Lisca [30] showed that if M has a positive scalar curvature metric, then a semi-filling is automatically a one-component filling (there is also a related, but weaker, result in Ohta-Ono [34]).

Lisca [31] went further to show (among other things) that:

Theorem 2.2 (Lisca [31]). *Let M be a Seifert fibered space over S^2 with Seifert invariants $(-\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$ or $(-\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$. The manifold M does not carry a weakly symplectically semi-fillable contact structure.*

In Lisca's paper, the Seifert fibered space with invariants $(-\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$ is described as the boundary M of the 4-manifold X obtained by plumbing disk bundles over S^2 according to the positive E_7 diagram (left-hand side of Figure 1). It is an easy exercise in Kirby calculus [20] to show that M is orientation-preserving diffeomorphic to the manifold shown on the right-hand side of Figure 1, which is a presentation for a Seifert fibered space. Similarly, the Seifert fibered space with invariants $(-\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$ corresponds to the positive E_6 diagram.

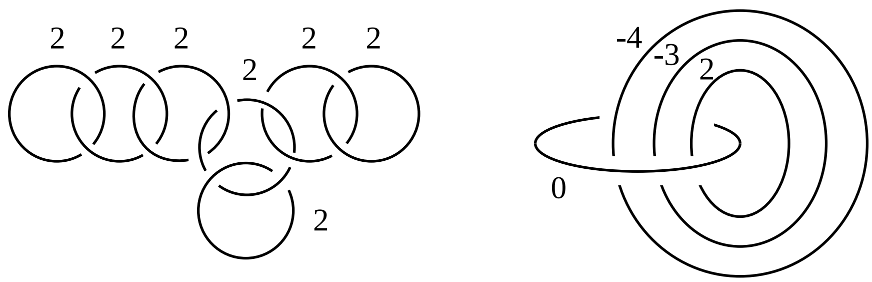

FIGURE 1. The plumbed disk bundles (left) and the Seifert fibered space M (right).

2.3. **Legendrian surgeries.** Let us now describe a contact surgery technique called *Legendrian surgery.* We first give a description on the 3-manifold level. Given a Legendrian knot L in any contact 3-manifold (M, ξ), a *Legendrian surgery* on L yields the contact manifold (M', ξ'), where M' is obtained from M by $(t(L)-1)$-Dehn surgery on L and ξ' is obtained from ξ as follows: Let N be a *standard convex neighborhood* of L. Choose a framing on N so that $t(L) = 0$. This choice of framing allows us to make an oriented identification $-\partial(M \setminus N) \simeq \mathbf{R}^2/\mathbf{Z}^2$, where $(1,0)^T$ is the meridian of N and $(0,1)^T$ is the longitude of N corresponding to the framing. Now take an identical copy N' of N (with the same framing), and make an oriented identification $\partial N' \simeq \mathbf{R}^2/\mathbf{Z}^2$, where $(1,0)^T$ is the meridian and $(0,1)^T$ is the longitude. Then let $M' = (M \setminus N) \cup_\psi N'$ where $\psi : \partial N' \stackrel{\sim}{\to} -\partial(M \setminus N)$ is represented by the matrix $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \in SL(2, \mathbf{Z})$. Since $\psi(\Gamma_{\partial N'})$ and $\Gamma_{\partial(M \setminus N)}$ are isotopic, we may use Giroux's Flexibility Theorem to arrange the characteristic foliation on $\partial N'$ and isotop ψ so $\psi_*(\xi|_{N'}) = \xi|_{M \setminus N}$. Hence we may glue the contact structures on N' and $M \setminus N$.

Theorem 2.3. *Legendrian surgery is category-preserving for each category in the diagram of inclusions above, with the possible exception of the category of tight contact structures.*

Eliashberg [8] proved that Legendrian surgery is category-preserving for holomorphically fillable contact structures. Weinstein [35] proved Theorem 2.3 for strongly symplectically fillable contact structures.

On the 4-manifold level, Legendrian surgery is described as follows: Let (X, ω) be a symplectic 4-manifold with ω-convex boundary and L a Legendrian knot in the induced contact structure on ∂X. If X' is obtained from X

by adding a 2-handle to ∂X along L with framing $t(L)-1$, then ω extends to a symplectic form ω' on X' so that $\partial X'$ is ω'-convex. For many interesting applications of Theorem 2.3, we refer the reader to Gompf [19].

The case of Theorem 2.3 known to a few experts but surprisingly absent in the literature is for the category of weakly fillable contact structures. The proof of Theorem 2.3 in the strongly fillable case relies only on the symplectic structure on X in a neighborhood of L. Hence Theorem 2.3 in the weakly fillable case follows from:

Lemma 2.4. *Let (M,ξ) be a weakly symplectically fillable contact 3-manifold, (X,ω) one of its weak fillings, and L a Legendrian knot in (M,ξ). There is an arbitrarily small perturbation of ξ in a neighborhood N of L in M so that N is strongly convex. By this we mean there is a vector field v defined on X near N so that v points transversely out of X, $\mathcal{L}_v\omega=\omega$ and $\xi|_N = ker\ (\iota_v\omega)|_N$.*

Proof. Let (M',ξ') be any strongly fillable contact 3-manifold, (X',ω') one of its strong fillings and L' a Legendrian knot in (M',ξ'). It is not hard to find a neighborhood N of L in M and N' of L' in M' and a diffeomorphism $f: N \to N'$ such that $f(L)=L'$, $f^*\xi'=\xi$ along L, and $f^*(\omega'|_{N'})=\omega|_N$. One may then use standard arguments (see Exercise 3.35 in [33]) to extend f to a symplectomorphism $(U,\omega)\stackrel{\sim}{\to}(U',\omega')$, where $U\subset X$ is a neighborhood of N and $U'\subset X'$ is a neighborhood of N'. Finally note that the contact planes $f^*\xi'$ and ξ agree on L and are close together near L. By a small perturbation of ξ near L (small enough to keep ξ contact so we may use Gray's Theorem), we may therefore assume $f^*\xi'=\xi$ near L. Hence, if v is the expanding vector field for ω', then $f_*^{-1}(v)$ will be the desired vector field for ω. □

We now comment on Theorem 2.3 for the category of tight contact structures. In [25], it was shown that there exists a tight contact structure on a handlebody of genus 4 and a Legendrian surgery yielding an overtwisted contact structure. (Also see [5] for an example of a universally tight contact structure which does not survive an *admissible transverse surgery*, a procedure akin to Legendrian surgery.) It is currently not known whether Legendrian surgery preserves tightness for *closed* 3-manifolds.

3. The proof of the main result for the Seifert fibered space with invariants $(-\frac{1}{2},\frac{1}{4},\frac{1}{4})$

3.1. **Seifert fibered spaces.** Let M be a Seifert fibered space over S^2 with three singular fibers F_1,F_2,F_3 and Seifert invariants $(\frac{\beta_1}{\alpha_1},\frac{\beta_2}{\alpha_2},\frac{\beta_3}{\alpha_3})$. We describe M explicitly as follows: Let V_i, $i=1,2,3$, be a tubular neighborhood of the singular fiber F_i. We identify $V_i\simeq D^2\times S^1$ and $\partial V_i\simeq \mathbf{R}^2/\mathbf{Z}^2$ by choosing $(1,0)^T$ as the meridional direction, and $(0,1)^T$ as the longitudinal direction given by $\{pt\}\times S^1$. We also identify $M\setminus(\cup_i V_i)$ with $\Sigma_0\times S^1$, where Σ_0 is a sphere with three punctures, and further identify

$-\partial(M \setminus V_i) \simeq \mathbf{R}^2/\mathbf{Z}^2$, by letting $(0,1)^T$ be the direction of an S^1-fiber, and $(1,0)^T$ be the direction given by $\partial(M \setminus V_i) \cap (\Sigma_0 \times \{pt\})$. With these identifications we may reconstruct M from $(\Sigma_0 \times S^1) \cup (\cup_{i=1}^3 V_i)$ by gluing

$$A_i : \partial V_i \xrightarrow{\sim} -\partial(M \setminus V_i), \quad A_i = \begin{pmatrix} \alpha_i & \gamma_i \\ -\beta_i & \delta_i \end{pmatrix} \in SL(2, \mathbf{Z}).$$

Note we have some freedom in choosing our matrices A_i above. For example, in choosing A_i we can alter γ_i, δ_i by altering our choice of framing for V_i, which will result in post-multiplying a given A_i by $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$.

The twisting number of a Legendrian knot *isotopic* to a regular (*i.e.*, nonsingular) fiber of the Seifert fibration will be measured using the framing from the product structure $\Sigma_0 \times S^1$. (Whenever we say *isotopy* we will mean a smooth isotopy, as opposed to a *contact isotopy*.) This framing is well-defined for the following reason: Let N be a tubular neighborhood of a regular fiber. It is a well-known fact that "most" Seifert fibered spaces (in particular $M \setminus N$) admit a unique Seifert fibration up to isotopy. See, for example, Hatcher [22]. The Seifert fibration on $M \setminus N$, unique up to isotopy, restricts to a fibration on ∂N which is unique up to isotopy. Therefore, the framing induced from the product structure is well-defined.

On the other hand, a Legendrian knot isotopic to a singular fiber in a Seifert fibration will be measured with respect to the framing on V_i chosen in the description for M. The proof of the well-definition of this framing is identical to that of the regular fiber.

Let us now specialize to the case where M is given by Seifert invariants $(-\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. We make the following choices:

$$A_1 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = A_3 = \begin{pmatrix} 4 & 1 \\ -1 & 0 \end{pmatrix}.$$

From now on M will refer to this particular Seifert fibered space.

3.2. **Description as a torus bundle.** To define the contact structure ξ in Theorem 1.1 and prove tightness, we need a description of M as a torus bundle over S^1. Recall a torus bundle over S^1 can be described as the quotient of $T^2 \times \mathbf{R}$ by equivalence relation $\sim$ given by the diffeomorphism

$$\Psi_{\mathbb{A}} : T^2 \times \mathbf{R} \to T^2 \times \mathbf{R},$$
$$(x, t) \mapsto (\mathbb{A}x, t-1),$$

with $\mathbb{A} \in SL(2, \mathbf{Z})$, called the *monodromy* of the torus bundle.

Lemma 3.1. *The manifold M is a torus bundle over S^1 with monodromy* $\mathbb{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Proof. The map $\mathbb{A}$ has order four with two fixed points and two points of order two (which are interchanged under $\mathbb{A}$). One may thus conclude that the torus bundle is a Seifert fibered space over S^2 with Seifert invariants

$(\pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{4})$. To determine the sign of invariants, let $D \subset T^2$ be a small disk about one of the fixed points with $\mathbb{A}(D) = D$, and consider $S = D \times [0,1]/\sim$. If $x \in \partial D$, then a regular fiber in the Seifert fibered structure will be given by $(\{\mathbb{A}^i(x) | i = 0,1,2,3\} \times [0,1])/\sim$. One may pick a product structure on S so that a regular fiber will be a $(-1,4)$-curve on ∂S. From this we see that the gluing matrix A_i associated to this singular fiber is $\begin{pmatrix} 4 & 1 \\ -1 & 0 \end{pmatrix}$. Thus two of the Seifert invariants are $\frac{1}{4}$. Similarly, one may check that the third invariant is $-\frac{1}{2}$. □

3.3. **Tight contact structures on toric annuli.** In this section, we collect some results on the classification of tight contact structures on toric annuli.

Throughout this paper, if we have a toric annulus $T^2 \times [a,b]$, we will abbreviate $T_t = T^2 \times \{t\}$. If T_t is convex with parallel essential dividing curves, then we write the slope of Γ_{T_t} as s_t.

3.3.1. The results in this part can be found in [23] (as well as in [18], given in slightly different terms). A convenient way to think of the classification is in terms of the Farey tessellation of the standard hyperbolic disk $\mathbb{H}^2 = \{(x,y) | x^2 + y^2 \leq 1\}$ (with the S^1 at infinity added). See Figure 2.

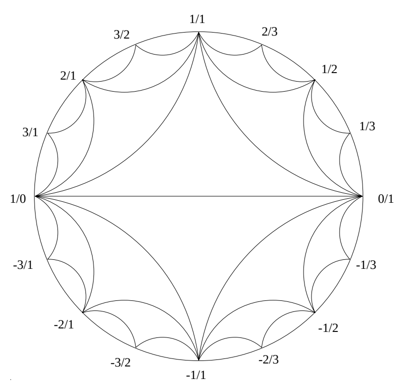

FIGURE 2. The Farey tessellation of the hyperbolic disk.

To construct the tessellation, first label $(1,0)$ as $0 = \frac{0}{1}$ and $(-1,0)$ as $\infty = \frac{1}{0}$. We now inductively label points on $\partial\mathbb{H}^2$ with $y > 0$ as follows: if two points have been labeled with $\frac{q}{p}$ and $\frac{q'}{p'}$ (where p, q and p', q' are relatively prime and nonnegative) and (p,q) and (p',q') form an integer basis for $\mathbf{Z}^2$, then we label the point halfway between these points as $\frac{q+q'}{p+p'}$. (When $y < 0$, one starts with $\infty = \frac{-1}{0}$ and similarly labels points — we now require that all $\frac{q}{p}$ satisfy $q \leq 0$ and $p \geq 0$.) Now, any time the labels on two points correspond to an integer basis for $\mathbf{Z}^2$, we connect them with a hyperbolic geodesic in $\mathbb{H}^2$.

Given two labeled points $a \neq b$ on $\partial\mathbb{H}^2$, we define a *path from a to b* to be a sequence $c_0 = a, c_1, \cdots, c_k = b$, where each pair (c_i, c_{i+1}), $i = 0, \cdots, k-1$, corresponds to a geodesic edge in the tessellation. We say path is *signed* if there is an assignment of a sign ($+$ or $-$) to each edge. A path is *clockwise*, if, for $i = 0, \cdots, k-1$, c_{i+1} lies on the clockwise arc along $\partial\mathbb{H}^2$ from c_i to $c_k = b$. A clockwise path is *minimal* if there exists no other clockwise path with the same endpoints and fewer geodesic edges. It is clear from the construction of the Farey tessellation that there is a unique minimal clockwise path from a to b. Finally define a *shuffling* of a clockwise signed path to be an interchange of signs between two consecutive edges c_i to c_{i+1} and c_{i+1} to c_{i+2} in case the corresponding shortest integral vectors v_i, v_{i+1}, v_{i+2} satisfy $\det(v_i, v_{i+2}) = \pm 2$. We now have:

Theorem 3.2 (Classification in the minimally twisting case). *Fix a convex foliation $\mathcal{F}_i$, $i = 0, 1$, on T_i with $\#\Gamma_{T_i} = 2$ and $s_0 \neq s_1$. The set isotopy classes (rel boundary) of minimally twisting tight contact structures on $T^2 \times [0,1]$ inducing $\mathcal{F}_i$ on T_i, is in 1-1 correspondence with the set of minimal clockwise signed paths from s_0 to s_1, modulo shuffling.*

Recall that, for a tight contact structure on $T^2 \times [0,1]$ to be *minimally twisting*, every convex surface parallel to T_0 (or T_1) must have a dividing set with slope which lies on the clockwise path along $\partial\mathbb{H}^2$ from s_0 to s_1 (provided $s_0 \neq s_1$). We also remark that the minimal clockwise path only depends on s_0 and s_1 (not on the contact structure), but the assignment of signs depends on the tight contact structure. Clockwise signed paths represent factorizations of $T^2 \times [0,1]$ into thinner toric annuli with convex boundary and slopes c_i and c_{i+1}.

We now describe a gluing theorem for gluing two toric annuli. Keeping the previous notation, we say a clockwise signed path from a to b can be *shortened* if there are two adjacent geodesic edges c_i to c_{i+1} and c_{i+1} to c_{i+2} with the same sign and a geodesic edge in the hyperbolic tessellation connecting c_i to c_{i+2}. Moreover, a *shortening* of such a signed path is obtained by replacing $\cdots, c_i, c_{i+1}, c_{i+2}, \cdots$ by $\cdots, c_i, c_{i+2}, \cdots$, and labeling the path from c_i to c_{i+2} with the common sign of the edges c_i to c_{i+1} and c_{i+1} to c_{i+2}. Note this results in a new clockwise signed path.

Theorem 3.3 (Gluing toric annuli). *Let ξ be a contact structure on $T^2 \times [0,2]$ such that (i) its restrictions ξ_1 to $T^2 \times [0,1]$ and ξ_2 to $T^2 \times [1,2]$ are both tight and minimally twisting, (ii) T_i, $i = 0,1,2$, are convex with $\#\Gamma = 2$, and (iii) $s_0 \neq s_2$, and s_1 lies on the clockwise arc on $\partial\mathbb{H}^2$ from s_0 to s_2. Denote the minimal clockwise signed path associated to ξ_j, $j = 1,2$, by γ_j. Then ξ is tight and minimally twisting if and only if a minimal clockwise signed path from s_0 to s_2 can be obtained by a sequence of shortenings of $\gamma_1 \cup \gamma_2$.*

In particular, the gluing is not tight if there exist adjacent geodesic edges c_i to c_{i+1} and c_{i+1} to c_{i+2} with opposite signs and a geodesic edge in the hyperbolic tessellation connecting c_i to c_{i+2}.

3.3.2. We now treat the case of tight contact structures on $T^2 \times [0,1]$ where $\#\Gamma_{T_0}$ or $\#\Gamma_{T_1}$ is ≥ 2. Details of Theorem 3.5 are provided here for the following reason — there are some mistakes at the end of [23] (to be fixed in the corrigendum [26]) which affect the original proof of Proposition 3.6 which appeared in [24].

Recall a tight contact structure ξ on $T^2 \times [0,1]$ with convex boundary is *nonrotative* if all the convex tori parallel to T_0 (or T_1) have the same slope; otherwise ξ is called *rotative*. Without loss of generality we may assume that the characteristic foliation $\mathcal{F}_i$, $i=0,1$, on T_i are in *standard form* with rulings by parallel closed Legendrian curves which intersect each dividing curve exactly once. A *horizontal annulus* A for a nonrotative ξ is then a convex annulus with boundary $\partial A = \delta_0 \sqcup \delta_1$, where δ_i, is a Legendrian ruling curve on T_i. When discussing nonrotative tight contact structures, we assume $s_i = \infty$ and the slope of the horizontal annulus is 0.

The following theorem of [23] characterizes nonrotative tight contact structures:

Theorem 3.4. *There exists a 1-1 correspondence between isotopy classes (rel boundary) of nonrotative tight contact structures on a toric annulus with boundary condition $\mathcal{F}_1 \sqcup \mathcal{F}_2$ and isotopy classes of dividing sets on A (rel ∂A) with $\#\Gamma_{T_i}$ fixed endpoints on δ_i for $i=0,1$, and at least two nonseparating dividing curves which are arcs connecting between δ_0 and δ_1.*

Suppose $\#\Gamma_{T_0} > 2$ and $(T^2 \times [0,1], \xi)$ is rotative. Then, after possibly changing the product structure, we find a nonrotative layer $T^2 \times [0,\varepsilon]$ with $\#\Gamma_{T_\varepsilon} = 2$ — using the Imbalance Principle in [23], we find enough bypasses to reduce $\#\Gamma_{T_0}$ to 2 via repeated attachments. Such a nonrotative layer will correspond to a horizontal annulus A with $\partial A = \delta_0 \sqcup \delta_\varepsilon$ with Γ_A consisting of two nonseparating arcs and the rest separating arcs connecting between points on δ_0. The following theorem answers the question of the uniqueness of these *nonrotative outer layers*, *i.e.*, nonrotative toric annuli $L \subset T^2 \times [0,1]$ with $\partial L = T_0 \sqcup T_\varepsilon(L)$, where $T_\varepsilon(L)$ is a convex torus in the interior of $T^2 \times [0,1]$, with $\#\Gamma = 2$.

Theorem 3.5. *Let $T^2 \times [0,1]$ be a rotative toric annulus with $\#\Gamma_{T_0} > 2$. Suppose L is a nonrotative outer layer of $T^2 \times [0,1]$ (on the side of T_0) with corresponding horizontal annulus A. If L' is another nonrotative outer layer (on the side of T_0) with horizontal annulus A', then A and A' are disk-equivalent.*

Write $\partial A = \delta_0 \sqcup \delta_\varepsilon$ and $\partial A' = \delta_0' \sqcup \delta_\varepsilon'$. By sliding the boundary component of δ_0' up or down along T_0, we may assume that $\delta_0 = \delta_0'$ and the endpoints of Γ_A and $\Gamma_{A'}$ coincide.

We now explain the notion of *disk-equivalence*. Let D be a disk and $\phi : A \hookrightarrow D$ be an embedding so that $\phi(\delta_0) = \partial D$, and connect the two endpoints $\phi(\Gamma_A \cap \delta_\varepsilon)$ by an arc a in D to obtain a multicurve $\Gamma_D = \phi(\Gamma_A) \cup a$. Similarly embed $\phi' : A' \hookrightarrow D$ with $\phi|_{\delta_0} = \phi'|_{\delta_0}$ to obtain Γ_D'. Now, we say A and A' are *disk-equivalent* if Γ_D and Γ_D' are isotopic on D rel ∂D.

Proof. First observe that a nonrotative outer layer L can be embedded inside the standard I-invariant neighborhood of the convex surface $T_\varepsilon(L)$ with 2 dividing curves. This is because L can be obtained using a standard model by performing *folds* (see [23]) along the Legendrian divides of $T_\varepsilon(L)$.

We prove Theorem 3.5 using the *method of templates*. That is, we consider the set $\mathcal{S}$ of isotopy classes of nonrotative tight contact structures ζ on $K = T^2 \times [-\varepsilon, 0]$ (with fixed boundary characteristic foliation) for which the contact structure on $K \cup L$ is I-invariant, *i.e.*, isomorphic to an I-invariant neighborhood of a convex torus with parallel essential curves. In particular, it follows that the contact structure on $K \cup (T^2 \times [0,1])$ is tight. We will see that the condition that $K \cup L'$ be tight for all $K \in \mathcal{S}$ forces the disk-equivalence. Note that $K \in \mathcal{S}$ corresponds to an isotopy class of dividing curves Γ_B on the horizontal annulus B with $\partial B = \delta_{-\varepsilon} \sqcup \delta_0$, where Γ_B has $\#\Gamma_{T_0}$ endpoints on δ_0, 2 endpoints on $\delta_{-\varepsilon}$, and exactly two nonseparating dividing arcs, and $\Gamma_{A\cup B}$ consists of two parallel nonseparating arcs from $\delta_{-\varepsilon}$ to δ_ε.

We induct on $\#\Gamma_{T_0} = 2n$. When $n = 1$, there is nothing to prove. Assume Theorem 3.5 is true for $\#\Gamma_{T_0} = 2n$. Now suppose $\#\Gamma_{T_0} = 2(n+1)$. There are two cases. First suppose there exist at least two ∂-parallel dividing curves of A along δ_0. Note that if γ is an arc of B with endpoints on $\Gamma_A \cap \delta_0$, such that the endpoints of γ are not (1) the same as the endpoints of ∂-parallel dividing curves of A, and not (2) the same as the endpoints of the two nonseparating curves of Γ_A from δ_ε to δ_0 on A, then γ can be extended to a complete dividing set Γ_B on B such that the combined dividing set on $A \cup B$ is the union of two nonseparating arcs. It is clear that any Γ_B which has a ∂-parallel dividing curve satisfying (1) corresponds to an overtwisted gluing, and Γ_B satisfying (2) may or may not give an overtwisted gluing. Since there is at most one position for (2), under our hypothesis of at least two ∂-parallel dividing curves of A, given A', we can identify at least one of the ∂-parallel dividing curves of A' to be in the exact same position as that of A. Now we can induct on $\#\Gamma_{T_0}$.

On the other hand, suppose there is only one ∂-parallel dividing curve of A along δ_0. This implies that one of the two regions of A cut open by the two nonseparating arcs contains no other dividing curve, and in the other region all the arcs are nested concentrically about the one ∂-parallel arc. Hence there are at most two positions for ∂-parallel dividing curves on B which could possibly give an overtwisted gluing. It is easy to conclude that there are only two possibilities for $\Gamma_{A'}$: either it is the original Γ_A or a configuration with the positions of the endpoints of the ∂-parallel curve and the endpoints of the nonseparating curves switched. At any rate, the two possible $\Gamma_{A'}$ are disk-equivalent. □

3.4. **The tight contact structure.** We now define the tight contact structure ξ on M, using the description of M as a torus bundle. For this we first describe a tight contact structure on $T^2 \times [0,1]$ with coordinates (x, y, t).

Start with a contact structure given by the 1-form $\alpha = \sin(\frac{\pi t}{2})dx + \cos(\frac{\pi t}{2})dy$. Perturb the boundary so that T_0, T_1 are convex with $\#\Gamma_{T_i} = 2$, and slopes $s_0 = 0$, $s_1 = \infty$. Now we identify T_0 and T_1 via $\Psi_{\mathbb{A}}$ to obtain a contact structure on the quotient $M = (T^2 \times [0,1])/\sim$. This is possible since $\Psi_{\mathbb{A}}(\Gamma_{T_1})$ is isotopic to Γ_{T_0} — we may apply Giroux's Flexibility Theorem to ensure that the characteristic foliations agree.

Proposition 3.6. *The contact structure ξ is a virtually overtwisted contact structure on M.*

In fact, ξ is the *unique* virtually overtwisted contact structure on M (see [24]). The uniqueness is not required in this paper.

Proof. We first show the existence of a double cover $\pi : M' \to M$ for which $\pi^*\xi$ is overtwisted. Let $M' = (T^2 \times [0,2])/\sim$, where $(\mathbb{A}^2 x, 0) \sim (x, 2)$. We may assume $\#\Gamma_{T_i} = 2$, $i = 0, 1, 2$, and $s(\Gamma_{T_0}) = s(\Gamma_{T_2}) = 0$, $s(\Gamma_{T_1}) = \infty$. If the relative Euler class on $(T^2 \times [0,1], \pi^*\xi)$ is $e(T^2 \times [0,1], \pi^*\xi) = (0,1) - (1,0) = (-1,1)$, then $e(T^2 \times [1,2], \pi^*\xi) = (1,1)$, and $e(T^2 \times [0,2], \pi^*\xi) = (0,2)$, which is not a possible relative Euler class for a tight contact structure with the given boundary slopes. (See [23] for a discussion of the relative Euler class.) Therefore $\pi^*\xi$ is overtwisted. We leave it as an exercise for the reader to explicitly find an overtwisted disk in $T^2 \times [0,2]$.

We now prove the tightness of ξ. The proof can be found in [24], but we reproduce it here, with more details, for completeness and the convenience of the reader. The proof will be given in several steps over the next several pages.

Step 1. We first explain the general strategy, called *state traversal*. To construct ξ on M, we start with a tight contact structure $\xi|_{T^2\times[0,1]}$ which we know is tight and glue the two boundary components via $\Psi_{\mathbb{A}}$. If there is an overtwisted disk $D \subset M$, then necessarily $D \cap T_0 \neq \emptyset$. Since D is contractible, there exists a torus $T \subset M$ isotopic to T_0 in M for which $D \cap T = \emptyset$. If we can prove that $\xi_{M \setminus T}$ is tight, then this would contradict the initial assumption of the existence of D.

Let Σ be a 2-dimensional torus and $\phi_t : \Sigma \to M$, $t \in [0,1]$, be the isotopy for which $\phi_0(\Sigma) = T_0$, $\phi_1(\Sigma) = T$, and $\phi_t(\Sigma)$ is an embedded surface for all $t \in [0,1]$. Using Colin's Isotopy Discretization technique [3], we may assume that there exist $t_0 = 0 < t_1 < t_2 < \cdots < t_k = 1$ such that for each short time interval $[t_i, t_{i+1}]$, all the $\phi_t(\Sigma)$, $t \in [t_i, t_{i+1}]$, are mutually disjoint and foliate the embedded toric annulus between $\phi_{t_i}(\Sigma)$ and $\phi_{t_{i+1}}(\Sigma)$. Since every surface is C^∞-close to a convex surface, we may also assume that $\phi_{t_i}(\Sigma)$ are convex. Now, for every tight contact structure on a toric annulus with convex boundary, one can get from one boundary component to the other by a sequence of bypass moves — this follows from the proof of the classification of tight contact structures on toric annuli in [23] or from the convex movie picture of [18], where there is a finite number of retrogradient switches which correspond to bypass attachments. We may now assume without loss of generality that each subdivision $[t_i, t_{i+1}]$, in addition, represents a single bypass attachment along $\phi_{t_i}(\Sigma)$.

Therefore, the proof strategy is the following: Assume M has been cut open along $\phi_{t_i}(\Sigma)$ so that ξ on the resulting toric annulus $T^2 \times [0,1]$ with $\phi_{t_i}(\Sigma) = T_0 = T_1$ is tight. After possibly altering the product structure, let $\phi_{t_{i+1}}(\Sigma) = T_{1/2}$; also let $T^2 \times [\frac{1}{2}, 1]$ be the toric annulus layer generated by attaching exactly one bypass along $\phi_{t_i}(\Sigma)$. In order to show that ξ is tight on $M \setminus \phi_{t_{i+1}}(\Sigma)$, we remove $T^2 \times [\frac{1}{2}, 1]$ and reglue using the monodromy map $\Psi_{\mathbb{A}}$ to obtain $T^2 \times [-\frac{1}{2}, \frac{1}{2}]$. In other words, we "simply" need to show that this peeling and reattaching process always gives a tight toric annulus.

Step 2. We prove that, with the notation at the end of Step 1, if the contact structure $\xi|_{T^2 \times [0,1]}$ is tight, then $\xi|_{T^2 \times [-1/2, 1/2]}$ is tight. In this step, we only treat the case where $\#\Gamma_{T_1} = \#\Gamma_{T_{1/2}} = 2$. In Step 3, we will explain how to reduce to this situation.

Initially, we have a universally tight contact structure ξ on $T^2 \times [0,1]$ with $\#\Gamma_{T_0} = \#\Gamma_{T_1} = 2$ and $s_0 = 0$, $s_1 = \infty$. By construction, ξ is minimally twisting (see [23] for proof), and corresponds to the minimal clockwise signed path γ consisting of one (say $+$) edge from 0 to ∞. Moreover, any convex torus T in $T^2 \times [0,1]$ has slope $-\infty \leq s(T) \leq 0$. In particular, we may assume $-\infty < s_{1/2} = -\frac{p}{q} < 0$, since we effectively do not modify the contact structure if $s_{1/2} = 0$ or ∞. Now, the factorization of $T^2 \times [0,1]$ into $T^2 \times [0, \frac{1}{2}]$ and $T^2 \times [\frac{1}{2}, 1]$ corresponds to a *lengthening* (opposite of shortening) of γ into $\gamma_1 \cup \gamma_2$, where γ_1 (resp. γ_2) is the minimal clockwise signed path from 0 to $-\frac{p}{q}$ (resp. $-\frac{p}{q}$ to ∞) with all positive signs — this follows from Theorem 3.3. Now, ξ on $T^2 \times [-\frac{1}{2}, \frac{1}{2}]$ has convex boundary and slopes $s_{-1/2} = \frac{q}{p}$ and $s_{1/2} = -\frac{p}{q}$, and corresponds to the clockwise signed path $\gamma_{-1} \cup \gamma_1$, where $\gamma_{-1} = \mathbb{A}(\gamma_2)$ with signs reversed. The contact structure $\xi|_{T^2 \times [-1/2, 1/2]}$ is tight by Theorem 3.3, since the signed path $\gamma_{-1} \cup \gamma_1$ is minimal. To see the minimality, we easily compute that, if $0 < \frac{a}{b} < \infty$ and $-\infty < -\frac{a'}{b'} < 0$, then there are no edges of the tessellation between $\frac{a}{b}$ and $-\frac{a'}{b'}$, and hence no potential contractions.

In general, we inductively assume the following:

(1) $\#\Gamma_{T_0} = \#\Gamma_{T_1} = 2$.

(2) $s_0 = \frac{q}{p}$ and $s_1 = -\frac{p}{q}$, $p > 0$, $q \geq 0$.

(3) $\xi|_{T^2 \times [0,1]}$ is tight, and the corresponding minimal clockwise signed path is the union of γ_{-1} consisting of edges from $\frac{q}{p}$ to 0, all with the same sign, and γ_1 consisting of edges from 0 to $-\frac{p}{q}$, all with sign opposite that of γ_{-1}. (If $-\frac{p}{q} = -\infty$, then there will be no γ_{-1}.)

If it happens that $s_{1/2} = 0$, then after the state transition the new s_1 will be 0 — to get to the inductive step, we apply the diffeomorphism $\mathbb{A}$

fiberwise to $T^2\times[0,1]$ (there is no loss of generality, since the diffeomorphism commutes with the monodromy $\mathbb{A}$).

Also we remark that, under our hypotheses, the only time shuffling (interchange of signs) can occur between an edge of γ_{-1} and an edge of γ_1 is if the slopes of the edges were 1, 0, and -1 (easy determinant computation), *i.e.*, if $s_0=1$, $s_1=-1$.

Let us now peel off a $T^2\times[\frac{1}{2},1]$-layer corresponding to a single bypass move, with $s_{1/2}=-\frac{p'}{q'}$, $p'\geq 0$, $q'>0$. We take a clockwise signed path from s_0 to s_1 compatible with this factoring, *i.e.*, $\gamma_1'\cup\gamma_2'$ where γ_2' is divided into two subpaths γ_{21}' from 0 to $-\frac{p'}{q'}$ and γ_{22}' from $-\frac{p'}{q'}$ to $-\frac{p}{q}$. After acting by the monodromy $\mathbb{A}$, γ_{22}' goes to γ_{-1}', which is a clockwise signed path from $\frac{q'}{p'}$ to $\frac{q}{p}$ with sign opposite the sign for γ_{22}'. This means that, after possibly shortening, we obtain a minimal clockwise signed path satisfying Condition (3) of the inductive hypothesis.

Step 3. We now discuss the case when $T^2\times[\frac{1}{2},1]$ consists of a single bypass attachment with $s_{1/2}=s_1$ and $\#\Gamma_{T_{1/2}}=\#\Gamma_{T_1}\pm 2$. While we are performing state transitions which do not alter the slope, we use the following inductive hypotheses:

(1) There exist disjoint nonrotative toric annuli $N_1,N_2\subset T^2\times[0,1]$, with convex boundary and $\partial N_1=T_0'-T_0$ and $\partial N_2=T_1-T_1'$, where T_0',T_1' are convex surfaces parallel to T_0, T_1 with two dividing curves.

(2) $(T^2\times[0,1])\setminus(N_1\cup N_2)$ is a minimally twisting tight contact structure corresponding minimal clockwise signed path $\gamma_{-1}\cup\gamma_1$, where γ_{-1} from $\frac{q}{p}$ to 0 are of one sign and γ_1 from 0 to $-\frac{p}{q}$ are of another.

(3) $\Psi_{\mathbb{A}}(N_2)\cup N_1$ is I-invariant.

Note that since N_1 and N_2 are nonrotative outer layers, the tightness condition (2) implies the tightness on all of $T^2\times[0,1]$.

First, since $T^2\times[0,1]$ is rotative, we may find $N_1'=T^2\times[0,\varepsilon]$ and $N_2'=T^2\times[\frac{1}{2}-\varepsilon,1]$ (here N_2' extends $T^2\times[\frac{1}{2},1]$) which are nonrotative outer layers (hence we have condition (1)). This follows again from using the Imbalance Principle of [23]. Let A_i, A_i', $i=1,2$, be the horizontal annuli of N_i, N_i'. Then A_i and A_i' are disk-equivalent by Theorem 3.5. The condition (3) above is equivalent to saying that $S=A_1\cup\Psi_{\mathbb{A}}(A_2)$ consists of exactly two nonseparating arcs. (The boundary components of A_1 and $\Psi_{\mathbb{A}}(A_2)$ along T_0 can be made to agree without loss of generality.) Now embed $S\hookrightarrow S^2$ and note that Γ_{S^2}, obtained from Γ_S by connecting up the two endpoints from each component of ∂S inside $S^2\setminus S$, consists of exactly one closed curve. Now, the same is true for $S'=A_1'\cup\Psi_{\mathbb{A}}(A_2')\hookrightarrow S^2$, due to disk-equivalence. Removing the two disks, it is now clear that $\Gamma_{S'}$ consists of exactly two nonseparating arcs. This implies that condition (3) is preserved under peeling and reattaching.

It remains to prove that $(T^2 \times [0,1]) \setminus (N_1' \cup N_2')$ is contact isomorphic to $(T^2 \times [0,1]) \setminus (N_1 \cup N_2)$, which would prove (2). The tight contact structure on $(T^2 \times [0,1]) \setminus (N_1 \cup N_2)$ is contact isomorphic to the contact structure on $(T^2 \times [0,1]) \cup L_1 \cup L_2$, where L_i, $i = 1, 2$, is a nonrotative toric annulus with convex boundary which we glue onto $T^2 \times [0,1]$ along T_{i-1} so that $L_i \cup N_i$ is an I-invariant tight contact structure. As in the argument in the previous paragraph, the disk-equivalence implies that $L_i \cup N_i'$ is also an I-invariant tight contact structure. Therefore, we have the following contact diffeomorphisms:

$$(T^2 \times [0,1]) \setminus (N_1 \cup N_2) \simeq (T^2 \times [0,1]) \cup L_1 \cup L_2 \simeq (T^2 \times [0,1]) \setminus (N_1' \cup N_2').$$

This completes the proof of the tightness of ξ on M. □

3.5. **Maximizing the twisting number of a regular fiber.**

Lemma 3.7. *There exists a Legendrian knot F in (M, ξ) isotopic to a regular fiber of the Seifert fibered structure with $t(F) = 0$.*

Lemma 3.7 follows immediately from the following lemma, together with Proposition 3.6.

Lemma 3.8. *If ξ' is a tight contact structure on M and all Legendrian curves isotopic to a regular fiber have negative twisting number, then ξ' is universally tight.*

Proof. To prove ξ' is universally tight, we will first show that it can be made transverse to the S^1-fibers of the Seifert fibration. This then implies that (M, ξ') is covered by the standard tight contact structure on $\mathbf{R}^3$.

Step 1. (Normalization of contact structure ξ'.) Let F be a Legendrian curve isotopic to a regular fiber with $t(F) = n < 0$, which we take to be maximal among Legendrian curves isotopic to a regular fiber. Let L_i, $i = 1, 2, 3$, be Legendrian curves simultaneously isotopic to the singular fibers F_i with $t(L_i) = n_i < 0$, and let V_i be a standard convex neighborhood of F_i. We assume that $t(L_1), t(L_2), t(L_3)$ are *simultaneously maximal* among Legendrian knots isotopic to (F_1, F_2, F_3) with negative twisting numbers — by this we mean there is no L_j' for which (i) $-1 \geq t(L_j') > t(L_j)$, (ii) L_j' is disjoint from all $L_i \neq L_j$, and (iii) L_j' is isotopic to L_j in $M \setminus (\cup_{i \neq j} L_i)$.

After making the Legendrian ruling curves on V_i vertical (*i.e.,* parallel to the regular S^1-fibers), take a convex annulus $\mathcal{A}$ with Legendrian boundary for which one component of $\partial \mathcal{A}$ is a ruling curve on V_2 and the other component is a ruling curve on V_3. If not all dividing curves on $\mathcal{A}$ connect between V_2 and V_3, then the Imbalance Principle (see [23]) gives rise to a bypass along a ruling curve for, say, V_2. Provided $t(L_2) < -1$, the Twist Number Lemma (see [23]) implies the existence of a Legendrian curve isotopic to L_2 with larger twisting number. Therefore, we conclude that either $n_2 = n_3 < -1$ and $\mathcal{A}$ has no ∂-parallel dividing curves, or $n_2 = n_3 = -1$.

Assume $n_2 = n_3 \leq -1$ and $\mathcal{A}$ has no ∂-parallel dividing curves. If $N(\mathcal{A})$ is a convex neighborhood of $\mathcal{A}$, then $M' = V_2 \cup V_3 \cup N(\mathcal{A})$ will have a piecewise smooth convex torus boundary. Rounding the corners in the standard way (see [23]), M' will be a convex torus with boundary slope

$$\frac{-n_2}{4n_2+1}+\frac{-n_2}{4n_2+1}+\frac{-1}{4n_2+1}=-\frac{2n_2+1}{4n_2+1},$$

measured using the identification $\partial(M\setminus V_1)\simeq \mathbf{R}^2/\mathbf{Z}^2$. Now $M''=\overline{M\setminus M'}$ is a solid torus with convex boundary with slope $-\frac{2n_2+1}{4n_2+1}$, measured using $\partial(M\setminus V_1)$, which is equivalent to slope $\frac{1}{2n_2+1}$ measured using ∂V_1. This implies that the contact structure on M'' is the unique contact structure on the standard neighborhood of the Legendrian knot L_1 with $n_1=2n_2+1$.

Now assume $n_2=n_3=-1$ and $\mathcal{A}$ has ∂-parallel dividing curves. Then we use the corresponding bypasses to thicken V_2, V_3 to V_2', V_3' so that the boundary slopes (measured on $-\partial(M\setminus V_i)$) are $-\frac{1}{2},-\frac{1}{2}$ or $-1,-1$ and the dividing curves on $\mathcal{A}$ between V_2' and V_3' do not have ∂-parallel curves. The former case gives an overtwisted contact structure and the latter yields a Legendrian curve isotopic to a regular fiber with zero twisting number.

Summarizing, ξ' has been normalized so that V_2 and V_3 are standard neighborhoods of Legendrian curves with twisting number $n_2=n_3$, $\mathcal{A}$ has no ∂-parallel dividing curves, and $V_1=M\setminus(V_2\cup V_3\cup N(\mathcal{A}))$ is a standard neighborhood of a Legendrian curve with twisting number $n_1=2n_2+1$.

Step 2. (Making ξ' transverse to the fibers.) At this point, $K=\partial V_2\cup\partial V_3\cup\mathcal{A}$ has Legendrian rulings by vertical curves — in other words, ξ' is tangent to the fibers along K. We perturb K slightly so that the characteristic foliation becomes nonsingular Morse-Smale, and $V_2\cap\mathcal{A}$ and $V_3\cap\mathcal{A}$ become transverse to ξ'. Since ∂V_2, ∂V_3, and $\mathcal{A}$ are all convex in *standard form*, it is possible to perturb K along the Legendrian divides as in [24] to accomplish this.

Now, it is a question of isotoping ξ' so that ξ' is transverse to the fibers on each V_i. Let us consider V_2, for example, and use the identification $\partial V_2\simeq\mathbf{R}^2/\mathbf{Z}^2$ to measure slope. The regular fibers of the Seifert fibration have slope -4, and the nonsingular Morse-Smale characteristic foliation has dividing curves of slope $-\frac{1}{n_2}$. Since $-4<-\frac{1}{n_2}$, it is clearly possible to extend $\xi'|_{\partial V_2}$ so that the contact structure is transverse to the Seifert fibers. Moreover, this extension is contact-isotopic to ξ' rel ∂V_2. In this way, we isotop ξ' so that ξ' is transverse to the S^1-fibers of M.

Step 3. (Pulling back to $\mathbf{R}^3$ and universal tightness.) We pull back in two stages:

$$\mathbf{R}^3\stackrel{\pi_1}{\to}T^3\stackrel{\pi_2}{\to}M.$$

Since the monodromy of M as a torus bundle has order four, there exists a 4-fold cover $\pi_2:T^3=T^2\times S^1\to M$ with the property that $\{pt\}\times S^1\subset T^3$ are components of preimages of fibers of M. Next we take the standard projection $\pi_1:\mathbf{R}^3\to T^3=\mathbf{R}^3/\mathbf{Z}^3$, for which the fibers $x=y=const$ of $\mathbf{R}^3$ with coordinates (x,y,z) project to fibers $x=y=const$ of T^3. The pullback $\pi_1^*\pi_2^*\xi'$ is therefore transverse to the $\frac{\partial}{\partial z}$ direction. It is a simple exercise to see that any contact structure on $\mathbf{R}$ transverse to $\frac{\partial}{\partial z}$ is conjugate to the standard contact structure and hence is tight. $\square$

3.6. **Twisting number increase for singular fibers.** We need one more result before the proof of Theorem 1.1.

Proposition 3.9. *There is a Legendrian knot L in (M,ξ) isotopic to one of the singular fibers F_2 or F_3 with $t(L)=0$.*

Proof. Let F be a Legendrian knot isotopic to a regular fiber with $t(F)=0$ as in Lemma 3.7. Let V_i', $i=1,2,3$, be disjoint solid tori simultaneously isotopic to tubular neighborhoods of F_i, for which $\partial V_i'$ contains a contact-isotopic copy of F. By perturbing $\partial V_i'$ we may assume V_i' is convex with vertical (*i.e.,* parallel to the regular fibers) dividing curves, and, furthermore, we may assume that $\#\Gamma_{\partial V_i'}=2$, after possibly taking a smaller solid torus. In order to increase the twisting number of a Legendrian curve, we need to find a bypass. We will find a bypass along, say, V_3' by patching together meridional disks of V_1' and V_2' to obtain a punctured torus T and showing the existence of a ∂-parallel dividing curve on T.

Step 1. (Normalizing ξ on the complement.) Let us first normalize the tight contact structure on $\Sigma_0\times S^1 = M\setminus(V_1'\cup V_2'\cup V_3')$. Assume that the characteristic foliation on $\partial(\Sigma_0\times S^1)$ consists of rulings by closed Legendrian curves parallel to $\partial\Sigma_0$. This allow us to take Σ_0 to be convex with Legendrian boundary.

Lemma 3.10. *The contact structure on $\Sigma_0\times S^1$ is contactomorphic to a $[0,1]$-invariant tight contact structure on $T^2\times[0,1]$ with convex boundary, $\#\Gamma_{T_i}=2$, $i=1,2$, slopes $s_i=\infty$, and horizontal Legendrian rulings (i.e., the tight contact structure induced on $T^2\times[0,1]$, thought of as a neighborhood of a convex torus in standard form), with a standard (open) neighborhood of a vertical (i.e., isotopic to $\{pt\}\times S^1\subset T^2$) Legendrian curve with 0 twisting removed.*

Proof. We reproduce the proof of this lemma given in [16] (*cf.* [24]). We first claim there are no ∂-parallel dividing curves on Σ_0. If there existed a ∂-parallel arc of Γ_{Σ_0} along (for example) $\Sigma_0\cap\partial V_1'$, then we could use the corresponding bypass to thicken V_1' so that the slope of the new convex torus S parallel to $\partial V_1'$ is zero. We may further thicken V_1' using a bypass obtained by looking at a convex vertical annulus from S to $\partial V_2'$ (with Legendrian boundary, after the usual modifications using Giroux's Flexibility Theorem [17]). In the end we attach to V_1' a toric annulus with torsion exactly π. This would allow us to realize the meridional slope as a Legendrian divide of a convex torus and find an overtwisted disk in M.

Now, Γ_{Σ_0} has exactly two endpoints each on $\Sigma_0\cap\partial V_i'$, $i=1,2,3$. Since closed dividing curves would force ∂-parallel dividing curves to exist, there exist only two possibilities for Γ_{Σ_0}, modulo Dehn twists parallel to the boundary components, *i.e.*, spiraling. Cutting $\Sigma_0 \times S^1$ open along Σ_0 and applying edge-rounding [23], we obtain a tight contact structure on a genus two handlebody H with convex boundary. In addition, modulo modification of the boundary characteristic foliation using the Legendrian realization principle [23], it is easy to find two convex meridional disks D_1 and D_1 with Legendrian boundary and $tb(D_i) = -1$, $i = 1, 2$, for which $H \setminus (D_1 \cup D_2)$ is a 3-ball. Now, there is a unique choice for Γ_{D_1} and Γ_{D_2}. Hence, the tight contact structure on H is uniquely determined. From this we can conclude that there is a unique tight contact structure on $\Sigma_0 \times S^1$ with the given dividing curve data. Since the contact structure described in Lemma 3.10 also has this dividing curve data, our contact structure must be contactomorphic to it. □

Step 2. (Patching meridional disks.) If we measure slopes of $\Gamma_{\partial V_i'}$, $i = 1, 2, 3$, using the identification $\partial V_i \simeq \mathbf{R}^2/\mathbf{Z}^2$, then the slopes are 2, -4 and -4, respectively. After making the ruling curves on $\partial V_i'$ meridional, a convex meridional disk D_i for V_i' will have, respectively, $tb(\partial D_i) = -2$, -4, -4, and also 2, 4 and 4 dividing curves. We would like to patch copies of the meridional disks together to create a convex surface and moreover relate information about the dividing curves on this patched-together surface to the dividing curves on the meridional disks.

We view the $T^2 \times [0,1]$ (minus $D^2 \times S^1$) from Lemma 3.10 as the region between $\partial V_1'$ and $\partial V_2'$ (minus V_3'). Assume $T_0 = \partial V_2'$ and $T_1 = -\partial V_1'$. $T^2 \times [0,1]$ is foliated by parallel convex surfaces T_t, $t \in [0,1]$, due to the I-invariance. Now consider the 1-parameter family $\{(T_t)_+\}_{t \in [0,1]}$, where we recall $(T_t)_+$ is the positive part of $T_t \setminus \Gamma_{T_t}$. Also assume that the $D^2 \times S^1$ removed is the neighborhood of a Legendrian divide on $(T_{1/2})_-$. We may then isotop T_i, $i = 0, 1$, away from $(T_i)_+$ (*i.e.*, on $(T_i)_-$) to arrange the slopes of the Legendrian ruling curves so that the meridional disk D_i in V_i' has Legendrian boundary. Now, take one copy of D_2 and two copies D_{11}, D_{12} of D_1, and arrange them so that $D_2 \cap (T_0)_+ = \delta \times \{0\}$ and $(D_{11} \cup D_{12}) \cap (T_1)_+ = \delta \times \{1\}$, where δ is a union of Legendrian arcs on $(T^2)_+$ with endpoints on opposite edges of $\partial (T^2)_+$. Let $T = D_{11} \cup D_{12} \cup D_2 \cup (\delta \times [0,1])$, which is a torus with an open disk removed. See Figure 3.

After smoothing the corners using the "elliptic monodromy lemma" or the "pivot lemma" of [15] or [12], T will have smooth Legendrian boundary. Since $\partial T \subset \partial((T^2)_- \times [0,1])$, we shall think of T as having its boundary on $\partial V_3'$.

Step 3. (Combinatorics of D_1, D_2 and D_3.) Since D_1 has two dividing curves, $D_1 \setminus \Gamma_{D_1}$ either consists of two positive regions and one negative region, or one positive and two negative regions. We assume the former — the argument for the latter is identical.

The rotation number $r(\partial D_i)$, $i = 2, 3$, satisfies the formula $r(\partial D_i) = \chi((D_i)_+) - \chi((D_i)_-)$ in [28]. Therefore, $r(\partial D_i)$ can attain values $-3, -1, 1, 3$.

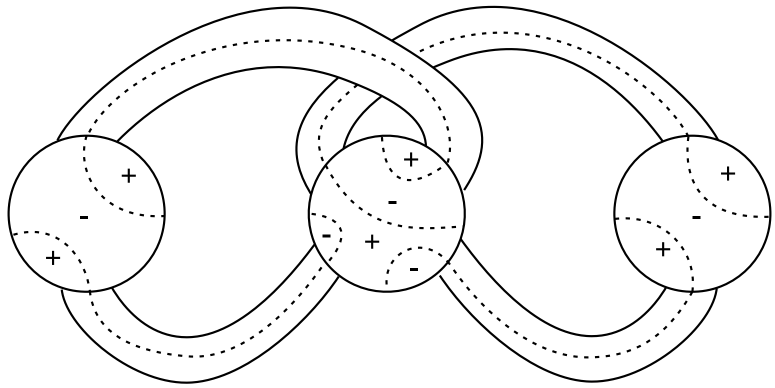

FIGURE 3. The punctured torus T with dividing curves (dashed lines).

Step 3A. Assume that at least one of D_2 or D_3 (say D_2, after possible relabeling) satisfies $r(\partial D_i) > -3$. We first show that D_2, after possibly isotoping rel ∂D_2, will have a positive ∂-parallel region. If $r(\partial D_2) = 3$ or 1, there is no problem. If $r(\partial D_2) = -1$, the dividing curves on D_2 may be either of the two types shown in Figure 4. If we have a configuration shown

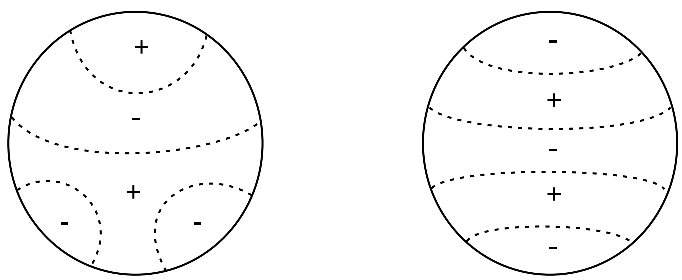

FIGURE 4. Possible dividing curves on D_2.

on the right-hand side of Figure 4, then we may isotop D_2 rel ∂D_2 so that the dividing curves on D_2 are as shown on the left-hand side of Figure 4. This follows from the classification of tight contact structures on solid tori in [23] or [18].

If $r(\partial D_2) = -1$, then we take the dividing curves on D_2 to be as shown on the left-hand side of Figure 4. The dividing curves on T will then be as shown in Figure 3. Note we have a ∂-parallel component and hence a bypass along ∂T. The cases $r(\partial D_2) = 1, 3$ are similar, by observing that any positive ∂-parallel component of D_2 must necessarily be connected to a positive ∂-parallel component on one of the copies of D_1, yielding a ∂-parallel component on T.

The slope of D_1 on $\partial(M \setminus V_1')$ is $-\frac{1}{2}$ and the slope of D_2 on $\partial(M \setminus V_2')$ is $\frac{1}{4}$. This implies that the slope of T on $-\partial(M \setminus V_3')$ is $-\frac{1}{4}$. We therefore have a bypass on $\partial V_3'$ attached along a ruling curve of slope $-\frac{1}{4}$ (as measured using $-\partial(M \setminus V_3'))$. Using this bypass, we may thicken V_3' to V_3'' with standard convex boundary having boundary slope 0. Thus, when measured from the product structure $D^2 \times S^1$ on V_3'', the slope is ∞, showing that V_3'' is the standard neighborhood of a Legendrian curve L isotopic to F_3 with twist number 0.

Step 3B. We are left with the case where $r(\partial D_2) = r(\partial D_3) = -3$. Now the dividing curves on the punctured torus T constructed from D_2 and two copies of D_1 will be as in Figure 5. Capping T off with D_3, we obtain a closed contractible dividing curve on the torus $T \cup D_3$ which contradicts tightness.

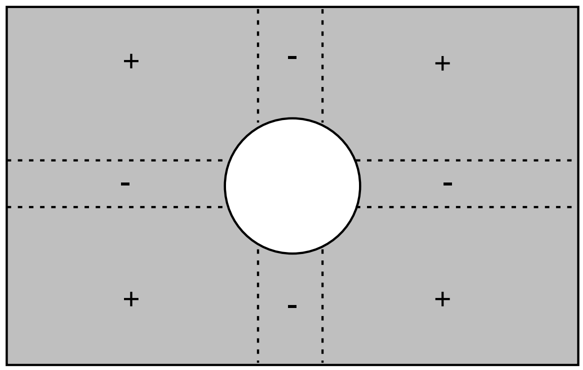

FIGURE 5. The punctured torus T (shaded).

This completes the proof of Proposition 3.9. □

3.7. **Proof of Theorem 1.1 for the Seifert fibered space with invariants** $(-\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$**.** We are now ready to prove our main theorem.

Proof. From Proposition 3.6 we have a tight contact structure ξ on M. Now assume that ξ is weakly symplectically semi-fillable. From Proposition 3.9 we have a Legendrian knot L in (M, ξ) that is isotopic to, say, F_3 with twist number 0. We may assume that the neighborhood V_3 was chosen so that $L = F_3$. As discussed above, if we perform Legendrian surgery on L, we remove a small neighborhood of L (take this neighborhood to lie in V_3) and re-glue it by $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. We easily see this has the same effect as changing the A_3 to

$$\begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Thus, after Legendrian surgery we obtain a weakly symplectically semi-fillable contact structure on M', the Seifert fibered space over S^2 with Seifert invariants $(-\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$. This contradicts Lisca's Theorem (Theorem 2.2), thus proving ξ is not weakly symplectically semi-fillable. □

3.8. Modifications for the Seifert fibered space with invariants $(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$.

The steps are almost identical for the Seifert fibered space M_2 over S^1 with 3 singular fibers and invariants $(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$. The manifold M_2 has a presentation as a torus bundle over S^1 with monodromy $\mathbb{A} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. By the classification in [24], there exist two virtually overtwisted contact structures on M_2 which are nonisotopic but isomorphic. Let ξ be one such contact structure. As before, we first find a Legendrian knot F isotopic to a regular fiber with $t(F) = 0$. We then find a Legendrian curve L isotopic to the $-\frac{2}{3}$-fiber F_1 with $t(L)$ large enough to perform a Legendrian surgery which modifies the Seifert invariants as follows:

$$\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) \rightsquigarrow \left(-\frac{1}{2}, \frac{1}{3}, \frac{1}{3}\right).$$

This again gives a contradiction of Theorem 2.2. The existence of such an L is proved by patching together meridional disks as in Proposition 3.9 — the only difference is that in one case we need to apply a "thinning before thickening" argument that is used in [16].

4. Further Questions

The obvious question raised in this paper is:

Question 1. *Are the contact structures on the Seifert fibered spaces over S^2 with Seifert invariants $(-\frac{1}{2}, \frac{1}{3}, \frac{1}{3})$ or $(-\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$ constructed above tight?*

We conjecture that these contact structures are tight. If the conjecture is true, we would have an example of a manifold that supported a tight contact structure but no symplectically fillable contact structures. Since this contact structure is constructed from a tight contact structure by Legendrian surgery, we are led to ask the following:

Question 2. *Is Legendrian surgery category-preserving for tight structures on closed 3-manifolds?*

If not, are there conditions which are sufficient to guarantee that Legendrian surgery on the contact structure yields a tight contact structure? For example,

Question 3. *Does Legendrian surgery on a universally tight contact structure on a closed 3-manifold produce a tight contact structure?*

Recall our tight but not symplectically fillable contact structure is virtually overtwisted. All other potential tight but not symplectically fillable contact structures known to the authors are also virtually overtwisted. (There are several candidates mentioned in [24].) So we ask:

Question 4. *Are all universally tight contact structures symplectically semi-fillable?*

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