An Inverse Problem Associated with Polynomials Orthogonal on the Unit Circle

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§0. Abstract

Polynomials orthogonal on the unit circle with random recurrence coefficients and finite band spectrum are investigated. It is shown that the coefficients are in fact quasi-periodic. The measures associated with these quasi-periodic coefficients are exhibited and necessary and sufficient conditions relating quasi-periodicity and spectral measures of this type are given. Analogs for polynomials orthogonal on subsets of the real line are also presented.

§1. Introduction

This paper is a companion to [GJ], in which we developed the essential theory of random orthogonal polynomials on the unit circle from a dynamical systems point of view. Our goal here is to formulate and solve an inverse problem for such random orthogonal polynomials. In addition to techniques of dynamical systems, we will use some elementary methods from the theory of algebraic curves, in particular the generalized Jacobian [F]. These methods permit us also to generalize some results obtained for polynomials orthogonal on the unit circle with periodic recurrence coefficients to the case when the recurrence coefficients are quasi-periodic.

To be more precise, let $T(z,n)$ be the following matrix:

$$T(z,n) = a_n \begin{pmatrix} z & \alpha_n \\ z\bar{\alpha}_n & 1 \end{pmatrix}, \tag{1.1}$$

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where $\{\alpha_n\}_1^\infty$ is a sequence of complex numbers with values in the unit disc in the complex plane, and z a complex number. The normalizing constant a_n is given by

$$a_n = (1 - |\alpha_n|^2)^{-1/2}.$$

Orthonormal polynomials on the unit circle $K = \{z \in \mathbb{C} \mid |z| = 1\}$ can be generated in the following way. Write

$$\begin{pmatrix} \phi_n(z) \\ \phi_n^*(z) \end{pmatrix} = T(z,n) \cdots T(z,1) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{1.2}$$

Then there is a unique probability measure $d\sigma$ on K with the property that $\{\phi_n(\cdot) \mid n = 0, 1, 2, \ldots\}$ is orthonormal with respect to $d\sigma$. The polynomials $\phi_n^*(z) = z^n \overline{\phi}_n(1/z)$ are called the dual polynomials. The relation between the quantities α_n and the polynomials ϕ_n is as follows:

$$\alpha_n = \frac{\phi_n(0)}{\phi_n^*(0)},$$

which easily follows from (1.1) and (1.2).

In section 2 of [GJ] the Weyl m functions, $m_\pm(z,n)$, were introduced ($m_+ = \tilde{m}_+$, $m_- = \frac{1}{\tilde{m}_-}$ in [GJ, section 2]). These were shown to be analytic for $|z| < 1$ with $|m_+| < 1$ and $|m_-| > 1$ in this region (in [GJ] these results where stated only for $m_\pm(z,0)$ however the proof carries over to $n \neq 0$). Furthermore it was shown (see [GJ], Eqs. (2.15) and (2.16)) that these functions satisfy the following relations

$$m_+(z,n) = -z \frac{\bar{\alpha}_{n+1} - m_+(z,n+1)}{1 - \alpha_{n+1} m_+(z,n+1)}, \tag{1.3}$$

and

$$m_-(z,n+1) = \frac{z\bar{\alpha}_{n+1} + m_-(z,n)}{z + \alpha_{n+1} m_-(z,n)}. \tag{1.4}$$

These functions will be introduced in a more geometric fashion below. From (1.3) and (1.4) we can evaluate $\lim_{z\to 0} m_\pm(z,n)$ to find,

$$m_+(0,n) = 0, \qquad m_-(0,n) = \frac{1}{\alpha_n}. \tag{1.5}$$

Likewise if these functions can be analytically continued to $|z| > 1$ then we find

$$m_+(\infty,n) = \infty, \qquad m_-(\infty,n) = \bar{\alpha}_n. \tag{1.6}$$

If we consider the functions

$$\tilde{m}_\pm(z) = \frac{1 + m_\pm(z,0)}{1 - m_\pm(z,0)}, \tag{1.7}$$

then (1.5) and (1.6) show that

$$\begin{aligned} \tilde{m}_-(0) &= \left[\frac{\alpha_0 + 1}{\alpha_0 - 1}\right] & \qquad \tilde{m}_+(0) &= 1 \\ \tilde{m}_-(\infty) &= \left[\frac{1 + \overline{\alpha_0}}{1 - \overline{\alpha_0}}\right] & \qquad \tilde{m}_+(\infty) &= -1 \end{aligned}$$

It now follows from the relation between Caratheodory functions and positive measures on the unit circle ([Akh]) that

$$\tilde{m}_+ = \int_K \left\{\frac{e^{i\theta} + z}{e^{i\theta} - z}\right\} d\sigma_+(\theta), \tag{1.8}$$

and

$$\tilde{m}_- = \frac{\overline{\alpha_0} - \alpha_0}{|1 - \alpha_0|^2} - \int_K \left\{\frac{e^{i\theta} + z}{e^{i\theta} - z}\right\} d\sigma_-(\theta), \tag{1.9}$$

where σ_+ and σ_- are positive measures supported on the circle such that $\int_K d\sigma_+ = 1$ and $\int_K d\sigma_- = \frac{1-|\alpha_0|^2}{|1-\alpha_0|^2}$, ($\sigma_+$ is the orthogonality measure σ).

The problem we pose and solve is the following. Let $\{\alpha_n\}$ be a stationary, ergodic sequence and consider the set Σ of non-isolated points of the topological support of the measure $d\sigma$. Thus Σ is a closed subset of K. We will characterize those stationary ergodic sequences $\{\alpha_n\}$ for which

(i) Σ is a finite union of arcs;

(ii) the Lyapounov exponent $\gamma(z)$ of the family $\{\phi_n(z) \mid n \geq 1\}$ is zero for $z \in \Sigma$.

Here $\gamma(z)$ measures the exponential rate of growth of the product in (1.1):

$$\gamma(z) = \lim_{n\to\infty} \frac{1}{n} \log \|T(z,n) \cdots T(z,1)\|. \tag{1.10}$$

In fact, we will show that the sequence $\alpha_1, \ldots, \alpha_n, \ldots$ can be obtained as the evaluation of an explicit rational function along a straight-line winding in a real subtorus of a generalized Jacobian variety.

These results are rather similar to those holding for the AKNS operator whose inverse spectral theory was discussed by [AKNS] and then, in a context similar to the one considered in this paper, by [DC-J]. That paper was in turn inspired by the work in the 1970's on the inverse spectral theory of the periodic Schrödinger operator [DMN]. The idea of regarding the set Σ as an "essential spectrum" and z as a "spectral parameter" is quite natural. Indeed the suspension technique [W] can be used to write the matrix product in (1.2) as the discretization of the fundamental matrix solution of an Atkinson-type spectral problem [A]; the details are carried out in [GJ]. Results of this paper will be used below.

This paper is organized as follows. In §2 we formulate the inverse problem in a precise way and review the basic results we will need to solve it. In §3 we carry out the solution of the inverse problem using methods of algebraic curve theory. We work out explicit formulas for those stationary ergodic sequences satisfying condition (i) and (ii).

The solution of the inverse problem mentioned above rests upon the Weyl m functions satisfying Equation (2.5). A similar property plays an important role for Jacobi operators associated with orthogonal polynomials on the real line ([BGHT], [GT]). Operators having this property have come to be called "reflectionless". In §4 we discuss the theory of polynomials orthogonal on the unit circle (which can also be associated with an operator [GT], [T], [GNV]) which have this reflectionless property. This allows us to make contact with the work in [PS]. Finally we utilize the Stahl-Totik theory of general orthogonal polynomials to obtain some stability results. Some analogs of these results for polynomials orthogonal on a finite number of segments of the real line are also given.

We finish this Introduction by discussing the concept of randomization as we use it and some basic definitions regarding random orthogonal polynomials.

Let (Ω, μ) be a probability space and let $A : \Omega \to D$ be a μ-measurable map of Ω to the open unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Let $\tau : \Omega \to \Omega$ be a bimeasurable bijection with respect to which μ is ergodic [W]. Suppose that there is a point $\omega \in \Omega$ such that

$$\alpha_n = A(\tau^{n-1}(\omega)).$$

In this situation, we say that the matrices $T(z, n)$ (or the orthogonal polynomials $\{\phi_n(z)\}$) have been *randomized*. Note that

$$\alpha_1 \equiv \alpha_1(\omega) = A(\omega);$$

to save notation we will write α_1 in place of A from now on. We have $\alpha_n(\omega) = \alpha_1(\tau^{n-1}(\omega))$ $(n \geq 1)$. Since τ is assumed bijective, α_n is also defined for negative n and for $n = 0$.

It will be convenient to replace the pair (Ω, α_1) with the path space which it generates. Thus for each $\omega \in \Omega$, let $s_\omega = (\alpha_1(\tau^{n-1}(\omega))_{-\infty<n<\infty}$ be the associated bi-infinite sequence. We view this sequence as an element of the bi-infinite product $X = \prod_{-\infty}^{\infty} D$ of copies of the unit disc. Give X the product (Tychonov) topology.

Unless otherwise stated we will assume that

$$\|\alpha_1\|_\infty < 1. \tag{1.11}$$

This will allow us to use the results of [GJ] and, as we will see, presents no loss of generality as far as the final solution of the inverse problem is concerned. An immediate consequence of (1.11) is that, for μ-a.a. ω, the sequence s_ω lies in the compact subset $X_1 = \prod_{-\infty}^{\infty}\{z \in D : |z| \leq \|\alpha_1\|_\infty\}$ of X. Let $\tilde{\Omega} = \text{cls}\{s_\omega \mid \omega \in \Omega_0\}$ where $\Omega_0 \subset \Omega$ is the set of ω for which s_ω lies in X_1. Then $\tilde{\Omega}$ is invariant under the homeomorphism $\tilde{\tau}$ of X_1 defined by left translation. Moreover the image of $\tilde{\mu}$ of the measure μ under the map $i : \Omega_0 \to \tilde{\Omega} : \omega \to s_\omega$ is ergodic with respect to the translation $\tilde{\tau}$. It is natural to identify α_1 with the map $\tilde{\alpha}_1 : \tilde{\Omega} \to D : (\cdots s_{-1}, s_0, s_1, \ldots) \to s_0$. In what follows we will identify $(\Omega, \tau, \mu, \alpha_1)$ with $(\tilde{\Omega}, \tilde{\tau}, \tilde{\mu}, \tilde{\alpha}_1)$. Note that Ω is then a compact metric space and α_1 is continuous.

Next let $0 \neq z \in \mathbb{C}$; and define

$$\begin{aligned} \Phi_z(\omega, n) &= T(z,n)\cdots T(z,1) \qquad (n \geq 1) \\ \Phi_z(\omega, 0) &= I \\ \Phi_z(\omega, -n) &= T(z,-n+1)^{-1}\cdots T(z,0)^{-1} \qquad (n \geq 1). \end{aligned} \tag{1.12}$$

Then we can check that Φ_z is a *cocycle* in the sense that

$$\Phi_z(\tau^n(\omega), m)\Phi_z(\omega, n) = \Phi_z(\omega, m+n).$$

We have

$$\begin{pmatrix} \phi_n(z;\omega) \\ \phi_n^*(z;\omega) \end{pmatrix} = \Phi_z(\omega, n)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (n \geq 1)$$

for each $\omega \in \Omega$.

We now repeat two definitions of basic importance in the theory of random differential and difference equations.

Definition 1.1. Let $0 \neq z \in \mathbb{C}$, and define

$$\gamma(z) = \lim_{n\to\infty} \frac{1}{n} \log \|\Phi_z(\omega, n)\|.$$

The limit exists and is constant for μ-a.a. $\omega \in \Omega$ [O]. This μ-almost everywhere limit is the *Lyapounov exponent* of the cocycle Φ_z.

Definition 1.2. [C] Let $0 \neq z \in \mathbb{C}$. We say that the cocycle $\Phi_z(\omega, n)$ has an *exponential dichotomy* over Ω if there are constants $K > 0$, $\beta > 0$ and a continuous projection-valued function $P(\cdot) : \mathbb{C}^2 \to \mathbb{C}^2$ on Ω such that

$$\|\Phi_z(\omega, n)P(\omega)\Phi_z(\omega, m)^{-1}\| \leq Ke^{-\beta(n-m)} \qquad (n \geq m)$$

$$\|\Phi_z(\omega, n)(I - P(\omega))\Phi_z(\omega, m)^{-1}\| \leq Ke^{\beta(n-m)} \qquad (n \leq m).$$

The range $\operatorname{Im} P(\omega)$ and the kernel $\operatorname{Ker} P(\omega)$ define a hyperbolic splitting of the cocycle Φ_z in the product space $\Omega \times \mathbb{C}^2$.

§2. The Inverse Problem

We formulate the problem to be solved in this paper. Let Ω be a compact metric space, $\tau : \Omega \to \Omega$ a homeomorphism, $\alpha_1 : \Omega \to D$ a continuous map of Ω to the open unit disc D, and let μ be a Radon probability measure on Ω which is ergodic with respect to τ. Let $\alpha_n(\omega) = \alpha_1(\tau^{n-1}(\omega))$ $(n \in \mathbb{Z})$, and let $\{\phi_n(z) \equiv \phi_n(z;\omega)\}_{n\geq 0}$ be the corresponding family of orthogonal polynomials. Thus

$$\begin{pmatrix} \phi_n(z;\omega) \\ \phi_n^*(z;\omega) \end{pmatrix} = T_z(\omega, n) \cdots T_z(\omega, 1) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where

$$T_z(\omega, n) = a_n(\omega) \begin{pmatrix} z & \alpha_n(\omega) \\ z\overline{\alpha_n(\omega)} & 1 \end{pmatrix} \tag{2.1}$$

and $a_n(\omega) = (1 - |\alpha_n(\omega)|^2)^{-1/2}$. Let $\gamma(z)$ be the Lyapounov exponent of the cocycle $\Phi_z(\omega, n)$ defined by $\{T_z\}$ (see (1.12)). For each $\omega \in \Omega$, let Σ be the set of non-isolated points in the topological support of the orthogonality measure $d\sigma_\omega$ of the polynomials $\{\phi_n(z;\omega) \mid n \geq 1\}$ (see [A], [G]). Then Σ is a closed subset of the unit circle K. It is known (e.g., [GJ, Thm. 5.4]) that Σ is independent of ω μ-a.e. in Ω.

We will suppose that

(i) $\Sigma = [z_0, z_1] \cup [z_2, z_3] \cup \cdots \cup [z_{2N-2}, z_{2N-1}]$ is a finite union of (necessarily non-degenerate) arcs in K;

(ii) $\gamma(z) = 0$ for Lebesgue-a.a. $z \in \Sigma$.

Here the points $z_0, \ldots, z_{2N-1}$ are ordered (say) counter-clockwise on K. Starting from assumptions (i) and (ii), we will devise an explicit description of (Ω, τ, μ) and a formula for α_1. Note that with these assumptions γ is the Green's function associated with the complex set Σ i.e., γ is harmonic on $\mathbb{C}/\Sigma$, $\gamma - \log|z|$ is harmonic at infinity and $\gamma(z) = 0$ Lebesgue - a.a. on Σ.

Our starting point will be the Weyl m-functions, discussed in detail in [GJ]. First of all, it is proved in [GJ, Thm. 4.3] that, if $z \notin \Sigma$, then the cocycle Φ_z admits an exponential dichotomy over the topological support of the measure μ in Ω. We henceforth assume that Ω *is* the topological support of μ; thus Φ_z has an exponential dichotomy over $\Omega \times \mathbb{C}^2$ if $z \notin \Sigma$.

We define stable and unstable bundles $W^{\pm} \subset \Omega \times \mathbb{C}^2$ as follows:

$$W^+ = \{(\omega, u) \mid u \in \operatorname{Im} P(\omega)\},$$
$$W^- = \{(\omega, u) \mid u \in \operatorname{Ker} P(\omega)\},$$

where $P(\omega)$ is the projection of Definition 1.2. As is shown in [GJ section 5], W^+ and W^- are both one-dimensional (line) bundles over Ω. For each $\omega \in \Omega$ and $0 \neq z$, $z \notin \Sigma$, let $m_{\pm}(z, \omega)$ be the unique extended complex numbers such that

$$W^{\pm} \cap (\{\omega\} \times \mathbb{C}^2) = \operatorname{Span}\begin{pmatrix} 1 \\ m_{\pm}(z,\omega) \end{pmatrix} = \left\{ c \begin{pmatrix} 1 \\ m_{\pm}(z,\omega) \end{pmatrix} : c \in \mathbb{C} \right\}.$$

Thus $m_{\pm}$ are just the projective coordinates of the complex lines $W^{\pm} \cap (\{\omega\} \times \mathbb{C}^2)$.

We now have

Theorem 2.1. *For each $\omega \in \Omega = \operatorname{Supp} \mu$, the functions $m_{\pm}(z) \equiv m_{\pm}(z, \omega)$ are branches of a single meromorphic function $M(z) \equiv M(z, \omega)$ on the Riemann surface C of the algebraic relation $w^2 = K(z) = \prod_{j=1}^{2N} (z - z_j)$.*

Here we have written $z_{2N} = z_0$; thus the points $z_1, z_2, \ldots, z_{2N}$ coincide with the points $z_1, z_2, \ldots, z_0$. The intervals $(z_1, z_2), \ldots, (z_{2N-1}, z_{2N})$ determine the complement $K - \Sigma$.

Proof. We first note that in section 4 of [GJ] it is shown that $m_\pm(z)$ are analytic for $|z|<1$ and that $|m_+|<1$, while $|m_-|>1$ for all such z. If we write

$$\begin{pmatrix} a \\ b \end{pmatrix} = T_z(\omega,1)\begin{pmatrix} 1 \\ m_-(z,\omega) \end{pmatrix}$$

and recognize that $\frac{b}{a} = m_-(z,\tau(\omega))$ (by the invariance of the bundle W^-), we obtain the equation of motion for m_- (see also (1.3)),

$$m_-(z,\tau(\omega)) = \frac{z\overline{\alpha_1(\omega)} + m_-(z,\omega)}{z+\alpha_1(\omega)m_-(z,\omega)}. \tag{2.2}$$

Likewise $\binom{a}{b} = T_z^{-1}(\omega,1)\binom{1}{m_+(z,\tau(\omega))}$ leads to the equation

$$m_+(z,\omega) = -z\frac{\overline{\alpha_1(\omega)} - m_+(z,\tau(\omega))}{1-\alpha_1(\omega)m_+(z,\tau(\omega))}. \tag{2.3}$$

The exponential dicotomy now allows an analytic continuation of $m_\pm$ to $K-\Sigma\cup\{z: |z|>1\}$. Equations (2.2) and (2.3) again allow us to evaluate $\lim_{z\to\infty} m_\pm$ and $\lim_{z\to 0} m_\pm$. We collect the needed values for future reference:

$$\begin{aligned} m_-(0) &= \frac{1}{\alpha_1(\tau^{-1}(\omega))} \qquad & m_+(0) &= 0 \\ m_-(\infty) &= \overline{\alpha_1(\tau^{-1}(\omega))} \qquad & m_+(\infty) &= \infty. \end{aligned} \tag{2.4}$$

Next we see the Kotani-type result proved in [GT, Thm. 5.7] and [GJ, Cor. 5.12] (take careful note of the change in notation used in the proofs in those papers) to conclude that the functions $m_\pm(z)$, extend holomorphically through each open subinterval $(z_0,z_1),\ldots,(z_{2N-2},z_{2N-1})$ of Σ. Moreover the extensions satisfy the following relations

$$\lim_{r\to 1^-}\overline{m_+(rz,\omega)} = \lim_{r\to 1^-}\frac{1}{m_-(rz,\omega)}, \tag{2.5}$$

for each $\omega\in\Omega$ and each z in the interior of Σ.

At this point we see that $m_\pm$ are branches of a function $M=M(\cdot,\omega)$ defined and meromorphic on C with the possible exception of the branch points on C, i.e., the inverse images in C of the points $z_1,\ldots,z_{2N}\in\mathbb{C}$ under the natural projection π mapping C to the extended complex plane. However, the Riemann theorem on isolated singularities together with the fact [GJ, Eq. (5.1-5.2)] that $m_\pm$ take values on K exactly when z belongs to a "resolvent interval" (z_{2i-1}, z_{2i}) $(1 \leq i \leq N)$ shows that M is meromorphic on the entire curve C, for all $\omega \in \Omega$. This also implies that $m_+(z_i, \omega) = \frac{1}{m_-(z_i,\omega)}, i = 1, \ldots, 2N$. $\square$

The Riemann surface C contains two preimages under π of $z = 0$ and of $z = \infty$ respectively. With an eye to (2.4), it is convenient to label these points as $0_\pm$ and $\infty_\pm$ where $M(0_+) = 0$, $M(\infty_+) = \infty$ and $M(0_-) = \frac{1}{\alpha_1(\tau^{-1}(\omega))}$, $M(\infty_-) = \overline{\alpha_1(\tau^{-1}(\omega))}$. Note that this labelling does not determine the values of the meromorphic function $w = \sqrt{K(z)}$ on C. We agree to define $\sqrt{K(z)}$ so that, as $z \to \infty_+$, $\sqrt{K(z)} = z^N + \cdots$. Then, as $z \to \infty_-$, $\sqrt{K(z)} = -z^N + \cdots$. The function $w = \sqrt{K}$ is now well-defined on C, however its value at 0_+ may turn out to be either of the square roots of $\prod_{i=1}^{2N} z_i$.

We will now derive an explicit formula for the meromorphic function M. It is convenient to change coordinates. Define

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix},$$

and consider the transfer matrix $\hat{T} = ATA^{-1}$ in the coordinate $\hat{u} = Au$ defined by A. We obtain

$$\hat{T}(z,1) = \frac{-ia_1}{2} \begin{pmatrix} i[z(1-\overline{\alpha_1}) + 1 - \alpha_1] & z(1-\overline{\alpha_1}) - (1-\alpha_1) \\ -[z(1+\overline{\alpha_1}) - (1+\alpha_1)] & i[z(1+\overline{\alpha_1}) + (1+\alpha_1)] \end{pmatrix} \tag{2.6}$$

where α_1 and a_1 are evaluated at $\omega \in \Omega$. Let $\hat{m}$ be the complex projective coordinate in the $\hat{u}$-space. We have

$$\hat{m} = i\left(\frac{1+m}{1-m}\right). \tag{2.7}$$

Using the transformation (2.7), we obtain the parameterizations $\hat{m}_\pm(z,\omega)$ of the bundles $W^\pm$ in the $\hat{u}$-coordinate. These satisfy (see (2.4)):

$$\begin{aligned} \hat{m}_-(0) &= i\left[\frac{\alpha_1(\tau^{-1}(\omega)) + 1}{\alpha_1(\tau^{-1}(\omega)) - 1}\right] & \hat{m}_+(0) &= i \\ \hat{m}_-(\infty) &= i\left[\frac{1 + \overline{\alpha_1(\tau^{-1}(\omega))}}{1 - \overline{\alpha_1(\tau^{-1}(\omega))}}\right] & \hat{m}_+(\infty) &= -i. \end{aligned} \tag{2.8}$$

The functions $\hat{m}_\pm$ are branches of a single meromorphic function $\hat{M}$ on C, which we fix by requiring that $\hat{M}(\infty_+) = -i$. Note that (2.5) implies that

$$\hat{m}_+(z) = \overline{\hat{m}_-(z)}, \tag{2.9}$$

for z in Σ.

Let us determine the function $\hat{M}$. In what follows, this function will be more convenient than the function M. First of all since $|m_+| < 1$, while $|m_-| > 1$ for $|z| < 1$ we find that $\operatorname{Im} \hat{m}_+ > 0$ and $\operatorname{Im} \hat{m}_- < 0$ if $|z| < 1$, and by the relation of $\hat{m}_\pm$ to positive measures on the circle (see (1.8) and (1.9)) it follows that if these can be continued to $|z| > 1$ then $\operatorname{Im} \hat{m}_+ < 0$ and $\operatorname{Im} \hat{m}_- > 0$ in this region. From this plus the fact that, from (2.9), $\hat{m}_+(z_i) = \hat{m}_-(z_i)$, $i = 1, \ldots, 2N$ it follows without difficulty, that for each $\omega \in \Omega$, the function $\hat{M}$ has exactly one simple pole $P_i(\omega)$ in each simple closed curve $\pi^{-1}([z_{2i-1}, z_{2i}]) \subset C$ $(1 \le i \le N)$. Here we use the fact that $\hat{m}_+$ has a real positive angular derivative and $\hat{m}_-$ has a real negative angular derivative at each point of a resolvent interval (z_{2i-1}, z_{2i}). Note that these curves determine exactly the set where $\operatorname{Im} \hat{M} = \infty$. In what follows, we will denote by p_i the images of the poles P_i, i.e., $p_i = \pi(P_i) \in [z_{2i-1}, z_{2i}]$. Note that $K^{\frac{1}{2}}(P_i) = \pm K^{\frac{1}{2}}(\pi(P_i)) = \pm K^{\frac{1}{2}}(p_i)$ where the sign taken depends upon which sheet P_i is located.

Next let $i : C \to C$ be the involution which interchanges the sheets of $C : i(z, w) = (z, -w)$. The branch points of C are exactly $z_1, \ldots, z_{2N}$ (or rather their inverse images in C). These are the zeroes of the function $\hat{M} - \hat{M} \circ i$. Combining this fact with the previous paragraph, we get

$$\hat{M} - \hat{M} \circ i = c \frac{\sqrt{K(z)}}{\prod_{i=1}^{N} (z - p_i)} \tag{2.10}$$

where c is a constant to be determined. We use the fact that the zeroes and poles of $\hat{M} - \hat{M} \circ i$ are all simple.

We further note that $\hat{M} + \hat{M} \circ i$ is a rational function of the complex variable z. Therefore

$$\hat{M} + \hat{M} \circ i = \frac{Q(z)}{\prod_{i=1}^{N} (z - p_i)} \tag{2.11}$$

where Q is a polynomial. Since $\hat{M}$ is finite and non-zero at $\infty_\pm$, the order of Q must be N:

$$Q(z) = q_N z^N + \cdots + q_1 z + q_0$$

Combining (2.10) and (2.11), we get

$$\hat{M} = \frac{Q(z) + c\sqrt{K(z)}}{2\prod_{i=1}^{N}(z - p_i)} \tag{2.12}$$

Lemma 2.2. *Let $\hat{M}$ be given by Eq. (2.12) then c, and the coefficients of Q are uniquely determined by the values of $\hat{M}$ at 0_+, ∞_+, and the poles $P_i, i = 1, \ldots, N$. Furthermore they are meromorphic functions on C of the poles P_i. The formula*

$$q_N(\omega) = \frac{2i\overline{\alpha_1(\tau^{-1}(\omega))}}{1 - \overline{\alpha_1(\tau^{-1}(\omega))}}, \tag{2.13}$$

shows that the retarded coefficients $\overline{\alpha_1(\tau^{-1}\omega)}$ is also a meromorphic function on C of the poles P_i.

Proof. Note that

$$\hat{M} \circ i = \frac{Q - c\sqrt{K}}{2\prod_{i=1}^{N}(z - p_i)}.$$

Since only one inverse image $P_i \in \pi^{-1}\pi(P_i)$ is a pole of $\hat{M}$, we must have

$$c\sqrt{K(P_i)} = Q(p_i) \qquad (1 \leq i \leq N), \tag{2.14}$$

where the sign taken by the square root depends on which sheet P_i is located. Note that Equation (2.14) still holds if P_i coincides with a ramification point. We also have $\hat{M}(0_+) = i$, which means that

$$2i\prod_{i=1}^{N}(-p_i) = c\sqrt{K(0_+)} + q_0 \tag{2.15}$$

In addition $\hat{M}(\infty_+) = -i$, hence

$$-2i = q_N + c \tag{2.16}$$

Equations (2.14) - (2.16) uniquely determine $q_0, \ldots, q_N$ and c.

Substituting ∞_- into the argument of $\hat{M}$, we have

$$2i\left[\frac{1 + \overline{\alpha_1(\tau^{-1}(\omega))}}{1 - \overline{\alpha_1(\tau^{-1}(\omega))}}\right] = q_N(\omega) - c(\omega), \tag{2.17}$$

which when combined (2.16) and (2.17), yields

$$
\begin{aligned}
c(\omega) &= \frac{-2i}{1-\overline{\alpha_1(\tau^{-1}(\omega))}} \\
q_N(\omega) &= \frac{2i\overline{\alpha_1(\tau^{-1}(\omega))}}{1-\overline{\alpha_1(\tau^{-1}(\omega))}}.
\end{aligned}
\tag{2.18}
$$

If Newton's interoplation formula is used with Equation (2.14) we find

$$
Q(z) = q_N \prod_{j=1}^{N} (z-p_j) + c \sum_{j=1}^{N} K^{\frac{1}{2}}(P_j) \prod_{k \neq j} \frac{z-p_k}{p_j-p_k}. \tag{2.19}
$$

If (2.17) and (2.18) are now used in the above equation and if D is defined by

$$
D = -K_0 \begin{vmatrix} 1 & p_1 & \dots & p_1^{N-1} \\ \vdots & & & \\ 1 & p_N & \cdots & p_N^{N-1} \end{vmatrix} + (-1)^N \begin{vmatrix} p_1 & \cdots & p_1^{N-1} & p_1^N + K_1 \\ \vdots & & & \\ p_N & \cdots & p_N^{N-1} & p_N^N + K_N \end{vmatrix},
$$

we can solve for q_N in the form

$$
q_N = \frac{2i \left\{ K_0 \begin{vmatrix} 1 & p_1 & \cdots & p_1^{N-1} \\ \vdots & & & \\ 1 & p_N & \cdots & p_N^{N-1} \end{vmatrix} + (-1)^{N+1} \begin{vmatrix} p_1 & \cdots & p_1^{N-1} & K_1 - p_1^N \\ \vdots & & & \\ p_N & \cdots & p_N^{N-1} & K_N - p_N^N \end{vmatrix} \right\}}{D}, \tag{2.20}
$$

where $K_i = \sqrt{K(P_i)}\ i = 1, \dots N$. An analogous formula for c is,

$$
c = \frac{(-1)^{N+1} 4i \begin{vmatrix} p_1 & \cdots & p_1^{N-1} & K_1 - p_1^N \\ \vdots & & & \\ p_N & \cdots & p_N^{N-1} & K_N - p_N^N \end{vmatrix}}{D} \tag{2.21}
$$

Since z can be considered as a meromorphic function on C of degree two we see that the coefficients of Q are meromorphic functions of the pole divisor $\{P_1(\omega), \dots, P_N(\omega)\}$ of $\hat{M}(\cdot, \omega)$. Equations (2.21) and (2.18), show that $c(\omega)$ and the retarded coefficient $\alpha_1(\tau^{-1}(\omega))$ (or rather its conjugate) are also meromorphic functions of $\{P_1(\omega), \dots, P_N(\omega)\}$. $\square$

If we combine the formula (2.18) for c with an explicit evaluation of $\hat{M}(0_+)$ using relations (2.8), we get

$$
-\frac{1-\overline{\alpha_1(\tau^{-1}(\omega))}}{1-\alpha_1(\tau^{-1}(\omega))} = \frac{K_0}{\prod_{i=1}^{N} -p_i(\omega)} = \frac{\sqrt{\prod_{i=1}^{2N} z_i}}{\prod_{i=1}^{N} -p_i(\omega)} \tag{2.22}
$$

This simple relation determines the argument of $1-\alpha_1(\tau^{-1}(\omega))$. The determination of the values of α_1 itself appears to require the previous calculations. Finally, from $\hat{M}(0_-)$ and (2.15) it follows that $q_0 = \overline{q_N}\prod_{i=1}^{N}(-p_i)$.

§3. Determination of the Pole Motion

As we have seen in §2, the (retarded) coefficient α_1 is determined by the poles $P_1, \ldots, P_N$ of $\tilde{M}$:

$$\overline{\alpha_1(\tau^{-1}(\omega))} = \frac{q_N(\omega)}{2i + q_N(\omega)}$$

where q_N is the function (2.20) of the poles $P_1(\omega), \ldots, P_N(\omega)$. We fix $\omega \in \Omega$ and write $P_i(n) = P_i(\tau^n(\omega))$ $(n \in \mathbb{Z})$. We will determine the pole motion $n \to \{P_1(n), \ldots, P_N(n)\}$.

The first step is to consider the behavior of the function $\hat{m}_+(z)$ near $z = 0$ and $z = \infty$. We already know that $\hat{m}_+(0) = i$ and $\hat{m}_+(\infty) = -i$ (see (2.8)). We also have

$$\hat{m}_+ = i\left(\frac{1+m_+}{1-m_+}\right) \tag{2.5}$$

where $m_\pm(z)$ are the original m-functions considered in Theorem 2.1.

Recall from Eq.(2.4) that $m_+(0) = 0$; hence we can write

$$m_+(z) = \beta z + O(z^2)$$

where β is to be determined. With,

$$m_1(z) = m_+(z, \tau(\omega)), \qquad m_0(z) = m_+(z, \omega),$$

we find by inverting (2.3) that,

$$m_1(z) = \frac{z\overline{\alpha_1(\omega)} + m_0(z)}{z + \alpha_1(\omega)m_0(z)}. \tag{3.1}$$

Substituting $m_0(z) = \beta z + O(z^2)$, we see that

$$m_1(z) = \frac{z[\overline{\alpha_1(\omega)} + \beta] + O(z^2)}{z[1 + \beta\alpha_1(\omega)] + O(z^2)}.$$

The requirement that $m_1(0) = 0$ forces $\beta = \overline{-\alpha_1(\omega)}$. Thus equation (2.7) gives

$$\hat{m}_+(z, \omega) = i\left[\frac{1 - \overline{\alpha_1(\omega)}z + O(z^2)}{1 + \overline{\alpha_1(\omega)}z + O(z^2)}\right] = i[1 - 2\overline{\alpha_1(\omega)}z + O(z^2)]. \tag{3.2}$$

In a similar way we can determine the behavior of $\hat{m}_+(z)$ in a neighborhood of $z = \infty$. Write $u = 1/z$, $\tilde{m}_0(u) = \frac{1}{m_+(z,\omega)}$, and $\tilde{m}_1(u) = \frac{1}{m_+(z,\tau(\omega))}$ in (3.1) to find,

$$\tilde{m}_1(u) = \frac{\alpha_1(\omega)u + \tilde{m}_0(u)}{\overline{\alpha_1(\omega)}\tilde{m}_0(u) + u}.$$

Since $\tilde{m}_0(0) = 0$, we can write $\tilde{m}_0(u) = \beta u + O(u^2)$, and hence

$$\tilde{m}_1(u) = \frac{[\beta + \alpha_1(\omega)]u + O(u^2)}{[\overline{\alpha_1(\omega)}\beta + 1]u + O(u^2)}.$$

Since $\tilde{m}_1(0) = 0$ we get $\beta = -\alpha_1(\omega)$, and so finally

$$\hat{m}_+(u,\omega) = i\left[\frac{1 - \alpha_1(\omega)u + O(u^2)}{-1 - \alpha_1(\omega)u + O(u^2)}\right] = -i[1 - 2\alpha_i(\omega)u + O(u^2)]. \tag{3.3}$$

Armed with formulas (3.2) and (3.3), we study the Wronskian of (1.2). Fix $\omega \in \Omega$ and $0 \neq z \in \mathbb{C}$. Let $\binom{v_0}{w_0}$ be a non-zero vector in the complex line in $\mathbb{C}^2$ determined by $\hat{m}_-(z)$. For example, we can choose $\binom{v_0}{w_0} = \binom{1}{\hat{m}_-(z)}$. Similarly, let $\binom{e_0}{f_0}$ be a non-zero vector in the complex line determined by $\hat{m}_+(z)$.

Write $\binom{v_n}{w_n}, \binom{e_n}{f_n}$ for the images of these vectors under the map $\hat{T}(z,n)\cdots\hat{T}(z,1) = \hat{T}_z(\omega,n)\cdots\hat{T}_z(\omega,1)$, thus

$$\frac{e_n}{e_0} = G_n(z) + H_n(z)\hat{m}_+(z,0), \ \frac{v_n}{v_0} = G_n(z) + H_n(z)\hat{m}_-(z,0), \tag{3.4}$$

where G_n and H_n are polynomials of degree n in z. The Wronskian relation gives

$$\begin{aligned} v_n f_n - e_n w_n &= W\left[\begin{pmatrix} v_n \\ w_n \end{pmatrix}, \begin{pmatrix} e_n \\ f_n \end{pmatrix}\right] = z^n W\left[\begin{pmatrix} v_0 \\ w_0 \end{pmatrix}, \begin{pmatrix} e_0 \\ f_0 \end{pmatrix}\right] \\ &= z^n[v_0 f_0 - e_0 w_0] = z^n v_0 e_0\left[\frac{f_0}{e_0} - \frac{w_0}{v_0}\right] = z^n v_0 e_0[\hat{m}_+(z) - \hat{m}_-(z)]. \end{aligned}$$

Thus

$$v_n e_n\left[\frac{f_n}{e_n} - \frac{w_n}{v_n}\right] = z^n v_0 e_0[\hat{m}_+(z,0) - \hat{m}_-(z,0)],$$

and so

$$\frac{e_n}{e_0}\frac{v_n}{v_0} = z^n\frac{\hat{m}_+(z,0) - \hat{m}_-(z,0)}{\hat{m}_+(z,n) - \hat{m}_-(z,n)}, \tag{3.5}$$

where we have written $\hat{m}_\pm(z,0) = \hat{m}_\pm(z,\omega)$, $\hat{m}_\pm(z,n) = \hat{m}_\pm(z,\tau^n(\omega))$. Writing $c_0 = \frac{-2i}{1-\overline{\alpha_1(\tau^{-1}(\omega))}}$, $c_n = \frac{-2i}{1-\overline{\alpha_1(\tau^{n-1}(\omega))}}$ for the constants appearing in the expressions (2.12) for $\hat{M}(\cdot,\omega)$ and $\hat{M}(\cdot,\tau^n(\omega))$, we get

$$\frac{e_n}{e_0}\frac{v_n}{v_0} = z^n\frac{c_0}{c_n}\frac{\prod_{i=1}^N(z - P_i(n))}{\prod_{i=1}^N(z - P_i(0))}. \tag{3.6}$$

We claim,

Lemma 3.1. *$\frac{e_n}{e_0}$ and $\frac{v_n}{v_0}$ are branches of a single meromorphic function h_n on the algebraic curve C.*

Proof. Choose $e_0 = v_0 = 1$. Using formula (2.6) for $\hat{T}(z,1)$, we have

$$e_1 = A_1(z) + B_1(z)\hat{m}_+(z,0)$$
$$v_1 = A_1(z) + B_1(z)\hat{m}_-(z,0)$$

where

$$A_1(z) = \frac{a_1}{2} \cdot [z(1-\overline{\alpha_1(\omega)}) + (1-\alpha_1(\omega))]$$
$$B_1(z) = \frac{-ia_1}{2} \cdot [z(1-\overline{\alpha_1(\omega)}) - (1-\alpha_1(\omega))].$$

Thus we see that $h_1 = A_1(z) + B_1(z)\hat{M}$, and so the claim follows for $n = 1$.

Next note that

$$\frac{e_n}{e_0} = \frac{e_n}{e_{n-1}} \frac{e_{n-1}}{e_{n-2}} \cdots \frac{e_1}{e_0}$$
$$\frac{v_n}{v_0} = \frac{v_n}{v_{n-1}} \frac{v_{n-1}}{v_{n-2}} \cdots \frac{v_1}{v_0}.$$

Since

$$\begin{aligned} \frac{e_k}{e_{k-1}} &= A_k(z) + B_k(z)\hat{m}_+(z,k-1) \\ \frac{v_k}{v_{k-1}} &= A_k(z) + B_k(z)\hat{m}_-(z,k-1), \end{aligned} \tag{3.7}$$

we see that

$$h_n = g_n \cdot g_{n-1} \cdot \cdots \cdot g_1 \tag{3.8}$$

where g_k is the meromorphic function on C determined by (3.7) $(1 \leq k \leq n)$. In particular $g_1 = f_1$. □

We consider more closely the meromorphic function h_1 on C. From the relation (3.6):

$$\frac{e_1}{e_0}\frac{v_1}{v_0} = z \cdot \frac{c_0}{c_1} \frac{\prod_{i=1}^{N}(z-P_i(1))}{\prod_{i=1}^{N}(z-P_i(0))} \tag{3.9}$$

we see that the poles of h_1 on C are contained among the points ∞_+, ∞_-, and the inverse images $\pi^{-1}\pi(P_i(0))$ $(1 \leq i \leq N)$ of the projections $\pi(P_i(0))$ of the points $P_i(0)$ to C.

Lemma 3.2. *The poles of h_1 are ∞_- and the points $P_1(0), \ldots, P_N(0)$, and all these poles are simple.*

Proof. To verify this claim, note first that the simplicity follows immediately from (3.9) and the fact that the $P_i(0)$ are distinct. Second, (3.9) together with the formula $h_1 = A_1 + B_1\hat{M}$ shows that the "finite" poles of h_1 are exactly $P_1(0), \ldots, P_N(0)$.

To show that ∞_- rather than ∞_+ is a pole of h_1, we use the equation of motion

$$\begin{pmatrix} e_1 \\ f_1 \end{pmatrix} = \frac{-ia_1}{2} \begin{pmatrix} i[z(1-\overline{\alpha_1}) + (1-\alpha_1)] & z(1-\overline{\alpha_1}) - (1-\alpha_1) \\ * & * \end{pmatrix} \begin{pmatrix} 1 \\ \hat{m}_+(z,0) \end{pmatrix}$$

together with (3.3). Writing $u = 1/z$, we have

$$\begin{aligned} e_1 &= \frac{-ia_1}{2u}\left\{u[i(1-\alpha_1) + i(1-\alpha_1) + 2i\alpha_1(1-\overline{\alpha_1})] + O(u^2)\right\} \\ &= \frac{-ia_1}{2} \cdot 2i[1-|\alpha_1|^2] + O(u) = (1-|\alpha_1(\omega)|^2)^{1/2} + O(u). \end{aligned}$$

This means that, at ∞_+, h_1 has a finite value which can be evaluated from

$$h_1(u) = (1-|\alpha_1(\omega)|^2)^{1/2} + O(u); \tag{3.10}$$

this result will be useful later. Combining (3.10) with (3.9), we see that indeed h_1 has a simple pole at $\infty_- \in C$. □

For later reference, we note that, for z near zero one finds from (3.2) and the equation of motion for $\binom{e_1}{f_1}$,

$$e_1(z) = (1-|\alpha_1(\omega)|^2)^{1/2} z + O(z^2), \tag{3.11}$$

which means that h_1 has a simple zero at 0_+ with principal part $(1-|\alpha_1(\omega)|^2)^{1/2}$. The fact that the principal parts in (3.10) and (3.11) are equal is not an accident and will be important later on.

Having shown that h_1 has simple poles in $P_1(0), \ldots, P_N(0)$ and in ∞_-, and no others, we now determine the zeroes of h_1 on C. We use again the equation of motion:

$$\begin{pmatrix} e_1 \\ f_1 \end{pmatrix} = \hat{T}(z,1)\begin{pmatrix} 1 \\ \hat{m}_+(z,0) \end{pmatrix} = \begin{pmatrix} A_1 + B_1\hat{m}_+(z,0) \\ C_1 + D_1\hat{m}_+(z,0) \end{pmatrix}$$

where A_1, B_1 are as above and

$$\begin{aligned} C_1 &= \frac{ia_1}{2}[z(1+\overline{\alpha_1}) - (1+\alpha_1)] \\ D_1 &= \frac{a_1}{2} \cdot [z(1+\overline{\alpha_1}) + (1+\alpha_1)]. \end{aligned}$$

The quantity α_1 is evaluated at $\omega : \alpha_1 = \alpha_1(\omega)$. It follows that

$$\hat{m}_+(z,1) = \frac{C_1(z) + D_1(z)\hat{m}_+(z,0)}{A_1(z) + B_1(z)\hat{m}_+(z,0)},$$

and hence the poles of $\hat{M}(\tau(\omega),\cdot)$ are either zeroes of $A_1 + B_1\hat{M}(\omega,\cdot)$ or poles of $C_1 + D_1\hat{M}(\omega,\cdot)$. But the finite poles of $C_1 + D_1\hat{M}(\omega,\cdot)$ are among the finite poles of $A_1 + B_1\hat{M}(\omega,\cdot)$, hence the poles of $\hat{M}(\tau(\omega),\cdot)$ are among the zeroes of $A_1 + B_1\hat{M}(\omega,\cdot)$.

Now, by (3.11), $h_1 = A_1 + B_1\hat{M}(\omega,\cdot)$ has a simple zero in 0_+, and from (3.9) the set of zeroes of h_1 includes the set of poles $P_1(1),\ldots,P_N(1)$ of $\hat{M}(\tau(\omega),\cdot)$. Since the zeroes and poles of a meromorphic function are equal in number, we conclude that the zeroes of h_1 are exactly $0_+, P_1(1),\ldots,P_N(1)$ and that these are all simple.

The reasoning just used extends, using (3.4), (3.6), and (3.8) to show,

Theorem 3.3. *For each $n \geq 1$, h_n has simple poles at $P_1(0),\ldots,P_N(0)$ and a pole of order n at ∞_-, while h_n has simple zeroes at $P_1(n),\ldots,$ $P_N(n)$ and a zero of order n at 0_+.*

The idea now is to use Abel's theorem and the Jacobi Inversion theorem to show that the motion $n \to \{P_1(n),\ldots,$ $P_N(n)\}$ is quasi-periodic which will imply from (2.13) and (2.20) that α_n is quasi-periodic. Since the poles of h_n include $P_1(0),\ldots,P_N(0)$, and the zeroes of h_n include $P_1(n),\ldots,P_N(n)$, it is clear that this idea is reasonable. There are two difficulties in carrying it to completion that do not arise in the case of the one-dimensional Schrödinger operator ([DNM, MM]; historically speaking the first to which the inverse method we are now using was applied). First of all, the poles are N in number while the genus of C is $N-1$. This means that the standard version of the Jacobi Inversion theorem [S] cannot be applied. This difficulty arises also in the case of the AKNS operator [P, DC-J]. It is circumvented in those papers by singularizing the curve C. Namely, one identifies two points on C; intuitively this raises the genus by one and makes it possible to apply a generalized version of the Jacobi Inversion theorem [F] to the poles. In our case we wish to identify the points 0_+ and ∞_+ which can be accomplished (see (3.9) and (3.10)) by "dividing by $\sqrt{z}$". This amounts to working on a curve $\tilde{C}$ which is a four-times ramified double cover of C. It turns out that a generalized Abel's theorem can be applied when we work on $\tilde{C}$. Now for the details.

We begin by introducing the Riemann surface $\tilde{C}$ of the algebraic relation $w^2 = K(u^2)$ (thus $u = \sqrt{z}$). The following facts may be verified.

(1) The genus of $\tilde{C}$ is $2N-1$. There is a projection $\tilde{\pi} : \tilde{C} \to C$ which is 2-1 except at the points $0_\pm, \infty_\pm \in C$, where $\tilde{\pi}$ is 1-1. We will label the inverse images in $\tilde{C}$ of $0_\pm, \infty_\pm$ with the same symbols $0_\pm, \infty_\pm$.

(2) There is a homology basis $\tilde{a}_1, \dots, \tilde{a}_{2N-1}; \tilde{b}_1, \dots, \tilde{b}_{2N-1}$ of $\tilde{C}$ with the following properties. First of all, the basis is normalized: $\tilde{a}_i \circ \tilde{a}_j = 0 = \tilde{b}_i \circ \tilde{b}_j$, $\tilde{a}_i \circ \tilde{b}_j = \delta_{ij}$ where $\circ$ denotes intersection number. Second, the family of projected curves $\{\tilde{\pi} \circ \tilde{a}_i, \tilde{\pi} \circ \tilde{b}_i \mid 1 \le i \le 2N-1\}$ contains a normalized homology basis $\{a_1, \dots, a_{N-1}, b_1, \dots, b_{N-1}\}$ of C. The reader is advised to draw pictures to verify these facts.

(3) A basis for the holomorphic differentials on $\tilde{C}$ is given by

$$\eta_i = \frac{u^{i-1} du}{\sqrt{K(u^2)}} \qquad (1 \le i \le 2N-1).$$

Observe that

$$2\eta_{2r} = \tilde{\pi}^* \omega_r \qquad (1 \le r \le N-1)$$

where

$$\omega_r = \frac{z^{r-1} dz}{\sqrt{K(z)}} \qquad (1 \le r \le N-1)$$

is a holomorphic differential on C. The forms $\{\omega_r \mid 1 \le r \le N-1\}$ form a basis for the holomorphic differentials on C.

Let us now fix a differential of the third kind ω_N on C with simple poles at 0_+ and ∞_+, with residues -1 and $+1$ respectively, and no other poles. Such a differential can be uniquely determined by requiring that $\int_{a_i} \omega_N = 0$ $(1 \le i \le N-1)$, but there seems no particular point in doing so here. Define

$$\eta_{2N} = \frac{1}{2} \tilde{\pi}^* \omega_N.$$

Then η_{2N} has simple poles exactly at 0_+ and $\infty_+ \in \tilde{C}$, with residues -1 and 1 respectively. (There is no guarantee that $\int_{\tilde{a}_i} \eta_{2N} = 0$ even if $\int_{a_i} \omega_N = 0$ for all i.)

We are now ready to use the theory of generalized Jacobians as developed in Fay [F, pp. 50-60]. Let $\tilde{\Lambda}$ be the lattice in $\mathbb{C}^{2N}$ generated by the $4N-1$ vectors

$$\left(\int_c \eta_1, \int_c \eta_2, \dots, \int_c \eta_{2N} \right).$$

Here c ranges over the curves $\tilde{a}_1, \ldots, \tilde{a}_{2N-1}, \tilde{b}_1, \ldots, \tilde{b}_{2N-1}$ together with a small circle $\tilde{\alpha}$ in $\tilde{C}$ centered at ∞_+. Then [F, Coro. 3.8] the rank of this lattice is $4N-1$. (In [F] one works with a normalized basis $\tilde{\eta}_1, \ldots, \tilde{\eta}_{2N-1}$ of holomorphic differentials together with a normalized differential $\tilde{\eta}_{2N}$ of the third kind, but one has

$$\begin{pmatrix} \eta_1 \\ \vdots \\ \eta_{2N} \end{pmatrix} = \tilde{A} \begin{pmatrix} \tilde{\eta}_1 \\ \vdots \\ \tilde{\eta}_{2N} \end{pmatrix}$$

for a nonsingular $2N \times 2N$ matrix $\tilde{A}$.) Furthermore, if f is a meromorphic function on $\tilde{C}$ such that $f(0_+) = f(\infty_+)$, and if $\mathcal{A}$ resp. $\mathcal{B}$ is the pole resp. zero divisor of f, then by the Theorem of Abel [F, p. 55]:

$$\left(\int_{\mathcal{A}}^{\mathcal{B}} \eta_1, \int_{\mathcal{A}}^{\mathcal{B}} \eta_2, \ldots, \int_{\mathcal{A}}^{\mathcal{B}} \eta_{2N} \right) \in \tilde{\Lambda}.$$

It is permitted that $\{0_+, \infty_+\} \subset \mathcal{A}$ or $\{0_+, \infty_+\} \subset \mathcal{B}$ as long as the leading-order coefficients of f at 0_+ and ∞_+ are equal.

Let $\Lambda \subset \mathbb{C}^N$ be the lattice generated by the vectors

$$\left(\frac{1}{2} \int_c \omega_1, \frac{1}{2} \int_c \omega_2, \ldots, \frac{1}{2} \int_c \omega_N \right),$$

where c ranges over the curves $a_1, \ldots, a_{N-1}; b_1, \ldots, b_{N-1}, \alpha$. Note that the image $\alpha = \tilde{\pi} \circ \tilde{\alpha}$ of the circle $\tilde{\alpha}$ centered at ∞_+ is a small circle in C centered at ∞_+, but *traversed twice.* This is because $\tilde{C}$ ramifies over C at ∞_+. This lattice has rank $2N-1$ in $\mathbb{C}^N$. This can be proved as follows. Since

$$\begin{pmatrix} \omega_1 \\ \vdots \\ \omega_N \end{pmatrix} = A_0 \begin{pmatrix} \omega'_1 \\ \vdots \\ \omega'_N \end{pmatrix}$$

where $(\omega'_1, \ldots, \omega'_{N-1})$ is a basis of the holomorphic differentials which is normalized with respect to $\{a_i, b_i \mid 1 \leq i \leq N-1\}$, ω'_N is the differential of the third kind with residues -1 and $+1$ at 0_+ and ∞_+ which is normalized, and A_0 is a nonsingular $N \times N$ matrix. Hence the lattice $\Lambda = A_0^{-1}\Lambda'$, where Λ' is the lattice constructed in [F, pp. 50-60] except that (due to the double traversal of α) the last vector in Λ' is twice the last vector in the lattice of [F].

Define a meromorphic function $\tilde{h_1}$ on $\tilde{C}$ by

$$\tilde{h_1}(u) = \frac{h_1(z)}{u} = \frac{h_1(u^2)}{u}$$

where we make the obvious abuse of notation.

Lemma 3.4. *The function $\tilde{h_1}$ has simple poles at the inverse images $P_1^{(1)}(0), P_1^{(2)}(0), \dots, P_N^{(1)}(0), P_N^{(2)}(0)$ of the points $P_1(0), \dots, P_N(0)$ with respect to $\tilde{\pi}$. It has simple zeroes at the inverse images $P_1^{(1)}(1), P_1^{(2)}(1), \dots, P_N^{(1)}(1), P_N^{(2)}(1)$ of the points $P_1(1), \dots, P_N(1)$. In addition, $\tilde{h_1}$ has simple poles at 0_- and ∞_-, and simple zeroes at 0_+ and ∞_+. The first-order terms in the Taylor expansions of $\tilde{h_1}$ at 0_+ and ∞_+ are both are equal, i.e., $(1-|\alpha_1(\omega)|^2)^{1/2}$. Finally*

$$\begin{aligned}\sum_{\substack{1\le i\le N\\ j=1,2}} &\left(\int_{P_i^{(j)}(0)}^{P_i^{(j)}(1)} \eta_1, \dots, \int_{P_i^{(j)}(0)}^{P_i^{(j)}(1)} \eta_{2N}\right)\\ &= \left(\int_{0_+}^{0_-} \eta_1 + \int_{\infty_+}^{\infty_-} \eta_1, \dots, \int_{0_+}^{0_-} \eta_{2N} + \int_{\infty_+}^{\infty_-} \eta_{2N}\right) \bmod \tilde{\Lambda},\end{aligned} \tag{3.12}$$

which implies

$$\sum_{i=1}^{N} \left(\int_{P_i(0)}^{P_i(1)} \omega_1, \dots, \int_{P_i(0)}^{P_i(1)} \omega_N\right) = v \bmod \Lambda, \tag{3.13}$$

where,

$$v = \left(\frac{1}{2}\int_{0_+}^{0_-} \omega_1 + \frac{1}{2}\int_{\infty_+}^{\infty_-} \omega_1, \dots, \frac{1}{2}\int_{0_+}^{0_-} \omega_N + \frac{1}{2}\int_{\infty_+}^{\infty_-} \omega_N\right). \tag{3.14}$$

The integrals involving η_{2N} are interpreted so that the logarithmic terms cancel, leaving a finite result (recall that η_{2N} has residues -1 and 1 at 0_+ and ∞_+).

Proof. The first two parts of the lemma follow from the properties of h_1 and Equations (3.9) and (3.10). To obtain (3.12) we apply Abel's theorem to $\tilde{h}_1$. Here

$$\begin{aligned}\mathcal{A} &= \left\{P_1^{(1)}(0), \dots, P_N^{(1)}(0), P_1^{(2)}(0), \dots, P_N^{(2)}(0), 0_-, \infty_-\right\}\\ \mathcal{B} &= \left\{P_1^{(1)}(1), \dots, P_N^{(1)}(1), P_1^{(2)}(1), \dots, P_N^{(2)}(1), 0_+, \infty_+\right\}.\end{aligned}$$

We now project (3.12) to the curve C. Consider the projection $P : \mathbb{C}^{2N} \to \mathbb{C}^N : (z_1, z_2, \dots, z_{2N}) \to (z_2, z_4, \dots, z_{2N})$. Recall that $\eta_{2r} = \frac{1}{2}\,\tilde{\pi}^*\omega_r$, and that the images under $\tilde{\pi}$of curves $\tilde{a}_1, \dots, \tilde{a}_{2N-1}, \tilde{b}_1, \dots, \tilde{b}_{2N-1}$ contain a normalized homology basis of C.

Next let $v \in \mathbb{C}^N$ be the vector

$$v = \left(\frac{1}{2} \int_{0_+}^{0_-} \omega_1 + \frac{1}{2} \int_{\infty_+}^{\infty_-} \omega_1, \ldots, \frac{1}{2} \int_{0_+}^{0_-} \omega_N + \frac{1}{2} \int_{\infty_+}^{\infty_-} \omega_N \right)$$

where we choose fixed paths on C joining 0_+ with 0_- and ∞_+ with ∞_-. Using (3.12), we conclude that

$$\sum_{i=1}^{N} \left(\int_{P_i(0)}^{P_i(1)} \omega_1, \ldots, \int_{P_i(0)}^{P_i(1)} \omega_N \right) = v \bmod \Lambda.$$

The factor of $1/2$ in front of the ω_i's is canceled by a factor of 2 due to the repetition of the poles on $\tilde{C}$. □

Now, we can apply the above argument to h_n for each $n \geq 1$ (or, alternatively, work with $h_1(\tau(\omega), \cdot), \ldots, h_1(\tau^{n-1}(\omega), \cdot))$ to find

$$\sum_{i=1}^{N} \left(\int_{P_i(0)}^{P_i(n)} \omega_1, \ldots, \int_{P_i(0)}^{P_i(n)} \omega_N \right) = nv \bmod \Lambda. \tag{3.15}$$

To finish the discussion, let Λ_0 be the lattice in $\mathbb{C}^N$ generated by the vectors $\left\{ \left(\int_c \omega_1, \ldots, \int_c \omega_N \right) \mid c = a_1, \ldots, a_{N-1}; b_1, \ldots, b_{N-1} \right\}$ together with the vector $\left(\frac{1}{2} \int_\alpha \omega_1, \ldots, \frac{1}{2} \int_\alpha \omega_N \right)$. This just means that we traverse once the small circle centered at ∞_+. Then by [F, Prop. 3.11], the *generalized Jacobian*

$$J_0(C) \stackrel{\text{def}}{=} \frac{\mathbb{C}^N}{\Lambda_0}$$

is birationally isomorphic to the set of unordered N-tuples $\{P_1, \ldots, P_N\}$ from C which contain neither 0_+ nor ∞_+. A birational isomorphism is given by the *Abel map* A constructed as follows: let $\{P_1^*, \ldots, P_N^*\}$ be the fixed divisor not containing 0_+ and ∞_-, and define

$$A(P_1, \ldots, P_N) = \sum_{i=1}^{N} \left(\int_{P_i^*}^{P_i} \omega_1, \ldots, \int_{P_i^*}^{P_i} \omega_N \right).$$

Let c_i be the inverse image in C of the resolvent interval $[z_{2i-1}, z_{2i}]$ $(1 \leq i \leq N)$. It can be checked directly that the restriction of A to the real N-torus $T^N = c_1 \times \cdots \times c_N$ is an analytic diffeomorphism onto its image. We now prove,

Theorem 3.5. *The sequence $\{q_N(n)\}_{n=-\infty}^{\infty}$ and hence $\{\alpha(n)\}_{n=-\infty}^{\infty}$ is a quasi-periodic sequence. That is $q_N(n)$ is determined by the evolution of the meromorphic function F along a quasi-periodic winding in the generalized Jacobian $J_0(C)$.*

Proof. We begin by noting that the function which defines q_N in (2.20) is symmetric and rational in the poles $P_1, \ldots, P_N \in C$. Therefore it induces a meromorphic function F on the complex variety $J_0(C)$ via the formula

$$F \circ A(P_1, \ldots, P_N) = q_N(P_1, \ldots, P_N).$$

Observe that the denominator D in (2.20) does not vanish on T^N, hence F is a real-analytic function on the image $A(T^N) \subset J_0(C)$. We now prove the result by making a connection between the formula (3.15) and the function F via the Abel map A. For this, note that the lattice Λ contains Λ_0, and indeed the quotient group Λ/Λ_0 contains 2^{N-1} elements. We identify Λ/Λ_0 with a set $\{v_i | 1 \le i \le 2^{N-1}\}$ of fixed vectors in $\mathbb{C}^N$. The complex manifold

$$J(\Lambda) \stackrel{\text{def}}{=} \frac{\mathbb{C}^N}{\Lambda}$$

is a finite quotient of $J_0(C)$. In fact the canonical projection

$$\eta : J_0(C) \to J(\Lambda) : (z_1, \ldots, z_N) + \Lambda_0 \to (z_1, \ldots, z_N) + \Lambda$$

is a holomorphic group homorphism. Consider the motion $(P_1(0), \ldots, P_N(0) \to (P_1(n), \ldots, P_N(n))$ defined by a fixed sequence $s_\omega = \{\alpha_k\}_{k=-\infty}^{\infty} \in \Omega$. The formula (3.15) shows that $\eta \circ A(P_1(n), \ldots, P_N(n)) = \eta \circ A(P_1(0), \ldots, P_N(0)) + nv \bmod \Lambda$ $(-\infty < n < \infty)$. That is, if the pole motion is transferred to $J_0(C)$ via A, then it descends to a *linear* motion on $J(\Lambda)$. From the discussion above one can show that the pole motion is also linear on $J_0(C)$, in fact from (1.11), (2.13), and (2.20) one has

$$A(P_1(n), \ldots, P_N(n)) = A(P_1(0), \ldots, P_N(0)) + n(v + v_i) \bmod \Lambda_0$$

for a unique choice of v_i independent of s_ω and n. The choice is unique because the flow on $\mathbb{C}^{2N}/\tilde{\Lambda}$ is completely defined by the spectral intervals and because $\mathbb{C}^{2N}/\tilde{\Lambda}$ contains the real part $A(T^N)$ of the generalized Jacobian $J_0(C)$ as an embedded subvariety. We can now combine (2.20) and (3.15) as follows;

$$\begin{aligned} q_N(n) \equiv q_N(\tau^n(\omega)) =& F \circ A(P_1(n), \ldots, P_N(n)) \\ =& F(A(P_1(0), \ldots, P_N(0)) + nv + nv_i), \end{aligned} \tag{3.16}$$

which gives the result. $\square$

§4. Quasi-periodic Recurrence Coefficients

In this section we discuss the theory of "Reflectionless" quasi-periodic recurrence coefficients for polynomials orthogonal on the unit circle. By Reflectionless recurrence coefficients (See [BGHT] for a more general formulation) we mean that $\{\alpha_n\}_{-\infty}^{\infty}$, $|\alpha_n| < 1$ are such that the topological support Σ of the orthogonality measure σ associated with the coefficients $\{\alpha_n\}_1^{\infty}$ is a finite number of disjoint intervals and that for almost every point in these intervals $m_{\pm}$ satisfy Equation (2.5). These form a special class of those studied by [PS]. A polynomial of degree N, $w(z)$ is said to be self-reciprocal if $z^N\bar{w}(\frac{1}{z}) = w(z)$. Note if $w(z)$ is a self-reciprocal polynomial of degree N then $e^{\frac{-iN\theta}{2}}w(e^{i\theta}) = \overline{e^{\frac{-iN\theta}{2}}w(e^{i\theta})}$ hence on the unit circle $e^{\frac{-iN\theta}{2}}w(e^{i\theta})$ can be represented as a real trigonometric polynomial (in $\frac{\theta}{2}$). We now examine the measures that are associated with $\hat{m}_{\pm}$.

Lemma 4.1. *Suppose that $\{\alpha_n\}_{-\infty}^{\infty}$, $|\alpha_n| < 1$ are a sequence of recurrence coefficients such that $\tilde{m}_{\pm}$ are branches of a meromorphic function $\tilde{M}$ of the form given by Equation (2.12) with $\tilde{M}(0_+) = i$, $\tilde{M}(\infty_+) = -i$, $\tilde{M}(0_-) = i\left[\frac{\alpha_0+1}{\alpha_0-1}\right]$, and $\tilde{M}(\infty_-) = i\left[\frac{1+\overline{\alpha_0}}{1-\overline{\alpha_0}}\right]$. Set $P(z) = \prod_{i=1}^{N}(z-p_i)$, $\tilde{P}(e^{i\theta}) = \frac{K_0^{\frac{1}{2}}P(e^{i\theta})}{cP(0)}$, $\tilde{Q}(e^{i\theta}) = \frac{K_0^{\frac{1}{2}}Q(e^{i\theta})}{cP(0)}$, Q given above and $\tilde{K}(e^{i\theta}) = \frac{K(e^{i\theta})}{K_0}$. Then $\tilde{P}$, $\tilde{Q}$, and $\tilde{K}$, are self-reciprocal polynomials. Where $K_0 = \sqrt{K(0_+)}$ and $K_0^{\frac{1}{2}}$ indicates a fixed square root.*

Proof. That $\tilde{K}$ is self-reciprocal follows since all its zeros are on the unit circle. That $\tilde{P}$ is self-reciprocal follows from the formula for c (see (2.18)), Equation (2.22) and the fact that all the zeros of P are on the unit circle. In order to see that $\tilde{Q}$ is self-reciprocal we use the fact that $\tilde{m}_+$ and $\tilde{m}_-$ have integral representations given by (1.8) and (1.9) respectively multiplied by the complex number i. These representations and (2.12) imply that $\Im\frac{Q(e^{i\theta})}{P(e^{i\theta})} = 0$ for almost every $e^{i\theta} \in K\backslash\Sigma$. Note that Q and P have the same degree. The result now follows using the relation between P and $\tilde{P}$, the fact that $\tilde{P}$ is self-reciprocal, and the linear independence of the set $\{e^{\frac{ij\theta}{2}}\}_{-N}^{N}$ on $K\backslash\Sigma$. $\square$

From the above lemma it is not difficult to show using the boundary properties of the

Poisson kernel that the measure associated with $\tilde{m}_+$ is (see [PS], Thm. 2.1),

$$d\sigma_+(\theta) = \frac{1}{2\pi} f(e^{i\theta})d\theta + \sum_{i=1}^{n} \sigma_{k_i}\delta(\theta - \theta_{k_i})d\theta, \tag{4.1}$$

where

$$f(e^{i\theta}) = \begin{cases} \frac{|c|\sqrt{|\tilde{K}(e^{i\theta})|}}{2|\tilde{P}(e^{i\theta})|} & \text{if } e^{i\theta} \in \Sigma \\ 0 & \text{otherwise,} \end{cases} \tag{4.2}$$

and

$$\sigma_{k_i} = \frac{1}{2i} \frac{-c\sqrt{\tilde{K}(p_{k_i})}}{p_{k_i}\frac{d\tilde{P}(p_{k_i})}{dz}} = \frac{1}{2} \frac{|c|\sqrt{|\tilde{K}(e^{i\theta_{k_i}})|}}{\left|\frac{d\tilde{P}(e^{i\theta_{k_i}})}{d\theta}\right|}, \tag{4.3}$$

where $p_j = e^{i\theta_j}$ and $e^{i\theta_{k_i}}, i = 1, \ldots, n \leq N$ are the zeros of P which are the images of the poles of $\tilde{M}$ on the first sheet. Note that (2.14) has been used to compute the residues, σ_{k_i}, given by (4.3).

From the relation between $\tilde{m}_+$ and $\tilde{m}_-$ we find that the measure associated with $\tilde{m}_-$, σ_-, (see (1.9)), has the same density as σ_+ i.e., that given by (4.2) while the masses are given by (4.3) evaluated at the images of the poles of $\tilde{M}$ located on the second sheet.

Before stating the next theorem we use (2.13) and (2.20) to obtain the following formula for α_0,

$$\alpha_0 = \frac{1}{2}\left(1 + \frac{K_0}{P(0)} \frac{1}{(1 + (-1)^N \sum_{i=1}^{N} \frac{K_i}{p_i\frac{dP(p_i)}{dz}})}\right) \tag{4.4}$$

We now prove,

Theorem 4.2. *Given any set of distinct arcs of the circle* $[z_0, z_1], \ldots, [z_{2N-2}, z_{2N-1}]$, *points* $p_i, i = 1, \ldots, N,\ p_i \in [z_{2i-1}, z_{2i}],\ i = 1, \ldots, 2N, z_0 = z_{2N}$, *and integers,* $0 \leq k_1 < \cdots < k_n \leq N$, *there exists a unique probability measure* σ_+ *given by* (4.1) - (4.3) *with* $\tilde{K}(z) = \frac{\prod_{i=1}^{2N}(z-z_i)}{(\prod_{i=1}^{2N} z_i)^{\frac{1}{2}}}$ *and* $\tilde{P}(z) = \frac{\prod_{i=1}^{N}(z-p_i)}{(\prod_{i=1}^{N} -p_i)^{\frac{1}{2}}}$ *and a unique sequence of reflectionless quasi-periodic recurrence coefficients* $\{\alpha_n\}_{n=-\infty}^{\infty}, |\alpha_n| < 1$. *Furthermore* $\limsup |\alpha_n| < 1$

Proof. The uniqueness of the measure follows from (4.2), (4.3), (2.21), and the fact that D is not equal to zero (since the points $p_1, \ldots, p_N, 0_+$, and ∞_+ are linearly independent). From this measure we construct $\tilde{m}_+(z) = \tilde{m}_+(z, 0)$, and $\tilde{M}$. This gives $\tilde{m}_-(z) = \tilde{m}_-(z, 0)$ and the reflectionless property follows. Since σ is a positive measure with an infinite number of points in its support there is a unique set of recurrence coefficients $\{\alpha_n\}_{n=1}^{\infty}$, $|\alpha_n| < 1$ associated with it. We now compute $m_{\pm}$ from $\tilde{m}_{\pm}$ inverting (1.7). Use (1.5) to compute α_0, invert (1.4) to compute $m_-(\cdot, -1)$ and then α_{-1} from (1.5). Repeating this procedure gives $\alpha_n, n \leq 0$. The fact that $|\alpha_n| < 1$ follows since $|m_-(z,n)| > 1$ for $|z| < 1$.

Equation (1.3) shows that each $\tilde{m}_+(\cdot, n)$, $(n \in \mathbb{Z})$ is a meromorphic function similar to $\tilde{m}_+(\cdot, 0)$ consequently there is be a measure σ_+^n associated with $\tilde{m}_+(\cdot, n)$ which will have the form given by (4.1) - (4.3) except with different poles $p_1(n), \ldots p_N(n)$ and c_n. Equation (1.4) indicates that there is an analogous measure σ_-^n associated with $\tilde{m}_-(\cdot, n)$. From the relation between $\tilde{m}_+$ and $\tilde{m}_-$ both σ_+^n and σ_-^n have the same absolutely continuous part. Since all the zeros of $P_n(z)$ lie on the unit circle for all n there exists a constant d such that $|P_n(z)| < d$ for all n and all $z \in K$. Consequently,

$$|1 - \overline{\alpha}_n| = \frac{1}{|c_n|} \geq \frac{1}{4\pi d} \int_{\Sigma} \sqrt{|K(e^{i\theta})|} d\theta,$$

and

$$\frac{1 - |\alpha_n|^2}{|1 - \alpha_n|^2} \geq \frac{1}{2\pi} \int_{\Sigma} f_n(e^{i\theta}) d\theta > 0.$$

The first of the above inequalities implies that α_n stays uniformly away from one, which when coupled with the second implies that $\limsup |\alpha_n| < 1$. The quasi-periodicity of the $\{\alpha_n\}$ follows from the previous section. $\square$

Note that a formula similar to (4.4) holds for each α_n if P and its zeros are replaced by P_n and its zeros.

We now examine what happens if a particular sequence of recurrence coefficients is almost periodic. Again we assume that the sequence of recurrence coefficients $\alpha = \{\alpha_n\}_{n=-\infty}^{\infty}$ is an almost periodic sequence that is not identically zero with $\limsup |\alpha_n| < 1$.

In this case we can take the hull associated with the above sequence as our probability space Ω and μ is the Haar measure (see [JM], [AS]). Note that if $\{\alpha_n\}$ is given only for $n \geq 1$, $\{\alpha_n\}$ for $n < 1$ can be obtained from the Fourier representation for α. As above Σ_ω, $\omega \in \Omega$ is the set of non-isolated points of the topological support of the orthogonality measure σ_ω associated with polynomials $\{\phi_n(z;\omega) | n \geq 1\}$. For am $\omega \in \Omega$ let H_ω be the unitary operator associated with the sequence α_ω (see [GT] Eq. (2.11)).

Lemma 4.3. *Let $\{\alpha_n\}_{n=-\infty}^{\infty}, |\alpha_n| < 1$ be an almost periodic sequence. Then the Weyl m functions, $\{m_\pm(z,n)\}_{n=-\infty}^{\infty}$ are almost periodic for $|z| < 1$ as is*

$$g(z,n) = i\frac{m_-(z,n) + m_+(z,n)}{m_-(z,n) - m_+(z,n)}.$$

Proof. Since $\{\alpha_n\}_{n=-\infty}^{\infty}$ is almost periodic we need to show ([JM]) that if α_{n+n_i} converges uniformly in n as i tends to ∞ then $m_\pm(\cdot, n+n_i)$ also converge uniformly as i tends to ∞. But this follows from the fact that for $|z| < 1$ $\{m_+(\cdot,n)\}$ and $\{\frac{1}{m_-(\cdot,n)}\}$ are uniformly bounded and are the unique solutions to (1.3) and (1.4) respectively satisfying (1.5). □

Remark 4.4. The results to follow can also be proved in the more general case when Ω admits a unique τ invariant measure

Lemma 4.5. *With the hypotheses of Lemma 4.3 there is a set $\Sigma \subset K$ such that $\Sigma_\omega = \Sigma$ for all $\omega \in \Omega$. Furthermore there exists a unique measure k with* $\operatorname{supp} \mathrm{k} = \Sigma$ *such that for any $f \in C(K)$, $C(K)$ the continuous functions on K*

$$\int_\Sigma f dk = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{|n|\leq N} < f(H_\omega e_n, e_n) > \tag{4.5}$$

for all $\omega \in \Omega$. Here $\{e_n\}$ is the standard basis in l^2.

Proof. Since $g_\omega(z,n) =< (H_\omega + zI)(H_\omega - zI)^{-1} e_n, e_n >$ (see [GT, Eq. (5.11)]), we see that the above sum with $f = \frac{(e^{i\theta}+z)}{(e^{i\theta}-z)}$ is the average of an almost periodic function which is independent of $\omega \in \Omega$. That the integral holds for all $f \in C(K)$ follows from the Stone-Weierstrass theorem. The rest of the theorem follows from [GT, Thm. 3.6] or [GJ, Thm. 5.1] which have their roots in [AS], and [JM]. □

Note that (4.5) also follows from [GT, Thm. 3.6] and the topology of almost periodic functions.

Lemma 4.6. *Let $\{\alpha_n\}_{n=-\infty}^{\infty}, |\alpha_n| < 1$ be an almost periodic sequence as described above. Then*

$$\lim_{n\to\infty} \frac{1}{n} \ln |\phi_n^*(z) + z\phi_n(z)| = \int_\Sigma \ln |z - e^{i\theta}| dk(\theta) + R, \tag{4.6}$$

uniformly on compact subsets of $z \in C\backslash K$*. Furthermore,*

$$\lim_{n\to\infty} \frac{1}{n} \ln |\phi_n^*(z)| = \int_\Sigma \ln |z - e^{i\theta}| dk(\theta) + R, \tag{4.7}$$

uniformly on compact subsets of $|z| < 1$*, while*

$$\lim_{n\to\infty} \frac{1}{n} \ln |\phi_n(z)| = \int_\Sigma \ln |z - e^{i\theta}| dk(\theta) + R, \tag{4.8}$$

uniformly on compact subsets of $|z| > 1$*. Here* $R = \lim \frac{1}{N} \sum_{n=1}^{N} \ln a_n$

Proof. If in Lemma 4.4 we take $f(e^{i\theta}) = \ln |z - e^{i\theta}|$ then (4.6) follows from the above lemma and Lemmas 2.1 and 4.3 in [GT]. Since ϕ_n^* has all its zeros outside the unit circle ([Sz]), Equation (4.7) follows from (4.6) and the fact that for $|z| < 1, |\frac{\phi_n(z)}{\phi_n^*(z)}| < 1$. Equation (4.8) follows from a similar analysis.

Let Σ_0 be a compact subset of the unit circle and let g_{Σ_0} be the Green's function associated with the set Σ_0 (see Section 2). We denote the capacity of Σ_0 by $\text{cap}(\Sigma_0)$ and we assume that $\text{cap}(\Sigma_0) > 0$. Note ([Tu]) that g_{Σ_0} has the representation

$$g_{\Sigma_0}(z) = -\log \text{cap}(\Sigma_0) + \int_{\Sigma_0} \ln |z - e^{i\theta}| d\nu,$$

where ν is the equilibrium measure associated with Σ_0. We say that

$$\lim_{n\to\infty} f_n(z) = h(z)$$

locally uniformly in an open set D if for every $z \in D$ and $z_n \to z$ as $n \to \infty$ we have $\lim_{n\to\infty} f_n(z_n) = h(z)$ (see [ST]).

Let

$$\gamma_n(\omega, z) = \frac{1}{n} \log \|T(z, n) \cdots T(z, 1)\|,$$

and

$$\gamma_{-n}(\omega, z) = \frac{1}{n} \log \|T(z, 0) \cdots T(z, -n+1)\|.$$

Which norm is used is not important but for convenience we will use the Hilbert-Schmidt norm. We now show,

Lemma 4.7. *Let $\{\alpha_n\}_{n=-\infty}^{\infty}, |\alpha_n| < 1$ be an almost periodic sequence as described above. Let $\{\phi_n\}_{n=0}^{\infty}, \phi_0 = 1$ be the sequence of orthonormal polynomials satisfying (1.1). If $\lim_{n\to\infty} |\phi_n(z)|^{\frac{1}{n}} = e^{g_{\Sigma_0}(z)}$ locally uniformly on $|z| > 1$, then $\Sigma = \Sigma_0$, $k = \nu$, and $R = -\log \operatorname{cap}(\Sigma_0)$. Furthermore $\lim_{n\to\infty} \gamma_n(\omega, z) = g_{\Sigma_0}(z) = \lim_{n\to\infty} \gamma(z, -n)$ for all $z \in C\backslash K$ and $\omega \in \Omega$ and $\limsup \gamma_{\pm n}(\omega, z) \le \gamma(z)$ for all $z \in C$and $\omega \in \Omega$.*

Proof. That $\Sigma = \Sigma_0$, $k = \nu$, and $R = -\log \operatorname{cap}(\Sigma_0)$ follows from Lemmas 4.5 and 4.6. That $\lim_{n\to\infty} \gamma_n(\omega, z) = g_{\Sigma_0}(z)$ for all $\omega \in \Omega$ and all $z \in C\backslash K$ follows from Lemma 4.4 in [GT] and analogous arguments using Lemma 2.1 in [GT] show $\lim_{n\to\infty} \gamma_{-n}(\omega, z) = g_{\Sigma_0}(z)$. That $\limsup \gamma_{\pm n}(\omega, z) \le \gamma(z)$ for all $\omega \in \Omega$ and all $z \in C\backslash K$ now follows since $\limsup \gamma_{\pm n}(\omega, z) \le \gamma(z)$ is submean while γ is subharmonic (see Theorem 2.3 in [CS] or [J]). □

Systems of orthogonal polynomials having the behavior described in Lemma 4.7 are said to have regular nth root asymptotic behavior (see [ST] p. 60).

Let $\{\Phi^{\pm}(z, n)\}_{-\infty}^{\infty}$ be solutions of the difference system determined by (1.1) such that $\lim_{n\to\infty} \|\Phi^+\|$ and $\lim_{n\to-\infty} \|\Phi^-\|$ are bounded. The existence of such solutions follow from exponential dicotomy results ([GJ]) or from the fact that for $z \in C\backslash K$, $(zI - H)^{-1}$ is a bounded operator in l^2. It now follows from the "deterministic" Oseledec Theorem ([CL]) and $\det T(z, n) = z$ that for all $|z| < 1$ and all $\omega \in \Omega$, ([GT]),

$$\lim \frac{1}{n} \log \|\Phi^+(z, n)\| = \log |z| - \gamma(z), \tag{4.9}$$

and

$$\lim \frac{1}{n} \log \|\Phi^+(z, -n)\| = -\gamma(z). \tag{4.10}$$

We now prove,

Theorem 4.8. *Let $\{\alpha_n\}_{n=-\infty}^{\infty}, |\alpha_n| < 1$ be an almost periodic sequence as above. Let $\{\phi_n\}_{n=0}^{\infty}, \phi_0 = 1$ be the sequence of orthonormal polynomials satisfying (1.1) with orthogonality measure σ_{ω_0}. If $\lim_{n\to\infty} |\phi_n(z)|^{\frac{1}{n}} = e^{g_{\Sigma_0}(z)}$ locally uniformly on $|z| > 1$, with Σ_0 a finite union of nondegenerate arcs of the unit circle then σ_ω has the form given by* (4.1) - (4.3) *for all $\omega \in \Omega$. Furthermore $\{\alpha_n\}$ is a quasi periodic sequence.*

Proof. From its definition $g_{\Sigma_0}(z) = 0$ for almost all $z \in \Sigma_0$. Equations (4.9) and (4.10) allow us to apply the Kotani theorem for these systems for almost all $\omega \in \Omega$ ([GJ, Thm. 5.9], [GT, Thm. 5.6]). The ω continuity of the $\tilde{m}$ functions allows the result to be extended to *all* ω ([GJ, Corollary 5.12]). Furthermore since Σ_0 is comprised of a finite number arcs we can conclude that σ_ω has an analytic density for all $\omega \in \Omega$ and that $m_\pm$ are reflectionless. Hence the results from the previous section show that $\{\alpha_n\}$ is in fact a quasi-periodic sequence. □

We now describe a result from orthogonal polynomials and potential theory.

Lemma 4.9. *Let $\{\alpha_n\}_{n=1}^{\infty}$, $|\alpha_n| < 1$ be a sequence of recurrence coefficients for* (1.1). *Let Σ be the non-isolated points in the topological support of σ. If $\lim \frac{1}{N}\sum_{n=1}^{N} \log(1 - |\alpha_n|^2)^{\frac{1}{2}} = \log \operatorname{cap}(\Sigma)$ then $\lim_{n\to\infty} |\phi_n(z)|^{\frac{1}{n}} = e^{g_\Sigma(z)}$ locally uniformly for $|z| > 1$.*

Remark 4.10. This is a special case of a much more general theorem that has been proved by Stahl and Totik ([ST] Thm. 3.1.1) which also applies to polynomials orthogonal on bounded subsets of the real line. In the case when $\{\alpha_n\}_{-\infty}^{\infty}$, $|\alpha_n| < 1$ is an almost periodic sequence as described above the result follows since $\gamma(z) \geq 0$ for all $|z| = 1$ (in particular for $z \in \Sigma$) while the equation below (1.1) and Lemma 4.6 show $\lim_{z\to\infty}(\gamma(z) - g_\Sigma(z)) = 0$.

With this we now show,

Theorem 4.11. *Let $\{\alpha_n\}_{-\infty}^{\infty}$, $|\alpha_n| < 1$ be a sequence of recurrence coefficients for* (1.1). *Let Σ (non-isolated points in the topological support of σ) be a finite set of nondegenerate arcs of the circle. Then $\{\alpha_n\}$ is a quasi periodic sequence with $\limsup |\alpha_n| < 1$ and $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \log(1 - |\alpha_n|^2)^{\frac{1}{2}} = \log \operatorname{cap}(\Sigma)$ if and only if σ has the form given by* (4.1) - (4.3).

Proof. Since $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \log(1 - |\alpha_n|^2)^{\frac{1}{2}} = \log \operatorname{cap}(\Sigma)$ we see from the above lemma that the orthonormal polynomials ϕ_n associated with σ have regular nth root asymptotics, i.e., $\lim_{n\to\infty} |\phi_n(z)| = e^{g_\Sigma}$ locally uniformly for $|z| > 1$. Thus Theorem 4.8 implies that σ_ω (for any $\omega \in \Omega$) has the form given by (4.1) - (4.3). From Theorem 4.2 we see that (4.1) - (4.3) imply that $\{\alpha_n\}$ is quasiperiodic with $\limsup |\alpha_n| < 1$. Since all the measures are absolutely continuous on the same segments the analog of the Pastur-Ishii Theorem for these systems ([GT, Thm. 5.7], [GJ, Thm. 5.8]) shows that $\gamma = g_\Sigma$. Comparison of these functions near infinity shows that $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \log(1-|\alpha_n|^2)^{\frac{1}{2}} = \log \operatorname{cap}(\Sigma)$. $\square$

Remark 4.12. In the case that $\Sigma = \Sigma_0$ the example of Chulaevesky and Sinai [ChS] indicates that $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \log(1-|\alpha_n|^2)^{\frac{1}{2}} = \log \operatorname{cap}(\Sigma_0)$ is needed.

Finally using the theory developed in Stahl and Totik we give the following stability result.

Theorem 4.13. *Let $\{\alpha_n\}_{-\infty}^{\infty}$, $|\alpha_n| < 1$ be an almost periodic sequence as described above. Let $\sigma = \sigma_{\omega_0}$ be the orthogonality measure associated with $\{\alpha_n\}_{n=0}^{\infty}$ and suppose there is an infinite set Σ_{ac} such that $\frac{d\sigma}{d\theta} > 0$ for almost every point in Σ_{ac}. Then for almost every $\omega \in \Omega$, $\Sigma_{ac}^{\omega} = \Sigma_{ac}$.*

Proof. Let $\hat{\sigma} = \sigma|_{\Sigma_{ac}}$. Note that Σ_{ac} is a compact subset of K. Then for almost every $\theta \in \Sigma_{ac}$,

$$\lim_{r\to 0} \frac{\log \hat{\sigma}(\Delta_r(e^{i\theta}))}{\ln r} < \infty,$$

where $\Delta_r(e^{i\theta})$ is the disk of radius r centered at $e^{i\theta} \in K$. Therefore $\hat{\sigma}$ is a regular measure by the Λ criterion of Stahl and Totik (see [ST, p. 108]) and from [ST] Theorem 5.3 $\limsup_{n\to\infty} |\phi_n(z)|^{\frac{1}{n}} \leq e^{g_{\Sigma_{ac}}(z)}$ locally uniformly for $|z| > 1$ and $\limsup_{n\to\infty} |\phi_n^*(z)|^{\frac{1}{n}} \leq e^{g_{\Sigma_{ac}}(z)}$ locally uniformly for $|z| < 1$. Since $\lim_{n\to\infty} \frac{1}{n}\log|\phi_n(z)| = \gamma(z)$ for $|z| > 1$ and $\lim_{n\to\infty} \frac{1}{n}\log|\phi_n^*(z)| = \gamma(z)$ for $|z| < 1$ we see $\gamma(z) = 0$ for almost all $\theta \in \Sigma_{ac}$ since γ is subharmonic. The result now follows the Kotani theorem for these systems ([GT, Thm. 5.7], [GJ, Thm. 5.9]) for almost all $\omega \in \Omega$. $\square$

We note that the above theorems have real line analogs. In this case the orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ satisfy the following three term recurrence formula

$$a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x) = x p_n(x) \tag{4.11}$$

with initial condition $p_0 = 1$ and $p_{-1} = 0$. Here $a_n > 0$ and $b_n \in \mathbb{R}$. If it is assumed that the sequences $\{a_n\}_{-\infty}^{\infty}$ and $\{b_n\}_{-\infty}^{\infty}$ are almost periodic (or quasi periodic where appropriate) with bounds $0 < r \leq a_n \leq M < \infty$ and $|b_n| \leq M$ $\forall n \in \mathbb{Z}$. In this case there is a unique measure σ such that $\int_{\mathbb{R}} p_n(x)p_m(x)d\sigma(x) = \delta_{m,n}$ and the above results carry over with little modification to the real line case (See [AK], [AS], [BGHT], [M], and [D]). Let $\Sigma = [x_0, x_1] \cup [x_2, x_3] \cup \cdots \cup [x_{2N-2}, x_{2N-1}]$ be a finite union of finite non-degenerate intervals of the real line, C be the Riemann surface of the algebraic relation $w^2 = K(x) = \prod_{j=0}^{2N-1}(x - x_j)$. We will use the same branch of the square root as earlier.

Theorem 4.14. *Let $\{a_n\}_{-\infty}^{\infty}$ and $\{b_n\}_{-\infty}^{\infty}$ be a sequence of recurrence coefficients for (4.11). Let Σ (the non isolated points in the topological support of σ) be a finite union of non-degenerate intervals as given above. Then $\{a_n\}$ and $\{b_n\}$ are quasiperiodic sequences with $0 < r \le a_n \le M < \infty$, $|b_n| \le M$ $\forall n \in \mathbb{Z}$, and $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \log a_n = \log \operatorname{cap}(\Sigma)$ if and only σ has the form*

$$d\sigma(x) = f(x)dx + \sum_{i_n} \sigma_{i_n} \delta(x - p_{i_n})dx,$$

where

$$f(x) = \frac{c\sqrt{|K(x)|}}{\pi |P(x)|},$$

and, if the above sum is not empty, then $1 \le i_1 < i_2 \ldots < i_m \le N - 1$ with

$$\sigma_{i_n} = \frac{2c\sqrt{|K(p_{i_n})|}}{|\frac{dP(p_{i_n})}{dx}|}.$$

Here $p_i \in [x_{2i-1}, x_{2i}], i = 1, \ldots N-1$, $P(x) = \prod_{i=1}^{N-1}(x - p_i)$, and ([BGHT, Eq. (3.11)])

$$c^{-1} = -\sum_{j=1}^{N-1} \frac{\sqrt{K(p_j)}}{2\frac{dP(p_j)}{dx}} - \frac{1}{4}\{(\sum_{j=1}^{N-1} p_j - \frac{1}{2}\sum_{j=0}^{2N-1} p_j)^2 + \sum_{j=1}^{N-1} p_j^2 - 1/2 \sum_{j=0}^{2N-1} p_j^2\}$$

.

Remark 4.15. Since the number of points p_i is the same as the genus of the surface the standard forms of Abel's theorem and the Jacobi Inversion theorem can be used ([AK]).

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