Compactly Supported, Piecewise Affine Scaling Functions on Triangulations

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Abstract

The methods of using intertwining MRA's to find orthogonal scaling functions have previously been applied to one-dimensional MRA's and are here extended to two-dimensional bases. Two examples are constructed from MRA's consisting of continuous, compactly supported, piecewise affine functions of two variables. The resulting scaling functions can be conveniently restricted to compact domains.

1. Introduction.

Wavelet bases have shown themselves to be useful in diverse areas such as image processing, numerical analysis, and statistics. This is partly due to the efficient decomposition and reconstruction algorithms associated with these bases. In a number of the applications above, it is necessary to have wavelet bases that span $L^2(\mathbf{R}^2)$ or $L^2(\Omega)$, where Ω is some compact subset of $\mathbf{R}^2$. One convenient method of obtaining such bases is to take tensor products of one-dimensional wavelets and scaling functions. Wavelet bases constructed in this manner do have their drawbacks. For instance, it is difficult to exploit diagonal symmetries on the square using tensor product bases. Furthermore, tensor product bases may not be well suited to handling problems on non-separable domains, such as hexagonal or triangular regions.

Some examples of non-separable compactly supported bivariate orthogonal or bi-orthogonal bases can be found in the work of Cohen and Daubechies [2], Belogay and Wang [1], Gröchenig and Madych[8], He and Lai [9], Kovačevič and Vetterli [11] or Riemenschneider and Shen [14]. For examples of orthogonal multi-wavelet bases see Donovan et al [4] and [7].

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The goal of this paper is to construct orthogonal, compactly supported, continuous scaling functions in two dimensions. In [5] the technique of intertwining multiresolution analyses (MRA) was developed for constructing such bases in one dimension, and it was shown that this technique could be applied to a broad class of MRA's on $\mathbf{R}$. In [6], a systematic approach was applied to spline MRA's to produce compactly supported C^0 and C^1 symmetrical orthogonal scaling functions with arbitrary approximation order. Here, we will extend these techniques to bivariate MRA's. The examples presented here begin with MRA's derived from the "hat" functions supported on $[-1,1]^2$ and the regular hexagonal region centered at the origin. As in the one-dimensional constructions, the resulting piecewise polynomial scaling functions are well suited to restriction to compact domains. In this paper, only the construction of the scaling functions is discussed. In a companion paper by Donovan [3], the problem of constructing the corresponding orthogonal wavelet bases will be addressed.

This paper is organized as follows: Section 2 sets the notation used in the subsequent sections; in Section 3, a construction is given using functions that are piecewise affine on a triangulation based on the Quincunx lattice; and in Section 4, a similar construction is given using functions piecewise affine on a regular triangulation.

2. Definitions and Notation:

Note that in what follows, sometimes the *support* of a function f is not as important a concept as the support on unit cells; i.e.,

$$\widetilde{\operatorname{supp}}\, f = \bigcup\{[j,j+1]\times[k,k+1]\colon\ j,k\in\mathbf{Z},\ (j,j+1)\times(k,k-1)\cap\operatorname{supp} f\neq\emptyset\}.$$

In this paper, we will sometimes use the term *unit cell support* to refer to the unit cells containing the support.

Suppose that $\Phi=(\phi^1,\ldots,\phi^n)^\top$ is a column vector whose components are compactly supported, square-integrable functions on $\mathbf{R}^2$. We will also use the symbol 'Φ' to refer to the set $\{\phi^1,\ldots,\phi^n\}$, as appropriate in context. Given Φ as above and lattice $L\subset\mathbf{R}^2$, the space $V=\operatorname{cl}_{L^2}\operatorname{span}\{\phi(\cdot-k)\colon\ \phi\in\Phi,\ k\in L\}$ is a *finitely generated, shift invariant (FSI)* space on $\mathbf{R}^2$. The shift invariance is *with respect to* L, and the space is *generated by* Φ. If, in addition, $\{\phi(\cdot - k)\colon\ \phi \in \Phi,\ k \in L\}$ is an orthogonal (orthonormal) set, then Φ is said to be an *orthogonal (orthonormal) generator*.

For convenience, we will use subscripts to denote dilations and translations of functions or sets of functions. Given $\phi \in L^2(\mathbf{R}^2)$, $p \in \mathbf{Z}$, and $k \in \mathbf{Z}^2$, denote $\phi_{p,k} = \phi(2^p \cdot - k)$. This notation is extended to sets as well, so that if $\Phi \subseteq L^2(\mathbf{R}^2)$, $P \subseteq \mathbf{Z}$, and $K \subseteq \mathbf{Z}^2$, then $\Phi_{P,K} = \{\phi(2^p \cdot - k)\colon\ \phi \in \Phi,\ p \in P,\ k \in K\}$.

In constructing scaling functions with various support properties, it will be useful to have notation for classes of support-restricted spaces. Suppose $V \subset L^2(\mathbf{R}^2)$ and $S \subseteq \mathbf{R}^2$ with nonempty interior. Denote the support-restricted space $V^S = \{\phi \in V\colon \operatorname{supp} \phi \subseteq S\}$. Note that except in trivial cases, $V^S \neq V|_S$.

As in the one-dimensional case, the construction of wavelets and multi-wavelets is based on the concept of *multiresolution analysis* (Mallat [12], Meyer [13]). Let L and Φ be defined as above, V_0 denote the FSI space generated by Φ with shifts in L, and $V_p = \{f_{p,0}\colon\ f \in V_0\}$ for $p \in \mathbf{Z}$. We say that Φ generates a multiresolution analysis $(V_p)_{p\in\mathbf{Z}}$ with lattice L if

1. $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$
2. $\Phi_{0,L}$ forms a Riesz basis for V_0.

Since Φ is compactly supported, it follows that $\bigcap_{p\in\mathbf{Z}} V_p = \{0\}$ and $\bigcup_{p\in\mathbf{Z}} V_p$ is dense in $L^2(\mathbf{R}^2)$ (Jia and Shen [10]). If there exists a compactly supported Φ such that $\Phi_{0,L}$ is an orthogonal basis for V_0, then (V_p) is said to be an *orthogonal MRA*.

In this paper, we use the notion of *intertwining MRA's*, which is based on the following observation. Suppose (V_p) is an MRA generated by Φ, and w is a compactly supported function in $V_1 \setminus V_0$ such that $w_{0,\mathbf{Z}^2}$ is a Riesz basis. Then a new MRA can be formed by letting V_0 "borrow" w from V_1. In other words, let W be the $\mathbf{Z}^2$-shift invariant space spanned by $w_{0,\mathbf{Z}^2}$, and take $\tilde{V}_0 = V_0 + \mathrm{cl}_{L^2} W$ and $\tilde{V}_p = \tilde{V}_0(2^p \cdot)$. The sequence of spaces $(\tilde{V}_p)$, generated by $\Phi \cup \{w\}$, is called *intertwining* because of the relationships

$$\cdots \subset V_{-1} \subset \tilde{V}_{-1} \subset V_0 \subset \tilde{V}_0 \subset V_1 \subset \cdots, \tag{1}$$

and it clearly satsfies the conditions of an MRA, also as a consequence of (1). Of course, this process can be repeated as many times as desired, producing other MRA's, all of which are mutually intertwining in some way similar to (1). It should be noted that an intertwining MRA will generally have nice properties such as smoothness, spline structure, compactly supported generators, and symmetry, when the original does.

The reason this concept is useful here is that sometimes an MRA which is not itself orthogonal has an intertwining MRA which is orthogonal. Thus, given an MRA with nice properties but lacking orthogonality, one can hope to find a similar MRA with a compactly supported, orthogonal generator. In [5], it was shown how such intertwining MRA's can be found for general MRA's on $\mathbf{R}$ with compactly supported generators. The procedure for finding orthogonal, intertwining MRA's on $\mathbf{R}^2$ is very similar in principle to the same on $\mathbf{R}$. As in the one-dimensional case, we classify the scaling functions by support and try to "borrow" functions from V_1, adding them to Φ and modifying members of Φ that do not satisfy the requisite orthogonality relations. In so doing, we also try to preserve whatever symmetries are present in V_0. Despite the analogy, two-dimensional generators, have more support classes than their 1-D counterparts, and using them to construct orthogonal generators is correspondingly more complex.

To obtain an orthogonal generator, we will choose the borrowed functions w in the following fashion. Let Φ be a scaling vector with shifts in $\mathbf{Z}^2$ and supported on $[-1,1]^2$, where $\phi^1 \perp \{\phi^2, \ldots, \phi^n\}_{0,\mathbf{Z}^2}$. However, suppose, for instance that $\phi^1 \not\perp \phi^1_{0,(1,0)}$, which precludes Φ from being an orthogonal generator. The intent here is to choose w so that adding the fuction to Φ yields generator $\tilde{\Phi}$, where the component corresponding to ϕ^1 satisfies $\tilde{\phi}^1 \perp \tilde{\phi}^1_{0,(1,0)}$. To this end, let w be chosen to satisfy

1. $\operatorname{supp} w \subset [0,1] \times [-1,1]$,
2. $w \perp \{\phi^2, \ldots, \phi^n\}_{0,\mathbf{Z}^2} \cup \phi^1_{0,\mathbf{Z}^2 \setminus \{(0,0),(-1,0)\}}$,
3. $w_{0,\mathbf{Z}^2}$ is an orthogonal set, and
4. $(I - P_w)\phi^1 \perp (I - P_w)\phi^1_{0,(1,0)}$.

Note that the fourth condition given above is equivalent to

$$\langle \phi^1, \phi^1_{0,(1,0)} \rangle \langle w, w \rangle - \langle \phi^1, w \rangle \langle \phi^1_{0,(1,0)}, w \rangle = 0.$$

Assuming w can be found to satisfy these conditions, a new scaling vector

$$\begin{aligned}\tilde{\Phi} &= (\tilde{\phi}^1, \ldots, \tilde{\phi}^{n+1})^\top \\ &= \big((I - P_{w_{0,\mathbf{Z}^2}})\phi^1, \phi^2, \ldots, \phi^n, w\big)^\top\end{aligned}$$

is formed, where $\{\tilde{\phi}^2, \ldots, \tilde{\phi}^{n+1}\}_{0,\mathbf{Z}^2}$ is an orthogonal set, $\tilde{\phi}^1 \perp \{\tilde{\phi}^2, \ldots, \tilde{\phi}^{n+1}\}_{0,\mathbf{Z}^2}$, and additionally $\tilde{\phi}^1 \perp \tilde{\phi}^1_{0,(1,0)}$. Thus if $\tilde{\phi}^1$ is orthogonal to its shifts by $(0,1)$, $(1,1)$, and $(-1,1)$, then $\tilde{\Phi}$ is an orthogonal generator. Otherwise, a similar process can be used for each of the shifts to obtain an orthogonal generator.

3. Intertwining MRA's on the Square Lattice

For the first example, we use the lattice $\mathbf{Z}^2$ with unit cells being the $\mathbf{Z}^2$-translates of $[0,1]^2$. Here, as in the 1-D construction, the concept of *minimal support* plays an important role. Given that Φ is as above, we say that Φ is *minimally supported on* $[-1,1]^2$ if it satisfies

1. $\operatorname{supp} \Phi \subset [-1,1]^2$
2. For all $\phi \in \Phi$, $\operatorname{supp} \phi$ is not contained in either $[-1,0] \times [-1,1]$ or $[-1,1] \times [-1,0]$.
3. The nonzero functions among $\chi_{[0,1]^2} \phi^i_{0,j}$, $i = 1, \ldots, n$, $j \in \{0,1\}^2$, are linearly independent.

It follows from this definition that any minimally supported generator, Φ, can be partitioned into four support classes, $a^1 = \{\phi \in \Phi\colon \widetilde{\operatorname{supp}}\, \phi = [0,1]^2\}$, $a^2 = \{\phi \in \Phi\colon \widetilde{\operatorname{supp}}\, \phi = [-1,1] \times [0,1]\}$, $a^3 = \{\phi \in \Phi\colon \widetilde{\operatorname{supp}}\, \phi = [0,1] \times [-1,1]\}$, and $a^4 = \Phi \setminus (a^1 \cup a^2 \cup a^3)$. In addition, we have the following useful property.

Lemma 1. Let Φ be minimally supported on $[-1,1]^2$ and Ω be a union of unit cells in $\mathbf{R}^2$. Then

$$V_0^\Omega = \operatorname{cl}_{L^2} \operatorname{span}\{\phi_{0,j}\colon\ \phi \in \Phi,\ j \in \mathbf{Z}^2,\ \operatorname{supp} \phi_{0,j} \subset \Omega\}.$$

In the constructions that follow, it will be convenient to define particular support restricted spaces from the shift-invariant space V_0. Let

$$\begin{aligned}
A^1 &= V_0^{[0,1]^2} \\
A^2 &= (I - P_{A^1_{0,\mathbf{Z}^2}})V_0^{[-1,1]\times[0,1]} \\
A^3 &= (I - P_{A^1_{0,\mathbf{Z}^2}})V_0^{[0,1]\times[-1,1]} \\
A^4 &= (I - P_{(A^1\cup A^2\cup A^3)_{0,\mathbf{Z}^2}})V_0^{[-1,1]^2},
\end{aligned}$$

Where P_X denotes the orthogonal projection onto the closed span of X. Since it is our intent to construct orthogonal generators, we state the following, which gives conditions for an FSI space to have an orthogonal (minimally supported) generator.

Lemma 2. Suppose V_0 is an FSI space. If V_0 has an orthonormal generator minimally supported on $[-1,1]^2$, then

1. V_0 is the closed span of A^1, A^2, A^3, A^4, and their $\mathbf{Z}^2$-translates.
2. $A^2 \perp A^2_{0,(1,0)}$
3. $A^3 \perp A^3_{0,(0,1)}$
4. $A^2 \perp A^3_{0,\{-1,0\}\times\{0,1\}}$
5. $A^4 \perp A^2_{0,\{-1,1\}\times\{-1,0\}} \cup A^3_{0,\{-1,0\}\times\{-1,1\}}$
6. $A^4 \perp A^4_{0,\{-1,0,1\}^2\setminus\{(0,0)\}}$.

Conversely, if conditions 1–6 hold, then V_0 has an orthonormal generator Φ, where $\Phi \setminus a^4$ is minimally supported on $[-1,1]^2$.

Proof: Suppose that Φ is an orthonormal generator for V_0 minimally supported on $[-1,1]^2$. Lemma 1 assures that A^1 is spanned by a^1. Furthermore, since A^2 is orthogonal to A^1 and its translates, it follows from Lemma 1 that A^2 is spanned by a^2. Likewise, $A^3 = \operatorname{span} a^3$, and $A^4 = \operatorname{span} a^4$. Thus, 2–6, above, are clearly satisfied. Condition 1 is also satisfied because $\Phi = \bigcup_{i=1}^4 a^i \subset \bigcup_{i=1}^4 A^i$.

Now, suppose conditions 1–6 are satisfied. Then we may choose orthonormal bases a^i for A^i, $i = 1,2,3,4$. From condition 1, it follows that $\Phi = a^1 \cup a^2 \cup a^3 \cup a^4$ generates all of V_0. Conditions 2–6 and the definitions of A^i, $i = 1,2,3,4$, imply that that this generator is orthogonal. Furthermore, the orthogonalities among A^1, A^2, and A^3 preclude most of the linear dependencies that might foil minimal support. The cases not accounted for are such as, for example, a dependency in $\chi_{[0,1]^2} a^2$. However, if $\chi_{[0,1]^2} \sum_{\phi \in a^2} \alpha_\phi \phi = 0$, then certainly $\sum_{\phi \in a^2} \alpha_\phi \phi_{0,(1,0)} \in A^1$, which contradicts the orthogonality implied in the definition of A^2. ■

The construction of the minimally supported generator in Lemma 2 does not actually rely on conditions 2–6 except for the orthogonality of the shifts. Thus given that a minimally supported generator exists, we can obtain one from A^i, $i = 1, \ldots, 4$, as above where

$$\begin{aligned} &a^2 \perp a^1_{0,\{(0,0),(-1,0)\}} \\ &a^3 \perp a^1_{0,\{(0,0),(0,-1)\}} \\ &a^4 \perp a^1_{0,\{-1,0\}^2} \cup a^2_{0,\{(0,0),(0,-1)\}} \cup a^3_{0,\{(0,0),(-1,0)\}}. \end{aligned} \tag{2}$$

If by adding functions to Φ, we destroy one or more of these orthogonalities, they can be restored easily by using appropriate projections. From Lemma 2, we have some additional requirements for Φ to be an orthogonal generator. In particular,

$$\begin{aligned} &a^2 \perp a^2_{0,\{(1,0),(-1,0)\}} \cup a^3_{0,\{-1,0\}\times\{0,1\}} \\ &a^3 \perp a^3_{0,\{(0,1),(0,-1)\}} \\ &a^4 \perp a^2_{0,\{-1,1\}\times\{-1,0\}} \cup a^3_{0,\{-1,0\}\times\{-1,1\}} \cup a^4_{0,\{-1,0,1\}^2\setminus\{(0,0)\}}. \end{aligned} \tag{3}$$

For this example, consider a triangulation of the plane with vertices $\mathbf{Z}^2 \cup (\mathbf{Z}^2 + (\frac{1}{2}, \frac{1}{2}))$. The triangular cells are the $\mathbf{Z}^2$-translates of $t_1 = \operatorname{conv}\{(0,0),(1,0),(\frac{1}{2},\frac{1}{2})\}$, $t_2 = \operatorname{conv}\{(1,0),(1,1),(\frac{1}{2},\frac{1}{2})\}$, $t_3 = \operatorname{conv}\{(1,1),(0,1),(\frac{1}{2},\frac{1}{2})\}$, and $t_4 = \operatorname{conv}\{(0,1),(0,0),(\frac{1}{2},\frac{1}{2})\}$, where "conv" denotes the convex hull. Let V_0 consist of the square-integrable, continuous functions that are affine on each of these triangles. This space is refinable, so its dilations form a multiresolution analysis (V_p). The symmetries associated with V_0 include $\mathbf{Z}^2$-translation (of course), horizontal, vertical, and diagonal reflection, and rotation by 90°. The intertwining MRA will be chosen in such a way that these symmetries are preserved. One generator for V_0 is $\Phi = \{\phi^1, \phi^2\}$, where $\phi^1 = h$ is the 2-D "hat" function,

$$h(x,y) = \begin{cases} 1-x & \text{if } (x,y) \in \operatorname{conv}\{(0,0),(1,-1),(1,1)\} \\ 1-y & \text{if } (x,y) \in \operatorname{conv}\{(0,0),(1,1),(-1,1)\} \\ 1+x & \text{if } (x,y) \in \operatorname{conv}\{(0,0),(-1,1),(-1,-1)\} \\ 1+y & \text{if } (x,y) \in \operatorname{conv}\{(0,0),(-1,-1),(1,-1)\} \\ 0 & \text{otherwise,} \end{cases}$$

and $\phi^2 = \phi^1_{1,(1,1)}$. Note that $\Phi = \{\phi^1, \phi^2\}$ is a refinable set whereas ϕ^1 by itself is not refinable. This is true despite the fact that ϕ^2 is simply a dilate of ϕ^1.

To obtain an orthonormal scaling vector, which we will denote by $\Phi^{\square}$, we want to take functions from V_1 and add them to Φ. However, we find, in attempting to do this, that V_1 is not large enough to allow the construction to succeed. This problem can be overcome by augmenting V_0 (and thereby augmenting V_1) by adding the four functions $\phi^2_{1,\{0,1\}^2} \subset V_1$ to Φ. This produces $\hat{\Phi}$, $\hat{V}_0$, and a new MRA $(\hat{V}_p)$, where, as we will see, it suffices to borrow only from $\hat{V}_1$. The intertwining relationship between the resulting orthogonal MRA and the original is therefore characterized by $V_0 \subset \tilde{V}_0 \subset V_2$.

We assume without loss of generality that $\hat{\Phi}$ satisfies (2), and that $\hat{a}^1$ is an orthogonal basis containing ϕ^2 as well as

$$\begin{aligned}
\phi^3 &= (I - P_{\phi^2})(\phi^2_{1,(0,0)} + \phi^2_{1,(0,1)} + \phi^2_{1,(1,0)} + \phi^2_{1,(1,1)}) \\
\phi^4 &= \phi^2_{1,(0,0)} - \phi^2_{1,(0,1)} + \phi^2_{1,(1,0)} - \phi^2_{1,(1,1)} \\
\phi^5 &= \phi^2_{1,(0,0)} + \phi^2_{1,(0,1)} - \phi^2_{1,(1,0)} - \phi^2_{1,(1,1)} \\
\phi^6 &= \phi^2_{1,(0,0)} - \phi^2_{1,(0,1)} - \phi^2_{1,(1,0)} + \phi^2_{1,(1,1)}.
\end{aligned}$$

Here, ϕ^1 has been modified to be orthogonal to $\hat{a}^1_{0,\{-1,0\}^2}$. Note that $\hat{a}^2$ and $\hat{a}^3$ are empty. Thus (3) reduces to

$$\hat{a}^4 \perp \hat{a}^4_{0,\{-1,0,1\}^2 \setminus \{(0,0)\}},$$

and consequently, the problem is much easier to solve than the general case.

The construction is done in two steps. First, we borrow functions ϕ^7 and ϕ^8 supported on $[-1,1] \times [0,1]$ and $[0,1] \times [-1,1]$, respectively. These functions should be chosen so that by the intertwining procedure outlined above, ϕ^1 is replaced with a function that is orthogonal to its horizontal and vertical shifts. Second, we borrow functions ϕ^9 and ϕ^{10} supported in $[0,1]^2$ so that ϕ^1 is replaced by a function orthogonal to its diagonal shifts.

For the first part of the construction, choose $\phi^7 \in \hat{V}_1^{[-1,1]\times[0,1]}$ and $\phi^8 \in \hat{V}_1^{[0,1]\times[-1,1]}$. The spaces from which these functions are taken both have 43 dimensions, but there are a number of constraints on the choices of ϕ^7 and ϕ^8. Since we want $\tilde{V}_0$ to be symmetrical with respect to horizontal and vertical reflections, ϕ^7 and ϕ^8 must have these symmetries. Also, if $\tilde{V}_0$ is to be symmetrical with respect to diagonal reflections, we must have

$\phi^8(x,y) = \phi^7(y,x)$; i.e., ϕ^8 is a diagonal reflection of ϕ^7. The symmetries restrict ϕ^7 to a 12-dimensional space, which we parametrize as linear combinations of dilates and translates of the hat function h in the following way:

$$\begin{aligned}\phi^7 = \alpha_1 {\textstyle\sum} h_{3,\{-7,7\}\times\{1,7\}} + \alpha_2 {\textstyle\sum} h_{3,\{-5,5\}\times\{1,7\}} + \alpha_3 {\textstyle\sum} h_{3,\{-3,3\}\times\{1,7\}} \\ + \alpha_4 {\textstyle\sum} h_{3,\{-1,1\}\times\{1,7\}} + \alpha_5 {\textstyle\sum} h_{3,\{-7,7\}\times\{3,5\}} + \alpha_6 {\textstyle\sum} h_{3,\{-5,5\}\times\{3,5\}} \\ + \alpha_7 {\textstyle\sum} h_{3,\{-3,3\}\times\{3,5\}} + \alpha_8 {\textstyle\sum} h_{3,\{-1,1\}\times\{3,5\}} + \alpha_9 {\textstyle\sum} h_{2,\{-3,3\}\times\{1,3\}} \\ + \alpha_{10} {\textstyle\sum} h_{2,\{-1,1\}\times\{1,3\}} + \alpha_{11} {\textstyle\sum} h_{1,\{-1,1\}\times\{1\}} + \alpha_{12} h_{1,(0,1)}.\end{aligned}$$

Of course, there is no additional freedom in choosing ϕ^8, as it is completely determined by the choice of ϕ^7. Furthermore, since ϕ^7 will become a member of a^2, we may as well choose $\phi^7 \perp A^1_{0,\{(0,0),(0,-1)\}}$, so that (2) continues to be satisfied. This orthogonality takes the form of three linear equations,

$$\begin{aligned}0 &= 3\alpha_1 + 24\alpha_9 + 64\alpha_{11} + 4\alpha_2 + 4\alpha_3 + 24\alpha_{10} + 20\alpha_{12} \\ &\quad + 3\alpha_4 + 4\alpha_5 + 11\alpha_6 + 11\alpha_7 + 4\alpha_8 \\ 0 &= 15\alpha_1 + 56\alpha_9 + 12\alpha_2 + 12\alpha_3 + 56\alpha_{10} + 36\alpha_{12} + 15\alpha_4 \\ &\quad + 12\alpha_5 - 9\alpha_6 - 9\alpha_7 + 12\alpha_8 \\ 0 &= 3\alpha_1 + 16\alpha_9 + 3\alpha_2 - 3\alpha_3 - 16\alpha_{10} - 12\alpha_{12} - 3\alpha_4 + 3\alpha_5 \\ &\quad + 3\alpha_6 - 3\alpha_7 - 3\alpha_8,\end{aligned}$$

which are obtained by taking inner products of ϕ^7 with ϕ^2, ϕ^3, and ϕ^5, respectively.

For a general minimally supported generator, (3) is not satisfied, but here, since we are free to choose the functions that will make up $\hat{a}^2$ and $\hat{a}^3$, we can do so judiciously so that $\hat{a}^2$ and $\hat{a}^3$ already satisfy the relevant relations. Thus, from (3) we obtain an additional linear constraint,

$$0 = 41\alpha_1 + 9\alpha_2 + 23\alpha_3 - 8\alpha_{12} - \alpha_4 + 37\alpha_5 - 19\alpha_6 - 5\alpha_7 - 5\alpha_8$$

from $\phi^7 \perp \phi^0_{0,(1,0)}$, plus two nonlinear ones,

$$0 = \langle \phi^7, \phi^7_{0,(1,0)} \rangle \quad \text{and} \quad 0 = \langle \phi^7, \phi^8 \rangle. \tag{4}$$

The other relations are redundant by reason of symmetry. One final requirement on ϕ^7 and ϕ^8 is that they perform the task for which they were intended. That is, $(I - P_{\phi^7})\phi^1 \perp (I - P_{\phi^7})\phi^1_{0,(0,1)}$, which is equivalent to

$$\begin{aligned} 0 &= \langle\phi^7, \phi^7\rangle\langle\phi^1, \phi^1\rangle - \langle\phi^7, \phi^1\rangle\langle\phi^7, \phi^1_{0,(0,1)}\rangle \\ &= \langle\phi^7, \phi^7\rangle\langle\phi^1, \phi^1\rangle - \langle\phi^7, \phi^1\rangle^2. \end{aligned} \tag{5}$$

With seven equations and twelve parameters, the system of equations is underdetermined, and there is much freedom in choosing ϕ^7. By first solving the linear equations for α_2, α_4, α_8, and α_{12}, we find

$$\begin{aligned} \alpha_2 &= -\alpha_1 - \tfrac{16}{3}\alpha_9 - \alpha_5 - \alpha_6 - 4\alpha_{11} \\ \alpha_4 &= -5\alpha_1 + 8\alpha_9 - 4\alpha_5 + 7\alpha_6 + 16\alpha_{11} - 4\alpha_3 - \tfrac{8}{3}\alpha_{10} + 3\alpha_7 \\ \alpha_8 &= 9\alpha_1 - \tfrac{40}{3}\alpha_9 - 4\alpha_{11} + 7\alpha_3 + \tfrac{8}{3}\alpha_{10} + 8\alpha_5 - 7\alpha_6 \\ \alpha_{12} &= -\alpha_1 + \tfrac{4}{3}\alpha_9 - \alpha_5 - 4\alpha_{11} - \alpha_3 - \tfrac{4}{3}\alpha_{10} - \alpha_7. \end{aligned}$$

Substituting these solutions into (4) and (5) yields three homogeneous quadratic equations in eight variables of the form

$$xE_1x^\top = xE_2x^\top = xE_3x^\top = 0, \tag{6}$$

where $x = (\alpha_1, \alpha_3, \alpha_5, \alpha_6, \alpha_7, \alpha_9, \alpha_{10}, \alpha_{11})$, and E_1, E_2, and E_3 are given in the Appendix.

The next step is to find $\phi^9, \phi^{10} \in \hat{V}_1^{[0,1]^2}$ such that ϕ^1 becomes orthogonal to its diagonal shifts. In principle, one function might suffice for this purpose, but projecting such a function out of ϕ^1 would destroy its orthogonality to its horizontal & vertical shifts. Thus, we choose two functions, one for each of the diagonal directions. The two should be orthogonal to each other (or we have the same problem as when taking only one), and moreover they should be chosen to maintain the symmetry of $\hat{V}_0$.

The defining characteristic for ϕ^9 is $(I - P_{\phi^9})\phi^1 \perp (I - P_{\phi^9})\phi^1_{0,(1,1)}$ which is equivalent to

$$\begin{aligned} 0 &= \langle\phi^9, \phi^9\rangle\langle\phi^1, \phi^1_{0,(1,1)}\rangle - \langle\phi^9, \phi^1\rangle\langle\phi^9, \phi^1_{0,(1,1)}\rangle \\ &= -\frac{5}{1792}\langle\phi^9, \phi^9\rangle - \langle\phi^9, \phi^1\rangle\langle\phi^9, \phi^1_{0,(1,1)}\rangle. \end{aligned} \tag{7}$$

Now, if ϕ^9 is symmetrical in the direction of $\pm(1,1)$, then this becomes

$$0 = -\frac{5}{1792}\langle\phi^9, \phi^9\rangle - \langle\phi^9, \phi^1\rangle^2,$$

which has no nontrivial solutions. We therefore choose ϕ^9 antisymmetrical in this direction and symmetrical in the direction of $\pm(1,-1)$. Note that as a consequence, $\phi^9_{0,(-1,0)} \perp \phi^1$. Of course, ϕ^{10} is of the opposite symmetry class, and is therefore orthogonal to ϕ^9 and ϕ^1, while $\phi^{10}_{0,(-1,0)} \not\perp \phi^1$. With this, (7) becomes

$$0 = -\frac{5}{1792}\langle \phi^9, \phi^9 \rangle + \langle \phi^9, \phi^1 \rangle^2, \tag{8}$$

which we will solve to find ϕ^9. Considering its symmetry type, ϕ^9 must be in the 5-dimensional space parametrized by

$$\begin{aligned} \phi^9 = \beta_1(h_{3,(5,1)} + h_{3,(1,5)} - h_{3,(7,3)} - h_{3,(3,7)}) + \beta_2(h_{3,(3,1)} + h_{3,(1,3)} - h_{3,(7,5)} - h_{3,(5,7)}) \\ + \beta_3(h_{3,(1,1)} - h_{3,(7,7)}) + \beta_4(h_{3,(3,3)} - h_{3,(5,5)}) + \beta_5(h_{2,(1,1)} - h_{2,(3,3)}). \end{aligned}$$

Orthogonality to $\hat{A}^1$ places one linear constraint on ϕ^9, which takes the form

$$0 = \tfrac{1}{2}\beta_2 + \tfrac{4}{3}\beta_5 + \tfrac{1}{4}\beta_3 + \tfrac{1}{4}\beta_4.$$

Orthogonality to ϕ^7 and ϕ^8 produces a bilinear constraint. Solving the above for β_5 and substituting into this constraint yields

$$y \begin{pmatrix} 84 & 60 & 60 & -96 & -12 & -208 & 16 & -144 \\ 99 & 75 & 75 & -72 & -21 & -86 & 22 & -84 \\ -78 & -54 & -54 & 84 & 30 & 95 & -31 & 168 \\ -6 & -6 & -6 & -12 & 6 & -1 & 1 & -24 \end{pmatrix} x^\top = 0 \tag{9}$$

where x is as in (6) and $y = (\beta_1, \beta_2, \beta_3, \beta_4)$. Finally, we write (8) in matrix form, which, after substituting for β_5, becomes

$$y \begin{pmatrix} 3328 & 3584 & 5376 & -1792 \\ 3584 & -968 & 3228 & -356 \\ 5376 & 3228 & 2382 & -1074 \\ -1792 & -356 & -1074 & -1202 \end{pmatrix} y^\top = 0. \tag{10}$$

The calculations above give the following result.

Theorem 3. Any real, nontrivial, simultaneous solution of (6), (9), and (10) in terms of x and y leads to an orthogonal generator $\Phi^\square$ and associated orthogonal MRA, $(\tilde{V}_p)$. Furthermore, $\Phi^\square$ is compactly supported, continuous, piecewise affine, and symmetrical.

In addition to the nice properties described in the preceeding theorem, the basis generated by $\Phi^{\Box}_{p,0}$ is easily modified to be a basis for $\tilde{V}_p|_{[0,1]^2}$. Indeed,

Theorem 4. Let $\bar{V}_p = \tilde{V}_p \chi_{[0,1]^2}$ for $p \geq 0$. Then $\Phi^{\Box}_{p,\mathbf{Z}^2}\chi_{[0,1]^2} \setminus \{0\}$ is an orthogonal basis for $\bar{V}_p$.

Proof: For the functions whose supports do not overlap, we clearly have orthogonality. For those whose supports do overlap, the overlap region consists of a union of unit cells scaled by 2^{-p}. However, it follows from the orthogonality and symmetry of $\Phi^{\Box}$ that the restrictions of its components to unit cells are orthogonal. ■

We now exhibit a particular solution to (6), (9), and (10) and the resulting orthogonal scaling functions. The first step is to solve (6) for x and hence ϕ^7 and ϕ^8. The solution given below was computed using `Maple` with thirty digits of accuracy. It was found by taking a 2-dimensional linear subspace satisfying (4) and then selecting from this space an element satisfying (5). The resulting parameter values are

$$x = (32.20722064, 41.98481451, 1, 0, 9.226747630, 0, -185.1269640, -9.056808054),$$

and the corresponding function ϕ^7 (normalized) has interpolation values some of which are given in Table 1. The remaining nonzero interpolation values can be obtained from these by symmetry.

1/2	2.775306897		1.324386068		−.1265347610		−.06326738048		0
3/8		2.633152849		−.5653911456		−.09490107076		−.01766245886	
1/4	1.387653448		−1.262065582		−.06326738048		−.06326738048		0
1/8		−.02373330758		−.04445323288		.01055959032		.4183408416	
0	0		0		0		0		0
	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1

Table 1. Interpolation values for one quadrant of $\phi^7 \in \Phi^{\Box}$, tabulated by x- and y-coordinates, on the bottom and left, respectively.

Having found ϕ^7 and ϕ^8 we now turn our attention to ϕ^9 and ϕ^{10}. Since x has already been set, (9) is now a linear condition on y. We find that

$$y = (1.025037163, -1.509877452, -2.313578430, 0, 1.0)$$

solves (9) and (10) simultaneously. This solution corresponds to the function ϕ^9 whose interpolation values (after normalization) are given in Table 2. Of course, ϕ^{10} is a reflection of ϕ^9 about the line $x = \frac{1}{2}$.

The complete set of orthonormal scaling functions $\Phi^\square$ is shown in Figure 1.

1	0		0		0		0		0
7/8		0		2.508701038		−2.471598790		−4.438596232	
3/4	0		0		0		2.447424472		0
5/8		−2.508701038		0		1.223712236		−2.471598790	
1/2	0		0		0		0		0
3/8		2.471598790		−1.223712236		0		2.508701038	
1/4	0		−2.447424472		0		0		0
1/8		4.438596232		2.471598790		−2.508701038		0	
0	0		0		0		0		0
	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$	1

Table 2. Interpolation values for ϕ^9 (from $\Phi^\square$), tabulated by x- and y-coordinates, on the bottom and left, respectively.

4. Intertwining on a Regular Triangulation.

For the second example, define the lattice $L = u_1\mathbf{Z} + u_2\mathbf{Z}$, where $u_1 = (1, 0)$, and $u_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Then L is the set of vertices of a regular triangulation of the plane with edge length 1. The cells are the L-translates of triangles $t_1 = \operatorname{conv}\{(0,0), u_1, u_2\}$ and $t_2 = \operatorname{conv}\{(0,0), u_1, (\frac{1}{2}, -\frac{\sqrt{3}}{2})\}$.

The notion of minimal support can be defined on this triangulation and is analogous to the case of square cells. In this example we work with generators supported in the hexagonal region

$$H = \operatorname{conv}\{\pm u_1, \pm u_2, \pm(u_1 - u_2)\}.$$

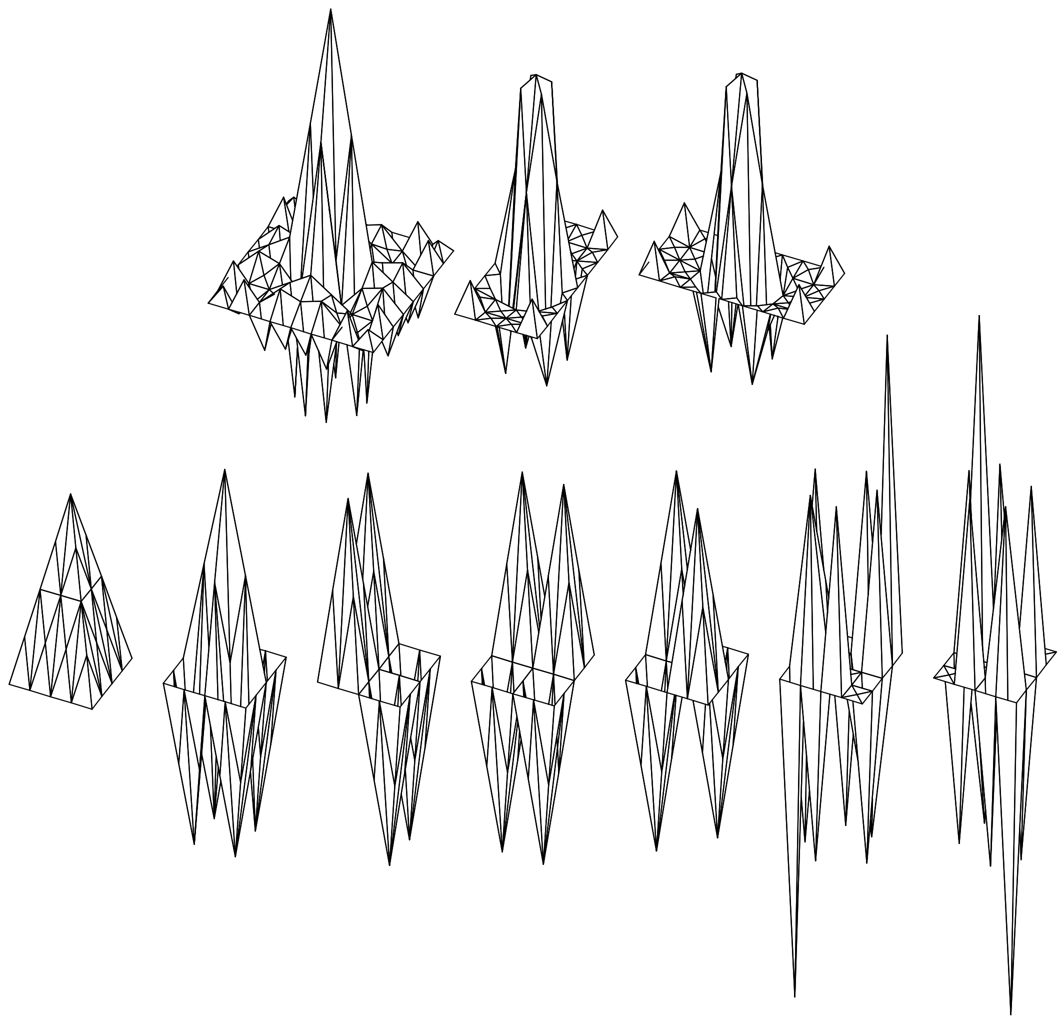

Figure 1. Piecewise linear, orthonormal scaling functions. From top to bottom and left to right, these are ϕ^1, ϕ^7, ϕ^8, ϕ^2, ϕ^3, ϕ^4, ϕ^5, ϕ^6, ϕ^9, and ϕ^{10}.

Thus, we say that a generator Φ is *minimally supported on H* if

1. $\operatorname{supp}\Phi \subset H$
2. For all $\phi \in \Phi$, $\operatorname{supp}\phi$ is not contained in one of the rhombi $(t_1 \cup t_2) - u_1$, $(t_1 - u_1) \cup (t_2 + u_2 - u_1)$, or $(t_1 - u_2) \cup (t_2 - u_1)$.
3. The nonzero functions among $\chi_{t_1}\phi^i_{0,j}$, $i = 1, \ldots, n$, $j \in \{0, -u_1, -u_2\}$, are linearly independent as are the nonzero functions among $\chi_{t_2}\phi^i_{0,j}$, $i = 1, \ldots, n$,

$j \in \{0, -u_1, u_2 - u_1\}$.

Here, we distinguish six support classes for scaling vectors that are minimally supported on H. These are

$$
\begin{aligned}
c^1 &= \{\phi \in \Phi \colon \operatorname{supp}\phi \subseteq t_1\} \\
c^2 &= \{\phi \in \Phi \colon \operatorname{supp}\phi \subseteq t_2\} \\
c^3 &= \{\phi \in \Phi \colon \operatorname{supp}\phi \subseteq t_1 \cup t_2\} \setminus (c^1 \cup c^2) \\
c^4 &= \{\phi \in \Phi \colon \operatorname{supp}\phi \subseteq t_1 \cup (t_2 + u_2 - u_1)\} \setminus c^1 \\
c^5 &= \{\phi \in \Phi \colon \operatorname{supp}\phi \subseteq t_2 \cup (t_1 - u_2)\} \setminus c^2 \\
c^6 &= \Phi \setminus (c^1 \cup c^2 \cup c^3 \cup c^4 \cup c^5).
\end{aligned}
$$

The first two classes, c^1 and c^2, have support on single cells and are analogous to a^1 of the previous example. The next three, c^3, c^4, and c^5, are each supported on two adjacent cells and are analogous to a^2 and a^3. The last one, c^6, has the full support and is analogous to a^4. We also have in this case the following result, which is analogous to Lemma 1.

Lemma 5. Let Φ be minimally supported on H and Ω be a union of triangular cells in $\mathbf{R}^2$. Then

$$V_0^{\Omega} = \operatorname{cl}_{L^2} \operatorname{span}\{\phi_{0,j} \colon \ \phi \in \Phi, \ j \in L, \ \operatorname{supp}\phi_{0,j} \subset \Omega\}.$$

As in the previous example, we use particular support restricted spaces defined in terms of the various support classes. Let

$$
\begin{aligned}
C^1 &= V_0^{t_1} \\
C^2 &= V_0^{t_2} \\
C^3 &= (I - P_{C^1 \cup C^2}) V_0^{t_1 \cup t_2} \\
C^4 &= (I - P_{C^1 \cup C^2_{0,u_2-u_1}}) V_0^{t_1 \cup (t_2 + u_2 - u_1)} \\
C^5 &= (I - P_{C^1_{0,-u_2} \cup C^2}) V_0^{(t_1 - u_2) \cup t_2} \\
C^6 &= (I - P_{C^*}) V_0^H,
\end{aligned}
$$

where $C^* = C^1_{0,\{0,-u_1,-u_2\}} \cup C^2_{0,\{0,-u_1,u_2-u_1\}} \cup C^3_{0,\{0,-u_1\}} \cup C^4_{0,\{0,-u_2\}} \cup C^5_{0,\{0,u_2-u_1\}}$. Given that V_0 is invariant with respect to L-shifts, we can write conditions for the existence of an orthonormal generator minimally supported on H. The proof of the subsequent lemma is similar to the proof given for Lemma 2.

Lemma 6. Suppose V_0 is an FSI space. If V_0 has an orthonormal generator minimally supported on H, then

1. V_0 is the closed span of $C^1, \ldots, C^6$, and their L-translates.
2. $C^3 \perp C^4_{0,\{0,u_1-u_2\}}$
3. $C^3 \perp C^4_{0,\{0,u_2\}}$
4. $C^4 \perp C^5_{0,\{u_2,u_2-u_1\}}$
5. $C^6 \perp C^3_{0,\{-u_2,u_2-u_1\}} \cup C^4_{0,\{-u_1,u_1-u_2\}} \cup C^5_{0,\{-u_1,u_2\}}$
6. $C^6 \perp C^6_{0,\{u_1,u_2,u_2-u_1\}}$.

Conversely, if conditions 1–6 hold, then V_0 has an orthonormal generator Φ, where $\Phi \setminus c^6$ is minimally supported on H.

For Φ minimally supported on H and having hexagonal symmetry, the relations it must satisfy to be an orthogonal scaling vector are

$$\begin{aligned} &c^1 \perp c^3 \cup c^6 \\ &c^3 \perp c^6, \end{aligned} \tag{11}$$

which can always be obtained by projection, and

$$c^3 \perp c^4 \cup c^6_{0,u_2} \tag{12}$$

$$c^6 \perp c^6_{0,u_1}, \tag{13}$$

which do not generally hold.

The initial multiresolution analysis is generated by a single scaling function $\phi^1 = h$, where h is the hexagonal "hat" function, linearly interpolating on the triangular cells with the value 1 at $(0,0)$ and 0 at all other points in L. The symmetries possessed by this MRA are L-translations, reflections about lines inclined at 0°, 30°, 60°, 90°, 120°, and 150° and rotations by 60°. As in the previous example, we will produce an intertwining MRA that has the same symmetries as the original.

Note that though L is different from $\mathbf{Z}^2$, we can apply a linear transformation T to "rectify" L, where $TL = \mathbf{Z}^2$. The resulting triangulation consists of the $\mathbf{Z}^2$-translates of $Tt_1 = \text{conv}\{(0,0),(1,0),(0,1)\}$ and $T(t_2+u_2) = \text{conv}\{(1,0),(0,1),(1,1)\}$. Together, these two triangles form $[0,1]^2$, which is the unit cell for the lattice $\mathbf{Z}^2$. An orthogonal MRA can be obtained by proceeding as in Section 3. However, unless one takes great care, the symmetry of the original MRA will most likely be destroyed in the process. We do not use the rectifying transformation here so that the hexagonal symmetry of this example may be preserved.

Initially, the only nonempty support class is c^6. However, as in the previous example, V^1 is not rich enough to allow for an orthogonal intertwining multiresolution analysis with $V_0 \subset \tilde{V}_0 \subset V_1$. We must therefore add a few more functions before getting started. In fact, to have enough freedom to find an orthogonal basis, we will "borrow" all of V_1 as well as some functions from V_2. From V_1, we obtain three functions,

$$\phi^2 = \phi^1_{1,u_1} \in c^3$$
$$\phi^3 = \phi^1_{1,u_2} \in c^4$$
$$\phi^4 = \phi^1_{1,u_1-u_2} \in c^5,$$

and we observe that the L-translates of $\{\phi^1, \dots, \phi^4\}$ span V_1. Then, from V_2, we take

$$\phi^5 = \phi^1_{2,u_1+u_2}$$
$$\phi^6 = \phi^1_{2,u_1+2u_2}$$
$$\phi^7 = \phi^1_{2,2u_1+u_2},$$

which constitute c^1, as well as

$$\phi^8 = \phi^1_{2,2u_1-u_2}$$
$$\phi^9 = \phi^1_{2,3u_1-u_2}$$
$$\phi^{10} = \phi^1_{2,3u_1-2u_2},$$

which constitute c^2. Before continuing, we modify $\phi^1, \dots, \phi^4$ so that (11) is satisfied. The resulting scaling vector is denoted $\hat{\Phi}$ and generates the MRA $(\hat{V}_p)$. As we will see, it suffices to borrow from $\hat{V}_1$ to obtain an orthogonal scaling vector, which will be denoted $\Phi^{\triangle}$. Thus the intertwining relationship between the orthogonal MRA, $(\tilde{V}_p)$ and the original is $V_1 \subset \tilde{V}_0 \subset V_3$.

As in the previous example, the construction is done in two parts. First, we choose a function ϕ^{11} supported on t_1 and its mirror-image ϕ^{12} supported on t_2, such that borrowing these two functions results in a generator satisfying (12). The next step is to find ϕ^{13} supported on $t_1 \cup t_2$, as well as ϕ^{14} and ϕ^{15} supported on $t_1 \cup (t_2 + u_2 - u_1)$ and $t_2 \cup (t_1 - u_2)$, respectively, such that borrowing these gives $\Phi^{\triangle} = (\phi^1, \ldots, \phi^{15})^\top$ satisfying (13).

The space from which ϕ^{11} is chosen is $\hat{V}_1^{t_1}$, which is a 15-dimensional space. However, we would like ϕ^{11} to be symmetrical with respect to reflections about lines (through the center of t_1) inclined at 30°, 90°, and 150° as well as rotation (about the center of t_1) by 120°. This restricts ϕ^{11} to a 4-dimensional subspace, which we parametrize as

$$\begin{aligned}\phi^{11} &= \alpha_1(h_{3,u_1+u_2} + h_{3,11u_1+u_2} + h_{3,u_1+11u_2}) \\ &\quad + \alpha_2(h_{3,3u_1+u_2} + h_{3,u_1+3u_2} + h_{3,9u_1+u_2} + h_{3,9u_1+3u_2} + h_{3,u_1+5u_2} + h_{3,3u_1+9u_2}) \\ &\quad + \alpha_3(h_{3,5u_1+3u_2} + h_{3,3u_1+5u_2} + h_{3,5u_1+5u_2}) + \alpha_4(\phi^2_{0,u_2} + \phi^3_{0,u_1} + \phi^4_{0,2u_2}).\end{aligned} \tag{14}$$

For ϕ^{11} and ϕ^{12} to have the desired effect, ϕ^{11} must satisfy two equations,

$$\begin{aligned}0 &= \langle\phi^{11},\phi^{11}\rangle\langle\phi^2,\phi^1_{0,u_2}\rangle - \langle\phi^{11},\phi^2\rangle\langle\phi^{11},\phi^1_{0,u_2}\rangle \\ 0 &= \langle\phi^{11},\phi^{11}\rangle\langle\phi^2,\phi^3\rangle - \langle\phi^{11},\phi^2\rangle\langle\phi^{11},\phi^3\rangle.\end{aligned} \tag{15}$$

An equivalent system which is more convenient to solve is obtained by taking linear combinations of the above,

$$\begin{aligned}0 &= \langle\phi^{11}, v\rangle \\ 0 &= \langle\phi^{11},\phi^{11}\rangle\langle\phi^2,w\rangle - \langle\phi^{11},\phi^2\rangle\langle\phi^{11},w\rangle,\end{aligned} \tag{16}$$

where v is a linear combination of ϕ^1_{0,u_2} and ϕ^3 chosen orthogonal to ϕ^2 and w is any other such combination. In this case we find $v = \phi^1_{0,u_2} - \frac{91942}{37249}\phi^3$, and $w = \phi^3$. With (14), the first of these becomes

$$0 = -40\alpha_1 - 40\alpha_2 + 40\alpha_3 - 29\alpha_4.$$

In addition, ϕ^{11} must be orthogonal to c^1, and hence

$$0 = 10\alpha_1 + 100\alpha_2 + 110\alpha_3 + 97\alpha_4.$$

Solving the above two linear equations for α_3 and α_4 and substituting into the bottom equation in (16) yields

$$0 = -124829\alpha_1^2 - 1266566\alpha_1\alpha_2 + 3757859\alpha_2^2. \tag{17}$$

Next, we choose $\phi^{13} \in V_1^{t_1 \cup t_2}$ satisfying

$$0 = \langle \phi^{13}, \phi^{13} \rangle \langle \phi^1, \phi^1_{0,u_1} \rangle - \langle \phi^{13}, \phi^1 \rangle \langle \phi^{13}, \phi^1_{0,u_1} \rangle. \tag{18}$$

Here, as in the previous example, we observe that since $\langle \phi^1, \phi^1_{0,u_1} \rangle < 0$, Equation 18 has no nontrivial solutions for symmetrical ϕ^{13}. Thus, we take ϕ^{13} symmetrical about the x-axis and antisymmetrical about the line $x = \frac{1}{2}$, which restricts it to a 7-dimensional subspace. To maintain the symmetry of V_p, ϕ^{14} and ϕ^{15} must be rotations of ϕ^{13}, and it behooves us to make these functions orthogonal to each other. Thus, in addition to (18), we have the condition,

$$0 = \langle \phi^{13}, \phi^{14} \rangle, \tag{19}$$

which is nonlinear because ϕ^{14} is determined by ϕ^{13}. Finally, we have two linear orthogonality constraints on ϕ^{13}, those being $\phi^{13} \perp c^1 \cup c^4$. The other orthogonalities are obtained from these two by symmetry. Let

$$\begin{aligned}\phi^{13} = \beta_1 & (h_{3,u_1+u_2} - h_{3,6u_1+u_2} + h_{3,2u_1-u_2} - h_{3,7u_1-u_2}) \\ & + \beta_2 (h_{3,2u_1+u_2} - h_{3,5u_1+u_2} + h_{3,3u_1-u_2} - h_{3,6u_1-u_2}) \\ & + \beta_3 (h_{3,u_1+2u_2} - h_{3,5u_1+2u_2} + h_{3,3u_1-2u_2} - h_{3,7u_1-2u_2}) \\ & + \beta_4 (h_{3,2u_1+3u_2} - h_{3,3u_1+3u_2} + h_{3,5u_1-3u_2} - h_{3,6u_1-3u_2}) \\ & + \beta_5 (h_{3,u_1+5u_2} - h_{3,2u_1+5u_2} + h_{3,6u_1-5u_2} - h_{3,7u_1-5u_2}) \\ & + \beta_6 (\phi^4_{1,u_2} - \phi^3_{1,u_1} + \phi^3_{1,u_1-u_2} - \phi^4_{1,2u_1-u_2}) + \beta_7 (\phi^2_{1,0} - \phi^2_{1,u_1}).\end{aligned} \tag{20}$$

Using the orthogonalities to eliminate β_1 and β_2, (18) and (19) take the form

$$x \begin{pmatrix} -4824000 & 878400 & 3240000 & 2045880 & -5661000 \\ 878400 & -5169600 & 4334400 & -1320360 & -3082920 \\ 3240000 & 4334400 & -2520000 & 10330680 & -5670600 \\ 2045880 & -1320360 & 10330680 & -11977535 & -7514399 \\ -5661000 & -3082920 & -5670600 & -7514399 & -1641215 \end{pmatrix} x^\top = 0 \tag{21}$$

and

$$x \begin{pmatrix} 100800 & -91200 & 1444800 & -198520 & 20360 \\ -91200 & 484800 & 484800 & 247880 & 5960 \\ 1444800 & 484800 & 484800 & 1182280 & 18760 \\ -198520 & 247880 & 1182280 & 1348983 & 16791 \\ 20360 & 5960 & 18760 & 16791 & 2487 \end{pmatrix} x^\top = 0, \tag{22}$$

where $x = (\beta_3, \beta_4, \beta_5, \beta_6, \beta_7)$. The derivation in this section implies

Theorem 7. Any real, nontrivial, simultaneous solution of (17), (21), and (22) in terms of α_1, α_2, and x leads to an orthogonal generator $\Phi^{\triangle}$ and associated orthogonal MRA, $(\tilde{V}_p)$. Furthermore, $\Phi^{\triangle}$ is compactly supported, continuous, piecewise affine, and symmetrical.

Like $\Phi^{\square}$, the basis generated by $\Phi^{\triangle}_{p,0}$ is easily modified to be a basis on a compact domain, in particular for $\tilde{V}_p|_{t_1}$. Indeed,

Theorem 8. Let $\bar{V}_p = \tilde{V}_p \chi_{t_1}$ for $p \geq 0$. Then $\Phi^{\triangle}_{p,L}\chi_{t_1} \setminus \{0\}$ is an orthogonal basis for $\bar{V}_p$.

Proof: For the functions whose supports do not overlap, we clearly have orthogonality. For those whose supports do overlap, the overlap region consists of a union of triangular cells scaled by 2^{-p}. However, it follows from the orthogonality and symmetry of $\Phi^{\triangle}$ that the restrictions of its components to triangular cells are orthogonal. ■

$\sqrt{3}/2$									0
$7\sqrt{3}/16$								0	
$3\sqrt{3}/8$							0		−2.173126807
$5\sqrt{3}/16$						0		−1.622602234	
$\sqrt{3}/4$					0		1.398887642		2.797775284
$3\sqrt{3}/16$				0		1.398887642		−2.396841399	
$\sqrt{3}/8$			0		−1.622602234		2.797775284		−2.396841399
$\sqrt{3}/16$		0		−2.173126807		−1.622602234		1.398887642	
0	0		0		0		0		0
	0	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{7}{16}$	$\frac{1}{2}$

Table 3. Interpolation values for one half of $\phi^{11} \in \Phi^{\triangle}$, tabulated by x- and y-coordinates, on the bottom and left, respectively. By symmetry, the values on the other half are the same.

We finish this section by exhibiting a solution to (17), (21), and (22). The first step in obtaining this solution is to solve (17) for α_2, taking the positive branch of the square root. This fixes ϕ^{11} (up to a scaling), with parameter values $\alpha_1 = 1$, $\alpha_2 = .4167517772$, $\alpha_3 = .5655469225$, $\alpha_4 = -1.174075662$. The interpolation values for the corresponding

$\sqrt{3}/2$									0
$7\sqrt{3}/16$								0	
$3\sqrt{3}/8$							0		0
$5\sqrt{3}/16$						0		.2683485316	
$\sqrt{3}/4$					0		0		0
$3\sqrt{3}/16$				0		−.5791276685		−.1064152147	
$\sqrt{3}/8$			0		−.7180923863		−1.158255337		0
$\sqrt{3}/16$		0		1.483315351		3.317382568		−.5791276685	
0	0		.5016069904		1.003213981		.5016069904		0
	0	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{3}{8}$	$\frac{7}{16}$	$\frac{1}{2}$

Table 4. Interpolation values for one quadrant of $\phi^{13} \in \Phi^{\triangle}$, tabulated by x- and y-coordinates, on the bottom and left, respectively. The other nonzero values can be found using the function's symmetry.

function (normalized) are given in Table 3. The final step is to solve (21) and (22), which is underdetermined. The solution given below was found using a program for Newton's method, written in `Maple` and executed with 30 digits of accuracy. The parameter values are $\beta_1 = -1.703992768$, $\beta_2 = -3.114499406$, $\beta_3 = 1$, $\beta_4 = .4306615503$, $\beta_5 = -.2546144050$, $\beta_6 = 1.098975619$, and $\beta_7 = -.9518693074$, and the interpolation values for the corresponding function (normalized) are given in Table 4. Selected functions from $\Phi^{\triangle}$ are shown in Figure 2.

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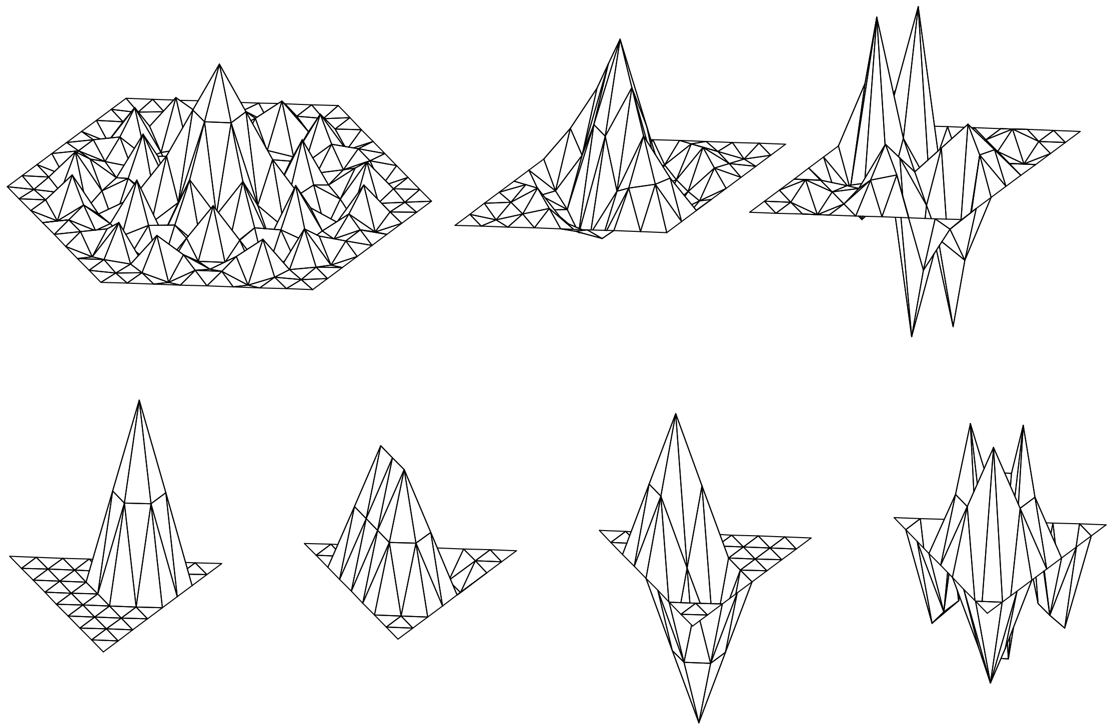

Figure 2. Seven functions from the orthonormal scaling vector $\Phi^{\triangle}$. Depicted are ϕ^1, ϕ^2, ϕ^{13}, ϕ^5, ϕ^6, ϕ^7, and ϕ^{11}.

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Appendix

Hessian Matrices for Nonlinear Equations

$$E_1 = \begin{pmatrix} -246 & -123 & 93 & 165 & 69 & 244 & -68 & 408 \\ -123 & 0 & 120 & -27 & 0 & -80 & 0 & -60 \\ 93 & 120 & 336 & -171 & -24 & -240 & 32 & -156 \\ 165 & -27 & -171 & 0 & 21 & 4 & -4 & -12 \\ 69 & 0 & -24 & 21 & 0 & 48 & 0 & 36 \\ 244 & -80 & -240 & 4 & 48 & -128 & -32 & 72 \\ -68 & 0 & 32 & -4 & 0 & -32 & 0 & -24 \\ 408 & -60 & -156 & -12 & 36 & 72 & -24 & 96 \end{pmatrix}$$

$$E_2 = \begin{pmatrix} 0 & 162 & 0 & -270 & -108 & -549 & 99 & -801 \\ 162 & 252 & 162 & -378 & -126 & -669 & 123 & -873 \\ 0 & 162 & 0 & -270 & -108 & -549 & 99 & -801 \\ -270 & -378 & -270 & 576 & 198 & 960 & -192 & 1296 \\ -108 & -126 & -108 & 198 & 72 & 291 & -69 & 423 \\ -549 & -669 & -549 & 960 & 291 & 1652 & -292 & 1986 \\ 99 & 123 & 99 & -192 & -69 & -292 & 68 & -438 \\ -801 & -873 & -801 & 1296 & 423 & 1986 & -438 & 2664 \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 245754 & 184149 & 208125 & -242091 & -60939 & -352980 & 72372 & -367632 \\ 184149 & 143856 & 157176 & -188811 & -47952 & -281496 & 57720 & -294372 \\ 208125 & 157176 & 181152 & -204795 & -50616 & -299256 & 61272 & -305028 \\ -242091 & -188811 & -204795 & 266400 & 72261 & 390276 & -77700 & 448884 \\ -60939 & -47952 & -50616 & 72261 & 26640 & 95016 & -20424 & 142524 \\ -352980 & -281496 & -299256 & 390276 & 95016 & 626336 & -111296 & 636696 \\ 72372 & 57720 & 61272 & -77700 & -20424 & -111296 & 29600 & -126984 \\ -367632 & -294372 & -305028 & 448884 & 142524 & 636696 & -126984 & 863136 \end{pmatrix}$$