Intertwining multiresolution analyses and the construction of piecewise polynomial wavelets

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Abstract

Let (V_p) be a local multiresolution analysis (MRA) of $L^2(\mathbf{R})$ of multiplicity $r \geq 1$, i.e., V_0 is generated by r compactly supported scaling functions. If the scaling functions generate an orthogonal basis of V_0 then (V_p) is called an *orthogonal MRA*. We prove that there exists an orthogonal local MRA (V_p') of multiplicity r' such that

$$V_q \subset V_0' \subset V_{q+n}$$

for some integers $q \geq 0$, $n \geq 1$ and $r' > 1$.

In particular, this shows that compactly supported orthogonal polynomial spline wavelets and scaling functions (of mulitplicity $r' > 1$) of arbitrary regularity exist and we give several such examples.

1 Introduction

The starting point for most wavelet constructions is a single function $\phi \in L^2(\mathbf{R})$ called a *scaling function* whose integer translates form a Riesz basis for a closed linear subspace $V_0 \subset L^2(\mathbf{R})$. If the scaling function is compactly supported and generates an orthogonal basis of V_0, then the associated wavelet will also be compactly supported and generate an orthogonal basis. Daubechies (cf. [5]) constructed scaling functions and associated wavelets that were compactly supported, generated orthogonal bases, and were continuous (or smoother). However, these wavelets do not have closed-form representations; they are defined via a limiting process. Also, it is known that these wavelets cannot have certain other desirable properties (e.g. symmetry). By giving up compact support (cf. [1]) or orthogonality (cf. [4]), symmetric wavelet bases have been constructed using piecewise polynomial splines. For some applications the symmetry and simple representation are more important than having both compact support and orthogonality.

Recently, wavelet constructions generated by a finite collection of scaling functions $\Phi = \{\phi^1, \ldots, \phi^r\}$ have been studied (cf. [10], [11], [8], [12], [13], [18], [22]). In

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[8] and [7] symmetric, compactly supported, continuous, and orthogonal scaling functions (and associated wavelets) were constructed using $r = 2$ scaling functions. As in the case of Daubechies wavelets, these functions do not have closed form representations. In this paper we construct, using $r > 2$ scaling functions, wavelets that not only have the three properties of compact support, arbitrary regularity, and orthogonality but are also symmetric and piecewise polynomial. In fact, we show that for any multiresolution analysis generated by compactly supported scaling functions there is an associated intertwining orthogonal multiresolution analysis also generated by compactly supported scaling functions.

More precisely, a *multiresolution analysis of multiplicity* r is a nested sequence of closed linear subspaces (V_p) in $L^2(\mathbf{R})$ satisfying the following:

1. $f \in V_p$ iff $f(2^{-p}\cdot) \in V_0$ for $p \in \mathbf{Z}$
2. $V_0 \subset V_1$
3. $\bigcap_{p\in\mathbf{Z}} V_p = \{0\}$
4. $\bigcup_{p\in\mathbf{Z}} V_p$ is dense in $L^2(\mathbf{R})$
5. There are r functions $\phi^1, \ldots, \phi^r$ such that the collection of integer translates $\{\phi^s(\cdot - n) \mid s = 1, \ldots, r \text{ and } n \in \mathbf{Z}\}$ is a Riesz basis of V_0.

The functions $\phi^1, \ldots, \phi^r$ are called scaling functions and are said to *generate* the multiresolution analysis (V_p). If there is a set of compactly supported scaling functions whose integer translates form an orthogonal basis of V_0 then we call (V_p) an *orthogonal* multiresolution analysis. The main theorem of this paper is

Theorem 1 *If (V_p) is a multiresolution analysis generated by compactly supported scaling functions then there is some pair of integers (q, n) and some orthogonal multiresolution analysis $(\tilde{V}_p)$ such that*

$$V_q \subset \tilde{V}_0 \subset V_{q+n}.$$

We say that (V_p) and $(\tilde{V}_p)$ are *intertwining multiresolution analyses.*

For the sake of completeness, we note that if the scaling functions are compactly supported then the intersection property (3) follows from (1) and the fact that V_0 is an FSI space (see section 2) [15, Theorem 2.2]. Furthermore the density property (4) follows from conditions (1) and (2) (see [3] for the single scaling function case which directly generalizes to the multiple scaling function case). The Riesz basis condition (5) may be relaxed: If V_0 is an FSI space with compactly supported generators then there exists a set of compactly supported generators whose collection of integer translates form a Riesz basis (cf. [2, Theorem 3.38] and [20, Theorem 3.7]). In practice we will take V_0 to be a classical spline space in which case it will be elementary to verify conditions (1)-(5) directly.

If (V_p) is an orthogonal multiresolution analysis then there is a general procedure (cf. [7], [22], [16], [23]) for calculating compactly supported wavelets $\psi^1, \ldots, \psi^r$ that generate an orthogonal basis of $W_0 = V_1 \ominus V_0$. As in the single scaling function case, the work is in finding orthogonal scaling functions.

The structure of this paper is this: In section 2 we show that, without loss of generality, we can assume that the scaling functions are supported in $[-1,1]$. The spaces V_p for $p \geq 0$ are examples of finitely generated shift-invariant (FSI) spaces and we develop a necessary and sufficient condition for an FSI space with generators supported in $[-1,1]$ to be orthogonal. In section 3 we give a construction that, under certain conditions, gives an orthogonal intertwining multiresolution analysis. We also provide an example based on continous piecewise linear splines.

In section 4 we first prove that if certain orthogonal projections are uniformly bounded below then the construction of section 3 works and Theorem 1 holds with $n = 1$. We then use this special case to prove Theorem 1. In section 5 we apply the general theory to the piecewise polynomial spline spaces $\mathcal{S}_{d,r}$ and give an example of orthogonal C^1 cubic spline scaling functions and wavelets.

2 Orthogonal FSI spaces

If Φ is a subset of $L^2(\mathbf{R})$, let $\tau(\Phi) = \{\phi(\cdot - n) \mid n \in \mathbf{Z}, \phi \in \Phi\}$ denote the set of integer translates of elements in Φ and let $\sigma(\Phi)$ denote the L^2-closure of the linear span of $\tau(\Phi)$. Following [2] we call a space $V \subset L^2(\mathbf{R})$ a *finitely generated shift-invariant space* (FSI) if $V = \sigma(\Phi)$ for some finite set Φ.

Suppose that (V_p) is a multiresolution analysis generated by r scaling functions. Then V_p, for $p \geq 0$, is an FSI space with $2^p r$ generators. In particular, if the scaling functions $\{\phi^1, \ldots, \phi^r\}$ are all supported in $[-L, 1]$, then V_1 is generated by the set of $2r$ scaling functions $\{\phi^s(2 \cdot -n), \mid s = 1, \ldots, r;\ n = 0, 1\}$ all supported in $[-L/2, 1]$. Hence, there is some q such that the multiresolution analysis (V'_p) defined by $V'_p = V_{p+q}$ is generated by $2^q r$ scaling functions all supported in $[-1,1]$.

Thus, we will be concerned with FSI spaces whose generators $\Phi = \{\phi^1, \ldots, \phi^r\}$ are supported in $[-1,1]$.

Remark 2.1. Certain linear dependencies can be removed as follows: If the restrictions $\phi^s|_{[-1,0]}$ and $\phi^{s'}|_{[-1,0]}$ are linearly dependent on $[-1,0]$ then, by taking linear combinations, we can replace one of the generators with a function supported on $[0,1]$. By making such replacements we can assume there are k

generators $\phi^1, \ldots, \phi^k$ supported in $[-1,1]$ such that

1. they are linearly independent on $[-1,0]$,
2. they are linearly independent on $[0,1]$,
3. the rest of the generators $\phi^{k+1}, \ldots, \phi^r$ are supported in $[0,1]$.

Let $H_h(\Phi) = \mathrm{span}\{\phi^s(\cdot - h)\chi_{[0,1]} \mid s = 1, \ldots, k\}$ for $h = 0, 1$. If, in addition to satisfying conditions 1–3 of the above remark, we have $H_0(\Phi) \cap H_1(\Phi) = \{0\}$ then

$$\phi^1, \ldots, \phi^k, \phi^1(\cdot - 1), \ldots, \phi^k(\cdot - 1), \phi^{k+1}, \ldots, \phi^r$$

are linearly independent on $[0,1]$ and we say $\phi^1, \ldots, \phi^r$ are *minimally supported on* $[-1,1]$.

Remark 2.2. In the following $k = k(\Phi)$ will always represent the number of scaling functions supported on $[-1,1]$ but not supported on $[-1,0]$ or $[0,1]$.

The following lemma, whose proof is delayed until section 4, states that any multiresolution analysis generated by compactly supported scaling functions can be converted (as in the above remark) so that it is generated by scaling functions that are minimally supported on $[-1,1]$.

Lemma 2.1 *Suppose (V_p) is a multiresolution analysis generated by compactly supported scaling functions. Then there is some n and some set of scaling functions minimally supported on $[-1,1]$ that generate the multiresolution analysis (V'_p) given by*

$$V'_p = V_{p+n}.$$

Let V be a finite-shift invariant space generated by a collection Φ of r scaling functions minimally supported on $[-1,1]$. We say that V is *orthogonal with respect to* $[-1,1]$ if there is some Φ' supported on $[-1,1]$ such that $\tau(\Phi')$ is an orthogonal basis for V' (which is some dilate of V). We will give a necessary and sufficient condition that V be orthogonal with respect to $[-1,1]$. Towards this end, let $\mathcal{A}(V), \mathcal{B}_0(V), \mathcal{B}_1(V) \subset L^2(\mathbf{R})$ be the subspaces defined by

$$\mathcal{A}(V) = \mathrm{span}\{\phi^s \mid s = k+1, \ldots, r\}$$

and

$$\mathcal{B}_h(V) = \mathrm{span}\left(\{\phi^s(\cdot - h)\chi_{[0,1]} \mid s = 1, \ldots, k\} \cup \mathcal{A}(V)\right) \text{ for } h = 0, 1$$

where $\chi_{[0,1]}$ denotes the characteristic function of $[0,1]$. Finally, define $\mathcal{C}_h(V)$ by $\mathcal{C}_h(V) = \mathcal{B}_h(V) \ominus \mathcal{A}(V)$, the orthogonal complement of $\mathcal{A}(V)$ in $\mathcal{B}_h(V)$.

Remark 2.3.

1. From Remark 2.1 (4) it follows that the spaces $\mathcal{A}(V), \mathcal{B}_0(V)$, and $\mathcal{B}_1(V)$ are independent of the choice of minimally supported generators. For instance, we have

$$\mathcal{A}(V) = \{f \in V \mid \operatorname{supp} f \subset [0,1]\}$$
$$\mathcal{B}_0(V) = \{f\chi_{[0,1]} \mid f\chi_{(-\infty,1]} \in V\}$$

and similarly for $\mathcal{B}_1(V)$.

2. $\mathcal{C}_0(V)$ and $\mathcal{C}_1(V)$ are k-dimensional.

Lemma 2.2 *Suppose that V is a finite shift invariant space generated by r functions $\phi^1, \ldots, \phi^r$ minimally supported on $[-1,1]$. Then V is orthogonal with respect to $[-1,1]$ if and only if $\mathcal{C}_0(V) \perp \mathcal{C}_1(V)$.*

Proof. If V is orthogonal, then it is easy to see directly that $\mathcal{C}_0(V) \perp \mathcal{C}_1(V)$.

Suppose $\mathcal{C}_0(V) \perp \mathcal{C}_1(V)$. By using the Gram-Schmidt procedure, if necessary, we can assume that

- $\phi^s(\cdot - h)$ is orthogonal to $\mathcal{A}(V)$ for $s = 1, \ldots, k$ and $h = 0, 1$.
- $\phi^s \perp \phi^{s'}$ for $s \neq s'$ and $s, s' = 1, \ldots, r$.

Since $\mathcal{C}_0(V) \perp \mathcal{C}_1(V)$, it follows that $\phi^s \perp \phi^{s'}(\cdot - 1)$ for $s, s' = 1, \ldots, k$ and hence $\{\phi^s(\cdot - n) \mid n \in \mathbf{Z}, s = 1, \ldots, r\}$ is an orthogonal basis for V. □

3 Constructing Orthogonal Multiresolution Analyses

The following lemma provides the basic idea we will use to construct orthogonal multiresolution analyses.

Lemma 3.1 *Let (V_p) be a multiresolution analysis generated by r functions $\phi^1, \ldots, \phi^r$ minimally supported on $[-1,1]$. Suppose there is a subspace $\mathcal{W}$ of $\mathcal{A}(V_1) \ominus \mathcal{A}(V_0)$ such that*

$$(I - P_{\mathcal{W}})\mathcal{C}_0(V_0) \perp (I - P_{\mathcal{W}})\mathcal{C}_1(V_0) \tag{1}$$

where $P_{\mathcal{W}}$ is the orthogonal projection onto $\mathcal{W}$. Let $w_1, \ldots, w_\kappa$ be a basis for $\mathcal{W}$ and let $(\tilde{V}_p)$ be the multiresolution analysis generated by $\phi^1, \ldots, \phi^r, w_1, \ldots, w_\kappa$. Then $(\tilde{V}_p)$ is an orthogonal multiresolution analysis and

$$V_0 \subset \tilde{V}_0 \subset V_1.$$

Proof. Since $\mathcal{W} \subset V_1$, we clearly have $V_0 \subset \tilde{V}_0 \subset V_1$. This implies $V_p \subset \tilde{V}_p \subset V_{p+1}$ for all integers p. It then follows that $(\tilde{V}_p)$ satisfies the first four conditions of a multiresolution analysis.

Observe that $\mathcal{C}_h(\tilde{V}_0) = (I - P_{\mathcal{W}})\mathcal{C}_h(V_0)$ for $h = 0, 1$ and so the result follows from Lemma 2.2. $\square$

Remark 3.1. Let P_h denote the orthogonal projection onto $\mathcal{C}_h(V_0)$ for $h = 0, 1$. The condition (1) can be rewritten as

$$P_0 P_1 = P_0 P_{\mathcal{W}} P_1. \tag{2}$$

Therefore the rank of $P_{\mathcal{W}}$ must be at least the rank of $P_0 P_1$ which is less than or equal to k. This gives a lower bound on κ, the dimension of $\mathcal{W}$ (which equals the rank of $P_{\mathcal{W}}$). In all of our examples $P_0 P_1$ has rank k and, it turns out, we can choose κ to be the lower bound k.

We will use the following notation in the rest of this paper:

$$\phi_{l,k}(x) = 2^{l/2}\phi(2^l x - k) \text{ for any } \phi \in L^2(\mathbf{R}). \tag{3}$$

3.1 Example: Piecewise linear orthogonal scaling functions

Let $H : \mathbf{R} \to \mathbf{R}$ be the "hat" function defined by

$$H(x) = \begin{cases} 1 - |x| & \text{if } |x| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\phi^1(x) = \sqrt{3}H(2x)$ and $\phi^2(x) = \sqrt{3}H(2x - 1)$ (note ϕ^1 and ϕ^2 have norm 1) and let (V_p) be the multiresolution analysis generated by $\Phi = \{\phi^1, \phi^2\}$. In this case, V_{-1} is the space of continuous piecewise linear splines with integer knots. Note that Φ is minimally supported on $[-1, 1]$ with $k = k(\Phi) = 1$. Also, $\mathcal{A}(V_0)$, $\mathcal{B}_0(V_0)$ and $\mathcal{B}_1(V_0)$ are spanned by $\{\phi^2_{0,0}\}$, $\{\phi^1_{0,0}\chi_{[0,1]}, \phi^2_{0,0}\}$ and $\{\phi^1_{0,1}\chi_{[0,1]}, \phi^2_{0,0}\}$, respectively. Using

$$c = \langle \phi^1_{0,0}, \phi^2_{0,0} \rangle = 1/4$$

gives the bases $\{\phi^1_{0,0}\chi_{[0,1]} - c\phi^2_{0,0}\}$ and $\{\phi^1_{0,1}\chi_{[0,1]} - c\phi^2_{0,0}\}$ for $\mathcal{C}_0(V_0)$ and $\mathcal{C}_1(V_0)$, respectively.

Note that $\{\phi^1_{1,1}, \phi^2_{1,0}, \phi^2_{1,1}\}$ is a basis for $\mathcal{A}(V_1)$. Direct calculation yields the basis $\{3\phi^1_{1,1} - 5\phi^2_{1,0}, 3\phi^1_{1,1} - 5\phi^2_{1,1}\}$ for $\mathcal{A}(V_1) \ominus \mathcal{A}(V_0)$. Let w be an arbitrary vector in $\mathcal{A}(V_1) \ominus \mathcal{A}(V_0)$:

$$w = a(3\phi^1_{1,1} - 5\phi^2_{1,0}) + b(3\phi^1_{1,1} - 5\phi^2_{1,1}),$$

where we will choose w so that Equation 1 is satisfied when $\mathcal{W}$ is the one dimensional subspace spanned by w. Then

$$\|w\|^2 = (53a^2 + 6ab + 53b^2)/2$$

and Equation 1 holds iff

$$\langle \phi^1_{0,0}\chi_{[0,1]} - c\phi^2_{0,0}, \phi^1_{0,1}\chi_{[0,1]} - c\phi^2_{0,0}\rangle = \langle \phi^1_{0,0}\chi_{[0,1]} - c\phi^2_{0,0}, w\rangle \langle w, \phi^1_{0,1}\chi_{[0,1]} - c\phi^2_{0,0}\rangle / \|w\|^2. \tag{4}$$

The left hand side is easily evaluated to get

$$\langle \phi^1_{0,0}\chi_{[0,1]} - c\phi^2_{0,0}, \phi^1_{0,1}\chi_{[0,1]} - c\phi^2_{0,0}\rangle = -c^2 = -1/16.$$

Using $H(x) = 1/2H(2x+1) + H(2x) + 1/2H(2x-1)$ we find

$$\phi^1_{0,0} = (1/2\sqrt{2})\phi^2_{1,-1} + (1/\sqrt{2})\phi^1_{1,0} + (1/2\sqrt{2})\phi^2_{1,0}$$

$$\phi^2_{0,0} = (1/2\sqrt{2})\phi^2_{1,0} + (1/\sqrt{2})\phi^1_{1,1} + (1/2\sqrt{2})\phi^2_{1,1}$$

which can be used to calculate

$$\langle \phi^1_{0,0}\chi_{[0,1]} - c\phi^2_{0,0}, w\rangle = (3/8\sqrt{2})(b - 9a)$$

and

$$\langle w, \phi^1_{0,1}\chi_{[0,1]} - c\phi^2_{0,0}\rangle = (3/8\sqrt{2})(a - 9b).$$

Thus, Equation 4 becomes

$$-4(53a^2 + 6ab + 53b^2) = 9(a - 9b)(-9a + b),$$

which has solutions

$$b/a = (-762 \pm 320\sqrt{5})/262.$$

We choose $b/a = (-762 + 320\sqrt{5})/262$. Set $\tilde{\phi}^2 = \phi^2$, $\tilde{\phi}^3 = \frac{w}{\|w\|}$,

$$y = \phi^1 - \langle \phi^1, \tilde{\phi}^2\rangle\tilde{\phi}^2 - \langle \phi^1, \tilde{\phi}^3\rangle\tilde{\phi}^3 - \langle \phi^1, \tilde{\phi}^2(\cdot+1)\rangle\tilde{\phi}^2(\cdot+1) - \langle \phi^1, \tilde{\phi}^3(\cdot+1)\rangle\tilde{\phi}^3(\cdot+1),$$

and $\tilde{\phi}^1 = y/\|y\|$ then from the above calculation we see that $\tilde{\Phi} = (\tilde{\phi}^1, \tilde{\phi}^2, \tilde{\phi}^3)$ generates the multiresolution analysis $(\tilde{V}_p)$ with $V_0 \subset \tilde{V}_0 \subset V_1$. These orthonormal scaling functions and associated wavelets in Figure 1. The wavelets have been constructed so that the techniques used in Theorem 4.4 of [7] lead to a wavelet basis for $L_2[0,1]$. The interpolation points for these functions are given in Table 1 of Appendix 1 and the dilation coefficients are give in Table 1 of Appendix 2

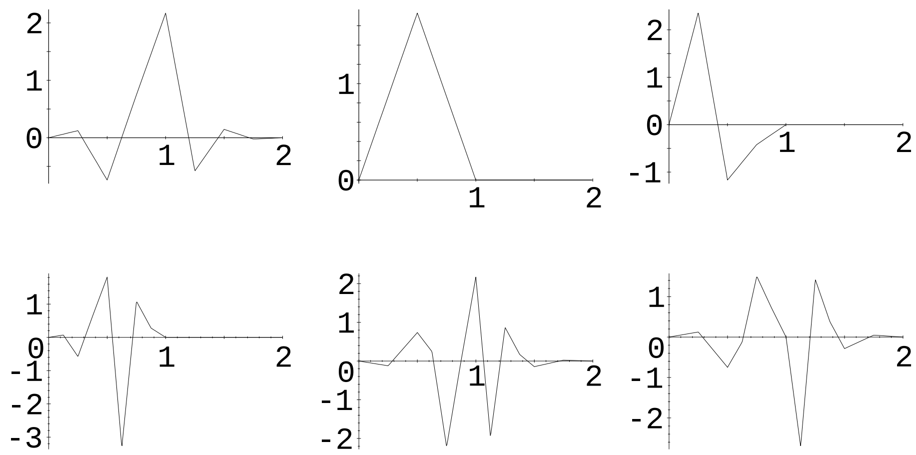


Figure 1. Piecewise linear orthogonal scaling functions (top, right-to-left: $\tilde{\phi}^1(\cdot - 1)$, $\tilde{\phi}^2$, and $\tilde{\phi}^3$) and wavelets (bottom, right-to-left: $\tilde{\psi}^1$, $\tilde{\psi}^2$, and $\tilde{\psi}^3(\cdot - 1)$).

3.2 Example: Symmetric piecewise linear orthogonal scaling functions

We now choose $\tilde{V}_0$ between V_1 and V_2. Then $\mathcal{A}(V_2) \ominus \mathcal{A}(V_1)$ is $7 - 3 = 4$ dimensional. By applying the intertwining technique twice, we can choose symmetric w_1 and antisymmetric w_2 to obtain scaling functions and wavelets that are either symmetric or antisymmetric as shown in Figure 2. In this case $\tilde{\Phi} = (\tilde{\phi}^1, \tilde{\phi}^2, \tilde{\phi}^3, \tilde{\phi}^4)$ where $\tilde{\phi}^2 = \phi^2$, $\tilde{\phi}_3 = w_1/\|w_1\|$, $\tilde{\phi}_4 = w_2/\|w_2\|$, and $\tilde{\phi}^1 = y/\|y\|$. Here

$$y = \phi^1 - \sum_{i=2}^{4} \langle \phi^1 \tilde{\phi}^i \langle \tilde{\phi}^i - \sum_{i=2}^{4} \langle \phi^1 \tilde{\phi}^i(\cdot + 1) \langle \tilde{\phi}^i(\cdot + 1).$$

Because of the symmetry, it is not difficult to generate a wavelet basis for $L_2[0,1]$ from the wavelets given below ([7, Theorem 4.4]).

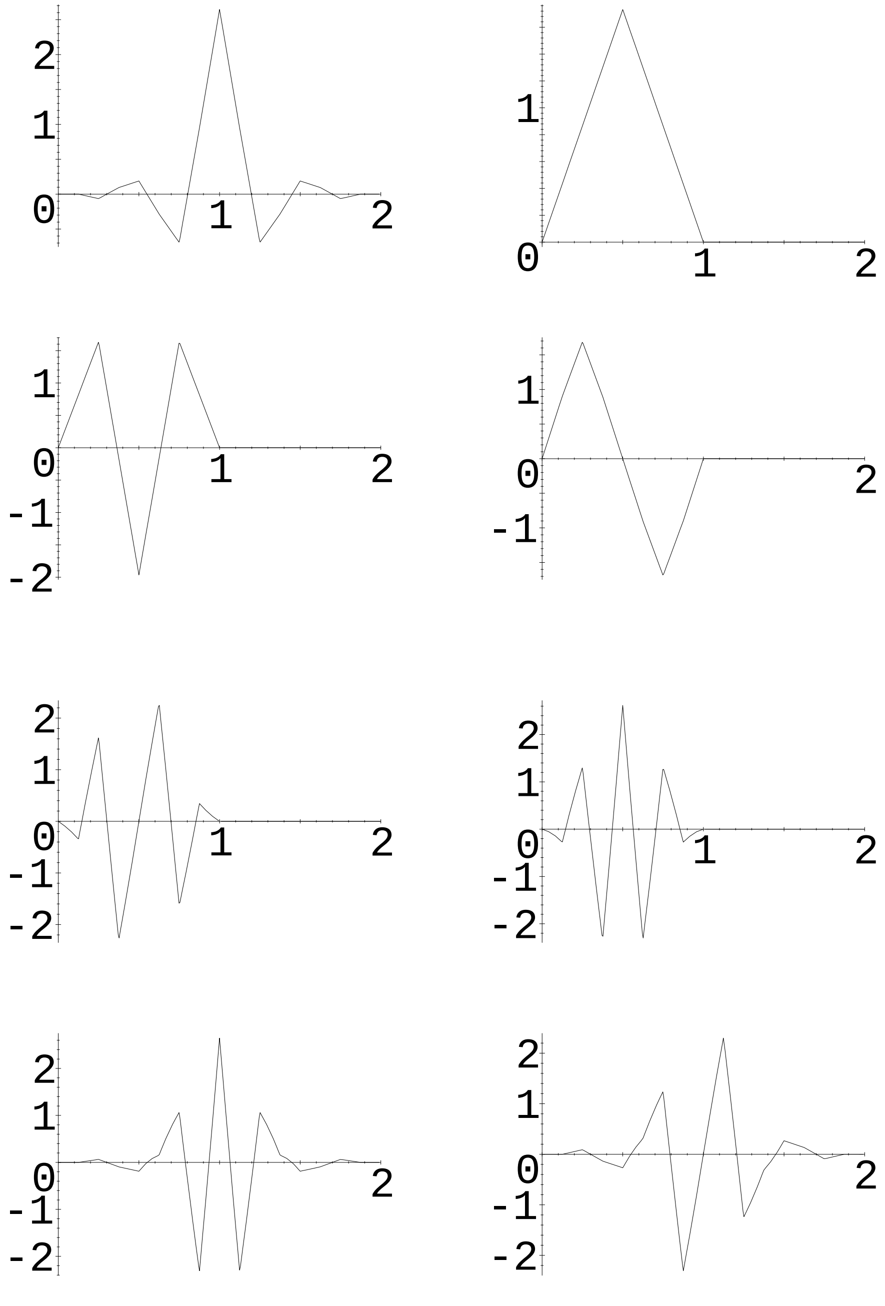


Figure 2. Symmetric (or antisymmetric) piecewise linear orthogonal scaling functions and wavelets, from top-to-bottom and left-to-right: $\tilde{\phi}^1(\cdot - 1)$, $\tilde{\phi}^2$, $\tilde{\phi}^3$, $\tilde{\phi}^4$, $\tilde{\psi}^1$, $\tilde{\psi}^2$, $\tilde{\psi}^3(\cdot - 1)$, and $\tilde{\psi}^4(\cdot - 1)$.

4 Main Theorem

We first prove that, with an additional hypothesis, Theorem 1 holds with $n = 1$.

Theorem 2 *Let* (V_p) *be a multiresolution analysis generated by* r *functions* $\phi^1, \ldots, \phi^r$ *minimally supported on* $[-1, 1]$. *Let* Q^l *denote the orthogonal projection onto* $\mathcal{A}(V_{l+1}) \ominus \mathcal{A}(V_l)$. *Suppose there are positive numbers* ϵ *and* l_0 *such that* $\|Q^l u\| \geq \epsilon \|u\|$ *for all* $l \geq l_0$ *and* u *in* $\mathcal{C}_0(V_l) \cup \mathcal{C}_1(V_l)$. *Then there is some integer* q *and some orthogonal multiresolution analysis* $(\tilde{V}_p)$ *such that*

$$V_q \subset \tilde{V}_0 \subset V_{q+1}.$$

Proof. As noted in the proof of Lemma 2.2 we can assume that

- $\phi^s(\cdot - h)\chi_{[0,1]}$ is orthogonal to $\mathcal{A}(V_0)$ for $s = 1, \ldots, k$ and $h = 0, 1$, and
- $\phi^s \perp \phi^{s'}$ for $s \neq s'$ and $s, s' = 1, \ldots, r$.

For $1 \leq s \leq r$ and $l, n \in \mathbf{Z}$ let $\phi^s_{l,n}(x) = 2^{l/2}\phi^s(2^l x - n)$. Then

$$\Lambda = \{\phi^s_{l,n} \mid 1 \leq s \leq k, 1 \leq n \leq 2^l - 1\} \cup \{\phi^s_{l,n} \mid k+1 \leq s \leq r, 0 \leq n \leq 2^l - 1\}$$

is a basis for $\mathcal{A}(V_l)$,

$$\{\phi^s_{l,0}\chi_{[0,1]} \mid 1 \leq s \leq k\} \cup \Lambda$$

is a basis for $\mathcal{B}_0(V_l)$ and

$$\{\phi^s_{l,2^l}\chi_{[0,1]} \mid 1 \leq s \leq k\} \cup \Lambda$$

is a basis for $\mathcal{B}_1(V_l)$. Observe that $\mathcal{A}(V_l)$, $\mathcal{B}_h(V_l)$ and $\mathcal{C}_h(V_l)$ have linear dimensions $2^l r - k$, $2^l r$ and k, respectively.

With $\Phi_{l,n} = (\phi^1_{l,n}, \ldots, \phi^k_{l,n})^\top$, we define the overlap matrix, $C = \langle \Phi_{0,0}, \Phi_{0,1} \rangle := \left(\langle \phi^s_{0,0}, \phi^{s'}_{0,1} \rangle\right)_{1 \leq s, s' \leq k}$. Let $l_2(\mathbf{R}^k)$ be the Hilbert space of sequences $y = \{y_n\}^\infty_{-\infty}$, $y_n \in \mathbf{R}^k$ with $\|y\|_{l_2} = \left(\sum (y_n, y_n)\right)^{1/2} < \infty$ where (y_n, y_n) is the Euclidean inner product. For $y, x \in l_2(\mathbf{R}^k)$ set $\langle x, y \rangle_{l_2} = \sum (x_n, y_n)$. We first prove the following three lemmas before proceeding with the remainder of the proof of Theorem 2.

Lemma 4.1 *Let* $x = \{x_n\}^\infty_{-\infty} \in l_2(\mathbf{R}^k)$ *and consider the infinite block tridiagonal matrix* $\mathcal{J} : l_2(\mathbf{R}^k) \to l_2(\mathbf{R}^k)$ *given by* $\mathcal{J}x = y$, $y = \{y_n\}^\infty_{-\infty}$ *with* $y_i = C^\top x_{i-1} + x_i + C x_{i+1}$ *for all* $i \in \mathbf{Z}$. *Then* $\mathcal{J}$ *is a bounded positive operator with* spectrum$\mathcal{J} \subset [a, b]$, $0 < a$. *Consequently* $\mathcal{J}_l = \Pi_l \mathcal{J} \Pi_l$, *where* $\Pi_l : l_2(\mathbf{R}^k) \to l_2(\mathbf{R}^k)$ *is the projection such that* $\Pi_l x = \{x_n\}_1^{2^l - 1}$, *is positive definite with* spectrum$\mathcal{J}_l \subset [a, b]$, *and* $|(\mathcal{J}_l)^{-1}_{i,j}| \leq K \lambda^{|i-j|}$. *Here* K *is independent of* $l, i,$ *and* j, *and* $\lambda = \left(\frac{\sqrt{b/a} - 1}{\sqrt{b/a} + 1}\right)$.

Proof. Since $\langle \phi^i, \phi^j \rangle = 0$, $i = 1, \ldots, k$, $j = k+1, \ldots, r$ and $\langle \phi^i, \phi^j \rangle = 0$, $i = k+1, \ldots, r$, $j = k+1, \ldots, r$, $i \neq j$, it follows from Theorem 3.2 in [8] (also see [10]) that $\{\phi^i\}_{i=1}^r$ forms a Riesz basis for V_0 if and only if $(I + C^\top e^{i\theta} + Ce^{-i\theta}) \geq a > 0$, $0 \leq \theta < 2\pi$. Now the symbol of y, $\hat{y} = (\widehat{\mathcal{J}x}) = (I + C^\top e^{i\theta} + Ce^{-i\theta})\hat{x}$ and we find $\langle x, \mathcal{J}x \rangle_{l_2} = \frac{1}{2\pi} \int_0^{2\pi} \hat{x}^*(I + C^\top e^{i\theta} + Ce^{-i\theta})\hat{x}d\theta \geq a \int_0^{2\pi} \hat{x}^*\hat{x}d\theta = a\|x\|_{l_2}^2$. The upper bound follows in an analogous manner. The min-max principle [19] now implies that spect $\mathcal{J}_l \subset [a, b]$ and the decay given in the lemma follows from the exponential decay of elements of inverses of banded matrices (cf. [6],[9]). $\square$

Since Q^l is orthogonal we can choose $\{u_{h,i}^l \mid 1 \leq i \leq k\}$ to be an orthonormal basis for $\mathcal{C}_h(V_l)$, $h = 0, 1$, such that $\{Q^l u_{h,i}^l \mid 1 \leq i \leq k\}$ is an orthogonal basis for $Q^l \mathcal{C}_h(V_l)$ for $h = 0, 1$. We also define the normalized basis elements $x_{h,i}^l = Q^l u_{h,i}^l / \|Q^l u_{h,i}^l\|$ for $1 \leq i \leq k$ and $h = 0, 1$.

Lemma 4.2 *Suppose there is some $\epsilon > 0$ such that $\|Q^l u\| \geq \epsilon \|u\|$ for all l sufficiently large and for u in either $\mathcal{C}_0(V_l)$ or $\mathcal{C}_1(V_l)$. Let $x_{h,i}^l$ and $u_{h,i}^l$ be as above. Then, for $1 \leq i, j \leq k$*

$$\lim_{l \to \infty} \langle x_{0,i}^l, x_{1,j}^l \rangle = 0 \tag{5}$$

and

$$\lim_{l \to \infty} \frac{\langle u_{0,i}^l, u_{1,j}^l \rangle}{\|Q^l u_{0,i}^l\| \, \|Q^l u_{1,j}^l\|} = 0. \tag{6}$$

Proof. Observe that

$$\langle \Phi_{l,n}, \Phi_{l,m} \rangle = \begin{cases} C & \text{if } m - n = 1 \\ I & \text{if } m = n \\ C^\top & \text{if } m - n = -1. \end{cases}$$

Let $f \in \mathcal{C}_0(V_l)$. Using the bases given above for $\mathcal{A}(V_l)$ and $\mathcal{C}_0(V_l)$, f may be expressed in the form

$$f = \gamma_0^\top \Phi_{l,0} \chi_{[0,1]} + \sum_{n=1}^{2^l - 1} \gamma_n^\top \Phi_{l,n} \tag{7}$$

where each γ_n is in $\mathbf{R}^k$. Since f is perpendicular to $\mathcal{A}(V_l)$ it follows that $\gamma = (\gamma_1^\top, \ldots, \gamma_{2^l-1}^\top)^\top$ satisfies the block tridiagonal system of equations:

$$\begin{aligned} \gamma_1 + C\gamma_2 &= -C^\top \gamma_0 \\ C^\top \gamma_{n-1} + \gamma_n + C\gamma_{n+1} &= 0, \qquad n = 2, 3, \ldots, 2^l - 2, \\ C^\top \gamma_{2^l-2} + \gamma_{2^l-1} &= 0. \end{aligned} \tag{8}$$

If we define $\mathcal{J}_l$ to be the matrix representation of the left hand side of the above equation we find $\gamma = \mathcal{J}_l^{-1}(-\gamma_0^\top C, 0, \ldots, 0)^\top$. Thus f is determined by γ_0 (and l) and we denote the dependence of f on γ_0 and l by $f_{\gamma_0,l}$. Let $\lambda = \left(\frac{\sqrt{b/a}-1}{\sqrt{b/a}+1}\right)$. It follows from Lemma 4.1 that there are positive numbers K and l_0 such that $\|\gamma_n\| \leq K\lambda^n$ for $n \geq 0$, $l \geq l_0$ and $\|f_{\gamma_0,l}\| = 1$. Hence,

$$\lim_{l\to\infty} \max\{\|f\chi_{[1/2,1]}\| \mid f \in \mathcal{C}_0(V_l), \|f\| = 1\} = 0. \tag{9}$$

Similarly,

$$\lim_{l\to\infty} \max\{\|g\chi_{[0,1/2]}\| \mid g \in \mathcal{C}_1(V_l), \|g\| = 1\} = 0. \tag{10}$$

Equations (9) and (10) then give

$$\lim_{l\to\infty} |\langle u_{0,i}^l, u_{1,j}^l\rangle| \leq \lim_{l\to\infty} \|u_{0,i}^l\chi_{[1/2,1]}\|\|u_{1,j}^l\| + \|u_{0,i}^l\|\|u_{1,j}^l\chi_{[0,1/2]}\| = 0.$$

Since Q^l is uniformly bounded below on $\mathcal{C}_h(V_l)$ for $h = 0, 1$ and l large enough, this proves the limit (6).

If $f \in \mathcal{C}_0(V_l)$ then $Q^l f = f - (I - P_{l+1})f$ where P_{l+1} is the orthogonal projection onto $\mathcal{A}(V_{l+1})$ and so $(I - P_{l+1})f$ is in $\mathcal{C}_0(V_{l+1})$. Therefore both f and $(I - P_{l+1})f$ decay exponentially away from 0. Thus, as above, the limit in (5) follows from Equations (9) and (10) and the uniform lower bound on Q^l. □

Let $\mathcal{D}_l$ denote $\mathrm{span}(Q^l\mathcal{C}_0(V_l) \cup Q^l\mathcal{C}_1(V_l))$.

Lemma 4.3 *There is some q such that for $l \geq q$ there exists a set $\Psi = \{z_{i,j}^\pm \mid 1 \leq i, j \leq k\}$ of $2k^2$ functions in $\mathcal{E}_l := (\mathcal{A}(V_{l+1}) \ominus \mathcal{A}(V_l)) \ominus \mathcal{D}_l$ such that if*

$$w_{i,j}^\pm = z_{i,j}^\pm + (x_{0,i}^l \pm x_{1,j}^l)/(4k)$$

then $\{w_{i,j}^\pm \mid 1 \leq i, j \leq k\}$ is an orthonormal set.

Proof. Note that $\dim(\mathcal{A}(V_{l+1}) \ominus \mathcal{A}(V_l)) = r2^l$. Let $\{e_i \mid 1 \leq i \leq r2^l\}$ be an orthonormal basis of $\mathcal{A}(V_{l+1}) \ominus \mathcal{A}(V_l)$. Represent $w_{i,j}^\pm, z_{i,j}^\pm, (x_{0,i}^l \pm x_{1,i}^l)/(4k)$ in terms of this basis as column vectors of size $r2^l$. Let W_l^+ be the $r2^l \times k^2$ matrix whose $(i-1)k + j$ column is $w_{i,j}^+$. Similarly define W_l^- and $Z_l^\pm$. Let $X_l^\pm$ be the $r2^l \times k^2$ matrix whose $(i-1)k + j$ column is $(x_{0,i}^l \pm x_{1,j}^l)/(4k)$. In block form we have

$$W_l := \begin{pmatrix} W_l^+ & W_l^- \end{pmatrix} = \begin{pmatrix} X_l^+ + Z_l^+ & X_l^- + Z_l^- \end{pmatrix} =: X_l + Z_l \tag{11}$$

Note that $\{w_{i,j}^\pm \mid 1 \leq i, j \leq k\}$ is an orthonormal set if and only if $W_l^\top W_l = I$ where I is the $2k^2 \times 2k^2$ identity. Since $z_{i,j}^\pm \in \mathcal{D}_l^\perp$ we have $X_l^\top Z_l = \mathbf{0}$. Hence, $W_l^\top W_l = I$ if and only if $Z_l^\top Z_l = I - X_l^\top X_l$. We can solve this for Z_l (using

Cholesky factorization, for instance) if the right hand side $I - X_l^\top X_l$ is positive definite (since it is clearly symmetric).

Let q be large enough so that $|\langle x_{0,i}^l, x_{1,j}^l\rangle| < 1/4$ for $l \geq q$. Then, for $l \geq q$, the absolute values of the elements of $X_l^\top X_l$ are of the form $|\langle x_{0,i}^l \pm x_{1,j}^l, x_{0,m}^l \pm x_{1,n}^l\rangle|/(16k^2) < 3/(16k^2)$. Thus any row or column sum of the absolute values of the elements of $X_l^\top X_l$ is less than $3/8 < 1$ implying that $I - X_l^\top X_l$ is diagonally dominant with positive terms on the diagonal and hence is positive definite.

□

Let q be as in Lemma 4.3 and $l \geq q$. For $1 \leq i,j \leq k$, let $w_{i,j}^l = \alpha_{i,j} w_{i,j}^+ + \beta_{i,j} w_{i,j}^-$ where $\alpha_{i,j}^2 + \beta_{i,j}^2 = 1$ and $\alpha_{i,j}, \beta_{i,j} \in [0,1]$. Let $\mathcal{W} = \operatorname{span}\{w_{i,j}^l \mid 1 \leq i,j \leq k\}$. Then

$$
\begin{aligned}
\langle u_{0,i}^l, P_{\mathcal{W}} u_{1,j}^l\rangle &= \sum_{m,n=1}^{k} \langle u_{0,i}^l, w_{m,n}^l\rangle\langle w_{m,n}^l, u_{1,j}^l\rangle \\
&= \|Q^l u_{0,i}^l\|\,\|Q^l u_{1,j}^l\| \sum_{m,n=1}^{k} \langle x_{0,i}^l, w_{m,n}^l\rangle\langle w_{m,n}^l, x_{1,j}^l\rangle \\
&= \frac{\|Q^l u_{0,i}^l\|\,\|Q^l u_{1,j}^l\|}{16k^2} \sum_{m,n=1}^{k} \langle x_{0,i}^l, (\alpha_{m,n}+\beta_{m,n})x_{0,m}^l + (\alpha_{m,n}-\beta_{m,n})x_{1,n}^l\rangle \cdot \\
&\quad \langle(\alpha_{m,n}+\beta_{m,n})x_{0,m}^l + (\alpha_{m,n}-\beta_{m,n})x_{1,n}^l, x_{1,j}^l\rangle \\
&= \frac{\|Q^l u_{0,i}^l\|\,\|Q^l u_{1,j}^l\|}{16k^2}\{\alpha_{i,j}^2 - \beta_{i,j}^2 + \\
&\quad \langle x_{0,i}^l, x_{1,j}^l\rangle(\sum_{n=1}^{k}(\alpha_{i,n}+\beta_{i,n})^2 + \sum_{m=1}^{k}(\alpha_{m,j}-\beta_{m,j})^2) + \\
&\quad \sum_{m,n=1}^{k}(\alpha_{m,n}^2 - \beta_{m,n}^2)\langle x_{0,i}^l, x_{1,n}^l\rangle\langle x_{o,m}^l, x_{1,j}^l\rangle\}
\end{aligned}
$$

Let $\gamma_{i,j} = \alpha_{i,j}^2$ and $\underline{\gamma} = (\gamma_{i,j}) \in [0,1]^{k\times k}$. Then $\alpha_{i,j} = \sqrt{\gamma_{i,j}}$ and $\beta_{i,j} = \sqrt{1-\gamma_{i,j}}$. We can rewrite the above calculation as:

$$\frac{16k^2}{\|Q^l u_{0,i}^l\|\,\|Q^l u_{1,j}^l\|}\langle u_{0,i}^l, P_{\mathcal{W}} u_{1,j}^l\rangle = 2\gamma_{i,j} - 1 + 2 f_{i,j}^l(\underline{\gamma}) \tag{12}$$

where $f_{i,j}^l(\underline{\gamma})$ is continuous for $\underline{\gamma} \in [0,1]^{k\times k}$. Furthermore, if $|\langle x_{0,i}^l, x_{1,n}^l\rangle| < \epsilon$ then $|f_{i,j}^l(\underline{\gamma})| < 4k\epsilon + k^2\epsilon^2$. Let $\theta_{i,j}^l = (1 + \frac{16k^2}{\|Q^l u_{0,i}^l\|\,\|Q^l u_{1,j}^l\|}\langle u_{0,i}^q, u_{1,j}^q\rangle)/2$. Then Equation (2) is equivalent to

$$\theta_{i,j}^l - f_{i,j}^l(\underline{\gamma}) = \gamma_{i,j} \tag{13}$$

or

$$\underline{\theta}^l - \underline{f}^l(\underline{\gamma}) = \underline{\gamma} \tag{14}$$

where $\underline{\theta}^l = (\theta^l_{i,j}) \in \mathbf{R}^{k\times k}$ and $\underline{f}^l = (f^l_{i,j}) : [0,1]^{k\times k} \to \mathbf{R}^{k\times k}$. Using Lemma 3.1, the proof will be finished once it is shown that Equation (14) has a solution.

By Lemma 4.2 we can find l large enough so that $\underline{\theta}^l \in [1/3, 2/3]^{k\times k}$ and so that $\underline{f}^l(\underline{\gamma}) \in [-1/3, 1/3]^{k\times k}$ for $\underline{\gamma} \in [0,1]^{k\times k}$. Then, for l large enough

$$\underline{g} := \underline{\theta}^l - \underline{f}^l$$

maps the compact, convex set $[0,1]^{k\times k}$ into itself continuously. By the Schauder Fixed Point Theorem Equation 14 has a solution $\underline{\gamma} \in [0,1]^{k\times k}$ and thus Theorem 2 is proved. □

By a suitable modification of the original multiresolution analysis a new multiresolution analysis $(\hat{V}_p)$ may be constructed with

$$V_0 \subset \hat{V}_0 \subset V_n$$

such that the hypotheses of Lemma 4.2 are satisfied. First note that by Lemma 2.1 any MRA generated by compactly supported functions may be assumed to have generators that are minimally supported in $[-1,1]$. Next, let $V_l[a,b] = \{\phi \in V_l : \text{supp}\phi \subset [a,b]\}, T^\infty = \{\phi \in V_0 : \phi \perp V_0(-\infty,0] \oplus V_0[0,\infty)\}, T_0^\infty = \{\phi\chi_{[0,\infty)} : \phi \in T^\infty\}$, and $T_1^\infty = \{\phi\chi_{(-\infty,0]} : \phi \in T^\infty\}$. If all the functions in T^∞ have bounded support, then for l chosen large enough so that $[-2^l, 2^l]$ supports T^∞, $\tilde{V}_0 = V_l$ gives rise to a compactly supported orthogonal multiresolution analysis. Consequently it is only those functions in T^∞ that have unbounded support that that need to be altered. To this end define $T_0^F = \{\phi \in T_0^\infty : \text{supp}\phi \text{ is bounded}\}, T_1^F = \{\phi \in T_1^\infty : \text{supp}\phi \text{ is bounded}\}, T_0^\omega = T_0^\infty \ominus T_0^F$, and $T_1^\omega = T_1^\infty \ominus T_1^F$ with $k_1 = \dim T_0^F = \dim T_1^F$ and $k_2 = \dim T_0^\omega = \dim T_1^\omega$, also set $\mathcal{C}_0^l = \mathcal{C}_0(V_l)$ and $\mathcal{C}_1^l = \mathcal{C}_1(V_l)$.

The uniform lower bound condition of Theorem 2 can be replaced by a simpler one by passing to limits. Using the notation in the proof of Lemma 4.2 and Lemma 4.1 it follows that $2^{-l/2} f_{\gamma_0,l}(2^{-l}\cdot)$ converges in $L^2(\mathbf{R})$ to some $f_{\gamma_0,\infty} \in T_0^\infty$. Furthermore, $2^{-l/2}(Q_l f_{\gamma_0,l})(2^{-l}\cdot)$ converges to $Q^\infty f_{\gamma_0,\infty}$, where Q^∞ is the orthogonal projection onto $V_1[0,\infty)$. We then have

Lemma 4.4 *If Q^∞ is nonsingular on T_0^∞, then Q^l is uniformly bounded below as in Theorem 2.*

We now complete the proof of Theorem 1 with

Lemma 4.5 *If (V_p) is a multiresolution analysis generated by compactly supported scaling functions then there is some integer n and some multiresolution analysis $(\hat{V}_p)$ with $V_0 \subset \hat{V}_0 \subset V_n$, such that for ϕ in the unit sphere of $\mathcal{C}_0(\hat{V}_0)$ or*

$\mathcal{C}_1(\hat{V}_0)$, $\|2^{-l/2}(\hat{P}^{l+1} - \hat{P}^l)\phi(2^l\cdot)\|$ is bounded away from zero for sufficiently large l. Here $\hat{P}^l$ is the othogonal projection onto $\mathcal{A}(\hat{V}_l)$.

Proof. To construct $\hat{V}_0$ we will "borrow" functions from $V_1, V_2, \ldots$, taking care not to disturb T_0^F or T_1^F. This will be accomplished first by working with functions in the spaces T_0^∞ and T_1^∞ then approximating these functions by members of $T_0^l = \{\phi(2^{-l}\cdot) : \phi \in \mathcal{C}_0^l\}$ and $T_1^l = \{\phi(2^l\cdot) : \phi \in \mathcal{C}_1^l(\cdot + 1)\}$ for sufficiently large l.

Given $\phi \in T_0^\omega$ of unit length, the fact that (V_p) forms a multiresolution analysis implies that $\lim_{l\to\infty} P_\infty^l \phi = \phi$, and hence that $\lim_{l\to\infty} \|P_\infty^l \phi\| = \|\phi\| = 1$ where P_∞^l is the orthogonal projection onto $V_l(-\infty, 0] \oplus V_l[0, \infty)$. Also, since these projections are onto successively larger spaces, this convergence is monotoncially nondecreasing in l. We have continuous functions converging monotonically to a continuous function (i.e., 1) on a compact set so the convergence must be uniform. Let n_0 be sufficiently large so that, $\|P_\infty^l \phi\| > \frac{1}{4}$ for all ϕ in the unit sphere of T_0^ω and all $l \geq n_0$. We may choose k_2 orthogonal functions $w_1, \ldots, w_{k_2} \in V_{n_0}[0, \infty)$, with span W_0, such that $\|P_{W_0}\phi\| > \frac{1}{4}$ for all unit functions $\phi \in T_0^\infty$. Furthermore, we may assume that these functions are compactly supported since if they were not, we could approximate them with arbitrary accuracy by functions that are. Observe that W_0 may be shifted as far to the right as desired without destroying its effectiveness. Indeed, given any $\phi \in T_0^\omega$, $\phi(\cdot + j)\chi_{[0,\infty)}$ is again in T_0^ω for any positive integer, j. Thus for $\phi \in T_0^\omega$, $\|P_{W_0(\cdot - j)}\phi\| = \|P_{W_0}\phi(\cdot + j)\| = \|P_{W_0}(\phi(\cdot + j)\chi_{[0,\infty)})\|$. This last expression is a continuous function of ϕ, which, by the construction of W_0, may not assume the value zero. Therefore, it is bounded away from zero on the unit sphere, say by ϵ_0 where the value of ϵ_0 may depend on j, but will always be positive.

In light of the above observation, shift W_0 so that $\text{supp}W_0 \cap \text{supp}T_0^F = \emptyset$ and $\text{supp}W_0 \cap \text{supp}W_0(\frac{1}{2}\cdot) = \emptyset$. The first condition guarantees that $W_0 \perp T_0^F$, and the second guarantees that "borrowing" the dilates of W_0 will not decrease the magnitudes of projections onto W_0. Similarly choose compactly supported $w_{-1}, \ldots, w_{-k_2} \in V_{n_1}$, with span W_1, so that $\text{supp}W_1 \cap \text{supp}T_1^F = \text{supp}W_1 \cap \text{supp}W_1(\frac{1}{2}\cdot) = \emptyset$ and $\|P_{W_1}\phi\| > \epsilon_1$ for unit $\phi \in T_1^\omega$. Let $n = \max\{n_0, n_1\}$ and $\epsilon = \min\{\epsilon_0, \epsilon_1\}$. Form a new multiresolution analysis $(\hat{V}_p)$ by taking

$$\begin{aligned}\hat{V}_0 = V_l &\oplus \left(\textstyle\bigoplus_{j\in Z} W_0(2^{l-1}\cdot - 2^{l-1}j)\right) \oplus \cdots \oplus \left(\textstyle\bigoplus_{j\in Z} W_0(2^{l-n}\cdot - 2^{l-n}j)\right)\\ &\oplus \left(\textstyle\bigoplus_{j\in Z} W_1(2^{l-1}\cdot - 2^{l-1}j)\right) \oplus \cdots \oplus \left(\textstyle\bigoplus_{j\in Z} W_1(2^{l-n}\cdot - 2^{l-n}j)\right),\end{aligned}$$

where l is chosen sufficiently large so that

1. $\text{supp}(W_0(2^{-1}\cdot) \oplus \cdots \oplus W_0(2^{-n}\cdot) \oplus W_1(2^{-1}\cdot) \oplus \cdots \oplus W_1(2^{-n}\cdot)) \subset [-2^{l-1}, 2^{l-1}]$, and consequently, $\text{supp}(T_0^F \oplus T_1^F) \subset [-2^{l-1}, 2^{l-1}]$ (we use 2^{l-1} here to assure that W_0 and W_1 do not interfere with each other);

2. T_0^l and T_1^l approximate T_0^∞ and T_1^∞ closely enough so that $\|P_{W_0}\phi\| > \frac{3}{4}\epsilon$ for unit $\phi \in T_0^l \ominus T_0^F$ and $\|P_{W_1}\phi\| > \frac{3}{4}\epsilon$ for unit $\phi \in T_1^l \ominus T_1^F$, which is possible because the scaling functions form a Riesz basis; and

3. $\|P_{W_0}(I - P_{W_1})\phi\| < \epsilon/2$ for unit $\phi \in \hat{\mathcal{C}}_1^0(\cdot - 1)$, and $\|P_{W_1}(I - P_{W_0})\phi\| < \epsilon/2$ for unit $\phi \in \hat{\mathcal{C}}_0^0$, which is possible because the functions in T_1^ω (respectively, T_0^ω) undergo exponential decay to the left (right), as shown in Lemma 4.1.

Thus we find

$$V_l \subset \hat{V}_0 \subset V_{l+n}.$$

In addition, $\|(\hat{P}^{i+1} - \hat{P}^i)(2^{i/2}\phi(2^i\cdot))\| > \epsilon/4$, for $i \geq 0$ and unit $\phi \in \mathcal{C}_0(\hat{V}_0)$, with an analogous condition holding for $\phi \in \mathcal{C}_1(\hat{V}_0)$. This is true because in the MRA $(\hat{V}_p)$, $W_0(2^l\cdot) \oplus W_1(2^l\cdot) \subset \hat{V}_1$, so $(W_0(2^l\cdot))_m \oplus (W_1(2^l\cdot))_m \subset \hat{V}_{m+1}$ for $m \geq 1$. If $u \in \mathcal{C}(\hat{V}_0)$ is a unit vector, then so is $u_{m,0}$, and hence

$$\begin{aligned}
\|\hat{P}^{m+1}(I - \hat{P}^m)u_{m,0}\| &\geq \|P_{(W_0(2^l\cdot))_m}(I - \hat{P}^m)u_{m,0}\| \\
&= \|P_{W_0(2^l\cdot)}(I - P_{\hat{V}_0[0,2^m]})u\| \\
&\geq \|P_{W_0(2^l\cdot)}u\| - \|P_{W_0(2^l\cdot)}P_{\hat{V}_0[0,2^m]}u\| \\
&\geq \tfrac{3}{4}\epsilon - \tfrac{1}{2}\epsilon = \epsilon/4,
\end{aligned}$$

for all $m \geq 1$. □

Remark 4.1.

1. In the proof of Theorem 2 we took $\mathcal{W}$ to be a k^2 dimensional subspace. In the remark following Lemma 3.1 it was noted that the rank of P_0P_1 (which is generically k) is a lower bound for dim $\mathcal{W}$. In all of our examples we achieve the lower bound k.

2. In the case $k = 1$ it is possible to show that the uniform lower bound condition of Theorem 2 always holds. In the next section we show that this is also true for the spline spaces $\mathcal{S}_{d,r}$. Thus Theorem 2 holds in, at least, the cases of most interest.

We conclude this section with the proof of Lemma 2.1. We restate the lemma for the reader's convenience.

Lemma 2.1

Suppose (V_p) is a multiresolution analysis generated by compactly supported scaling functions. Then there is some n and some set of scaling functions minimally supported on $[-1,1]$ that generate the multiresolution analysis (V_p') given by

$$V_p' = V_{p+n}.$$

Proof. Without loss of generality let V_0 be generated by scaling functions, $\Phi = (\phi^1, \ldots, \phi^r)$ as in Remark 2.1. If $f \in H_0(\Phi)$ then f can be expressed uniquely as

$$f = \sum_{s=1}^{k} a_s \phi^s \chi_{[0,1]}. \tag{15}$$

Let $p : H_0(\Phi) \to H_1(\Phi)$ be defined by

$$p(f) = \sum_{s=1}^{k} a_s \phi^s(\cdot - 1) \chi_{[0,1]}. \tag{16}$$

Let $E = H_0(\Phi) \bigcap H_1(\Phi)$ and E' be the maximal p invariant subspace of E i.e. $E' = \bigcap_{n=1}^{\infty} p^{-n}(E)$, (where for the purpose of computing E' the range of p is extended to the Minkowski sum of $H_0(\Phi)$ and $H_1(\Phi)$). Since p is nonsingular $p(E') = E'$. If we consider p as an operator on E' let $\lambda_1, \cdots, \lambda_j$ be the (possibly complex) eigenvalues of p with magnitude less than one and $\lambda_{j+1}, \cdots, \lambda_l$ be its (possibly complex) eigenvalues with magnitude greater than one. The eigenvalues are listed above including multiplicities. We claim that p has no eigenvalues that lie on the unit circle hence $l = \mathrm{Dim}(E') \leq k$. To see this suppose that λ is an eigenvalue of p with magnitude one and x an eigenvector associated with this eigenvalue. Let $y = \mathrm{Re}(x)$, then for λ real set $\kappa = \|y\| = \|p^n(y)\|$, otherwise set $0 < \kappa = \min_{|u|=1} \|x + u\bar{x}\| \leq \|p^n(y)\| \leq \|x\|$. We may define a sequence of functions $\{f_\nu\}_{\nu=0}^{\infty}$

$$f_\nu = \sum_{i=0}^{\nu} (-1)^i g_i, \tag{17}$$

where $g_i = (p^i y)(\cdot + i) + (p^{i+1} y)(\cdot + i + 1)$. Note

1. $g_i = \sum_{s=1}^{k} a_s^i \phi^s(\cdot + i)$,
2. $A \sum_{s=1}^{k} (a_s^i)^2 \leq \|g_i\|^2 \leq B \sum_{s=1}^{k} (a_s^i)^2$,
3. $\|g_i\|^2 = \|p^i y\|^2 + \|p^{i+1} y\|^2$, and
4. $f_\nu = y + (-1)^\nu (p^{\nu+1} y)(\cdot + \nu + 1)$.

Therefore $\|f_\nu\|^2 = \|y\|^2 + \|p^{\nu+1} y\|^2 \leq 2\|x\|^2$, and $\sum_{i=1}^{\nu} \sum_{s=1}^{k} (a_s^i)^2 \geq \sum_{i=1}^{\nu} \frac{2\kappa^2}{B} = \frac{2\kappa^2 \nu}{B}$ which shows that $\tau(\Phi)$ is not a Riesz basis and yields a contradiction.

Let $x_1, \ldots, x_l$ be the generalized eigenvectors associated with $\lambda_1, \cdots, \lambda_l$. In the event an eigenvalue is complex its conjugate is also an eigenvalue and we will replace the appropriate generalized eigenvectors by their real and imaginary parts. Note that $x_1, \ldots, x_l$ form a basis for E'. For $y = x_m$, $m = 1, \ldots, j$, we define a sequence of functions $\{f_\nu\}$ as above. In this case because of the choice of

y, $\|p^n y\| \to 0$ since $\lambda_1, \cdots, \lambda_j$ are in magnitude less than one. Therefore $f_\nu \xrightarrow{L^2} y$. For $y = x_m$, $m = j+1, \ldots, l$ choose the sequence $\{f_\nu\}$ as $f_\nu = \sum_{i=0}^{\nu}(-1)^i g_i$ where $g_i = (p^{-i}y)(\cdot - i) + (p^{-(i+1)}y)(\cdot - i - 1)$. Again we have $f_\nu \xrightarrow{L^2} y$ and so $x_1, \ldots, x_l$ are in V_0. By replacing $\phi^1, \ldots, \phi^k$ by appropriate linear combinations of $\phi^1, \ldots, \phi^k$ we may assume that the right halves (and hence also the shifted left halves) of $\phi^1, \ldots, \phi^l$, are in E'. Since $x_1, \ldots, x_l$ span E' we may replace the generators $\phi^1, \ldots, \phi^l$ by the equivalent set of generators $x_1, \ldots, x_l$. Since $x_1, \ldots, x_l$ have support in $[0,1]$ these may be removed to A. By the linear independence of $\tau(\Phi)$ none of the $\phi^{l+1}, \ldots, \phi^k$ can have their right halves or shifted left halves in E'. If $E \ominus E' = 0$ then $\Phi = \{x_1, \ldots, x_l, \phi^{l+1}, \ldots, \phi^r\}$ is minimally supported and spans V_0.

If there are non-zero functions in $E \ominus E'$ let $E'' = H_0(\Phi) \ominus E'$. For nonzero $x \in E''$ let n_x be such that $p^{n_x}(x) \notin E$ but $p^i(x) \in E$, $1 \le i < n_x$ and define $n_0 = \infty$. Note that n_x is finite for nonzero x since $x \notin E'$ and that $n_{\alpha x + \beta y} \ge \min\{n_x, n_y\}$ for scalars α, β, and $x, y \in E''$. Now for $i = 1, 2, \ldots$ set $E_i = \{x \in E'' : n_x \ge i\}$. Then E_i is a subspace of E'' and $E_1 = E''$. Also there is some $\tilde{n}$ such that $E_{\tilde{n}+1} = 0$ and $E_{\tilde{n}} \ne 0$. Let $X_{\tilde{n}}$ be a set of vectors that form a basis for $E_{\tilde{n}}$. By the definition of $\tilde{n}$ if $x \in X_{\tilde{n}}$, then $x \notin E$. Furthermore $p(X_{\tilde{n}})$ gives a linearly independent set of vectors in $E_{\tilde{n}-1} \bigcap E$. Beginning with the vectors $p(X_{\tilde{n}})$ and $X_{\tilde{n}}$ complete this set to a basis, $X_{\tilde{n}-1}$, for $E_{\tilde{n}-1}$. Now $p(X_{\tilde{n}-1})$ is a linearly independent set of vectors in $E_{\tilde{n}-2}$ and we add a sufficient number of independent vectors to form a basis, $X_{\tilde{n}-2}$, for $E_{\tilde{n}-2}$. Continue this processes until a basis, X_1, for E_1 has been formed. Since X_1 forms a basis for E'', $\{x + p(x)(\cdot + 1) : x \in X_1\}$ may be used to replace $\phi^{l+1}, \ldots, \phi^k, \in \Phi$. For $x \in X_1$ set $\phi_x = x + p(x)(\cdot + 1)$. We now consider $\tilde{\phi}_x = \phi_x(\cdot + i - 1)$ for $x \in X_i - X_{i+1}$ (where we set $X_{\tilde{n}+1} = 0$). Since these are just translates of ϕ_x, $x \in X_1$, we may replace ϕ_x by $\tilde{\phi}_x$. For each $x \in X_1$ let

$$\hat{\phi}_x = \sum_{i=0}^{n_x - 1} (-1)^i \tilde{\phi}_{p^i(x)}$$

which is a linear combination of elements from $\{\tilde{\phi}_x : x \in X_1\}$. Then $\{\hat{\phi}_x : x \in X_1\}$ is linearly independent and may be used to replace $\{\tilde{\phi}_x : x \in X_1\}$. Note that the support of $\tilde{\phi}_x$ is $[-n_x, -n_x + 1] \cup [0,1]$ and that $\operatorname{span}\{\hat{\phi}_x(\cdot - a_x)\chi_{[0,1]} x \in X_1\}$ and $\operatorname{span}\{\hat{\phi}(\cdot - b_x)\chi_{[0,1]} : x \in X_1\}$ are linearly independent spaces. The set $\{\phi_x, x \in X_1\} \cup \{x_i\}_{i=1}^{l} \cup \{\phi^i\}_{i=k+1}^{r}$ spans V_0. Consequently there is an integer h, $\log_2 \tilde{n} \le h < \log_2 \tilde{n} + 1$ such that scaling the above functions by 2^h, are all supported in $[-1,1]$ and there is a $\tilde{\Phi}$ formed from appropriate translates of these functions such that $\tau(\tilde{\Phi})$ forms a basis for V_h. From the above construction and by Remark 2.1 in section 2 appropriate linear combinations of the generators of $\tilde{\Phi}$ may be taken so as to produce a set of generators that is minimally supported.

It follows from Theorems 4.3 and 5.3 of [14] that $\tau(\tilde{\Phi})$ is a Riesz basis of V_h. $\square$

5 Orthogonal Spline Scaling Functions

For $h > 0$ and $1 \leq r \leq d+1$, let $\mathcal{S}_{d,r}(h)$ denote the space of piecewise polynomial functions on $\mathbf{R}$ of degree d with knots of multiplicity r on $h\mathbf{Z}$. Let $\lceil a \rceil$ denote the least integer greater than or equal to a.

We let (t_i) denote the sequence of knots with multiplicity r for $\mathcal{S}_{d,r}(h)$ given by:

$$t_i = h\lceil i/r \rceil, \ i \in \mathbf{Z}.$$

Also let N_i^d denote the B-spline in $\mathcal{S}_{d,r}(h)$ with knots $t_i, \ldots, t_{i+d+1}$. Thus N_i^d is supported on $[t_i, t_{i+d+1}] \subset [jh, (j+L)h]$ where $j = \lceil i/r \rceil$ and $L = \lceil (d+1)/r \rceil$. Observe that the support of N_0^d is exactly $[0, Lh]$.

Let q be such that $2^q + 1 \geq L$ and choose $h = 2^{-q}$. Let $R = 2^q r$ and define $\phi^s = N_{s-d-1}^d$ for $s = 1, \ldots, R$. Then $\Phi = (\phi^1, \ldots, \phi^R)^\top$ generates a multiresolution analysis (V_p) with $V_0 = \mathcal{S}_{d,r}(2^{-q}) \cap L^2(\mathbf{R})$ (Goodman and Lee, [11]). It follows from the choice of q that Φ is minimally supported in $[-1, 1]$. Observe that there are exactly $d+1-r$ values of i such that 0 is an interior point of $[t_i, t_{i+d+1}]$. Hence $k = k(\Phi) = d+1-r$.

In this section we will show that (V_p) satisfies the hypotheses of Theorem 2 and so there is an orthogonal $\tilde{V}_0$ between $V_{q'}$ and $V_{q'+1}$ for some integer q'. Towards this end, we first develop some results concerning the sign changes of spline functions. We say $(x_1, \ldots, x_\alpha)$ is a *sign change sequence* on (a, b) for $f : \mathbf{R} \to \mathbf{R}$ if f is not identically zero on any subinterval (x_j, x_{j+1}), $j = 0, \ldots, \alpha$ where $a = x_0 < x_1 < \ldots < x_\alpha < x_{\alpha+1} = b$ and, for either $k = 0$ or $k = 1$, we have $(-1)^{j+k} f(x) \geq 0$ for all $x \in (x_j, x_{j+1})$ and $j = 0, \ldots, \alpha$. Let i_0, i_1 be given integers and define $\mathcal{P}_{i_0,i_1} =$ span $\{N_i^d \mid i_0 \leq i \leq i_1\}$ and $\sigma_i := \{x \mid N_i^d(x) \neq 0\} = (t_i, t_{i+d+1})$ for $i = i_0, \ldots, i_1$. We will need the following result concerning sign changes of B-spline expansions.

Lemma 5.1 *Let $a, b \in \mathbf{R}$ and i_0, i_1 be integers such that $t_{i_0} < t_{i_1}$ and suppose that $r \leq d$ (and thus the B-splines are continuous).*

1. *If $f \in \mathcal{P}_{i_0,i_1}$ and $(x_1, \ldots, x_\alpha)$ is a sign change sequence for f on (a, b) then, for any open interval $G \subset (a, b)$, we have*

$$\text{card } \{x_j \in G\} \leq \text{card } \{i \mid \sigma_i \cap G \neq \emptyset\} - 1. \tag{18}$$

2. *If $a < x_1 < x_2 < \ldots < x_\alpha < b$ satisfies*

$$x_i \in \sigma_{i_0+i-1} \cap \sigma_{i_0+i}, \qquad \textit{for } i = 1, \ldots, \alpha$$

then there is some $f \in \mathcal{P}_{i_0,i_0+\alpha}$ for which $(x_1,\dots,x_\alpha)$ is a sign change sequence on (a,b). Furthermore f does not vanish on any subinterval of $\bigcup_{i=i_0}^{i_0+\alpha}\sigma_i$.

Proof. Part 1 follows from the variation diminishing property of B-spline expansions (cf. [21, Theorem 4.76]). Let $x_0 \in \sigma_{i_0}, x_0 \neq x_i$. By [21, Theorem 4.61] there is a unique $f \in \mathcal{P}_{i_0,i_1}$ such that

$$f(x_0)=1, \;\; f(x_i)=0, \; i=1,\dots,\alpha.$$

Suppose $\alpha=1$. Then $f = c_0 N_{i_0}^d + c_1 N_{i_0+1}^d$. Since $f(x_0)=1$, at least one of c_0 and c_1 are nonzero. Since $x_1 \neq t_k$ for all k it follows that both c_0 and c_1 are nonzero and so f does not vanish on any subinterval of $\sigma_{i_0}\cup\sigma_{i_1}$. Hence, by counting the zeros of f including their multiplicities if follows from [21, Theorem 4.56] that f changes sign at x_1. It then follows by induction on α that f does not vanish on any subinterval of $(t_{i_0}, t_{i_0+\alpha+d+1}) = \bigcup_{i=i_0}^{i_0+\alpha}\sigma_i$. Hence, [21, Theorem 4.56] shows that $(x_1,\dots,x_\alpha)$ must be all of the zeros of f in $(t_{i_0}, t_{i_0+\alpha+d+1})$ and that all of these zeros must be isolated, simple zeros. Thus $(x_1,\dots,x_\alpha)$ is a sign change sequence of f on (a,b). □

Lemma 5.2 *Let $f \in V_0$ be such that $f\chi_{[0,\infty)} \neq 0$. Then there is some $g \in V_1$ such that supp $g \subset [0,\infty)$ and $\langle f,g\rangle \neq 0$.*

Proof. If $r=d+1$ then $f\chi_{[0,\infty)} \in V_1$ and we can take $g = f\chi_{[0,\infty)}$.

Suppose $r \le d$. Let $s_i = t_i/2$ for all i. Note that $n_i^d := N_i^d(2\cdot)$ is the B-spline associated with the knots $s_i,\dots,s_{i+d+1}$. Let $\sigma_i' = (s_i, s_{i+d+1}) = \{x \mid n_i^d(x) > 0\}$ and $\mathcal{P}_{i_0,i_1}'$ be the span of the B-splines associated with the knots $s_{i_0},\dots,s_{i_1+d+1}$.

Let $\hat{f} = f\chi_{[0,\infty)}$. If there exists a σ_i' such that $\sigma_i' \subset \operatorname{supp}\hat{f}$ and f does not change sign on σ_i' then we can choose $g = n_i^d$. If $\operatorname{supp}\hat{f}$ is not of the form $[0,b]$ or $[0,\infty)$ then some component I of $\operatorname{supp}\hat{f}$ does not contain zero and $\hat{f}\chi_I \in V_1$. In this case choose $g = \hat{f}\chi_I$. If $\operatorname{supp}\hat{f} = [0,b]$ then by the first part of Lemma 5.1, on $[b-h,b]$, $\hat{f}$ is a polynomial with at most $r-1$ zeros. Thus on $[b-h,b]$, $\hat{f} = (x-b)^{d-r+1}p(x)$ where $p(x)$ is a polynomial of degree $r-1$. Consequently we can choose a function $g \in V_1$ such that on $[b-h,b]$, we have $g = (x-(b-h))^{d-r+1}p(x)$.

Finally suppose $\operatorname{supp}\hat{f} = [0,\infty)$ and each σ_i' contains a sign change. Suppose there are integers i_0, j_0 and $\alpha \ge 0$ such that

$$x_{j_0} \notin \sigma_{i_0}', \; x_{j_0+\alpha+1} \notin \sigma_{i_0+\alpha}', \text{ and } x_{j_0+i} \in \sigma_{i_0+i-1}' \cap \sigma_{i_0+i}' \text{ for } i=1,\dots,\alpha. \quad (19)$$

Then part 2 of Lemma 5.1 implies that there is some $g \in V_1$ with support $(a,b) := \bigcup_{i=i_0}^{i_0+\alpha}\sigma_i'$ such that $(x_{j_0+1},\dots,x_{j_0+\alpha})$ is a sign change sequence for g on (a,b). Furthermore, as g does not vanish on any subinterval of (a,b), we have $\langle f,g\rangle \neq 0$.

We now show that there must exist an i_0 and j_0 so that (19) holds. We first show that there do not exist integers i_0 and j_0 such that such that

$$x_{i_0+l} \in \sigma'_{j_0+l-1} \cap \sigma'_{j_0+l} \tag{20}$$

for all $l \geq 1$. Suppose not then

$$\operatorname{card}\{x_j \in \bigcup_{l=1}^{2m} \sigma'_{j_0+l}\} \geq 2m. \tag{21}$$

Since f is in V_0 it follows from part one of Lemma 5.1 that there exists a constant c such that

$$m + c > \operatorname{card}\{x_j \in \bigcup_{l=1}^{2m} \sigma'_{j_0+l}\} \tag{22}$$

for all m. This contradicts (21) for m sufficiently large. Thus the above shows that there does not exist an infinite sign change sequence that satisfies (20).

We now rule out the case where (19) fails for each finite α. Let $0 = x_0 < x_1 < \cdots$ be an infinite sign change sequence of f on $(0, \infty)$, $n_0 = 1 - r$ and for each $i \geq n_0$ set $k(i) = \min\{j : x_j \in \sigma'_i\} - 1$. Assume that (19) does not hold. Since we have already ruled out the case (20) it must be that for each i there exists an $n \geq 1$ which depends upon i such that $x_{k(i)+l} \in \sigma'_{i+l-1} \cap \sigma'_{i+l}$, $l = 1, \ldots, n-1$, but $x_{k(i)+n} \notin \sigma'_{i+n-1} \cap \sigma'_{i+n}$ and $x_{k(i)+n} \in \sigma'_{i+n-1}$. Thus $x_{k(i)+n} < \inf \sigma'_{i+n}$ and so from the definition of $k(i+n)$ we find that $k(i+n) > (k(i)+n)-1$ or $k(i+n) \geq k(i)+n$. Thus for each infinite sign change sequence not satisfying (19) there exists an infinite sequence $n_0 < n_1 < \cdots$ such that $k(n_{i+1}) \geq k(n_i) + n_{i+1} - n_i$. Since $k(n_0) = 0$ we find that $k(n_l) \geq n_l - n_0$ for all l. From (21) and (22) above we find that for l large enough this leads to a contradiction. $\square$

Thus Q_∞, the orthogonal projection onto $V_1[0, \infty)$, is nonsingular on T_0^∞. Then Lemma 4.4 implies that (V_p) satisfies the hypotheses of Theorem 2.

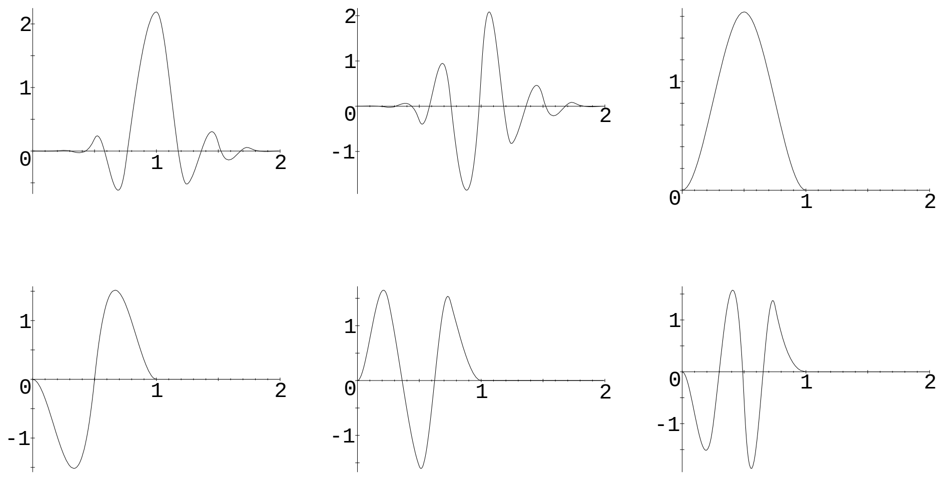
Theorem 3 *Let r, d, q be as above and let $V_0 = \mathcal{S}_{d,r}(2^{-q}) \cap L^2(\mathbf{R})$. Then there is some integer $\tilde{q}$ and some orthogonal multiresolution analysis $(\tilde{V}_p)$ such that*

$$V_{\tilde{q}} \subset \tilde{V}_0 \subset V_{\tilde{q}+1}.$$

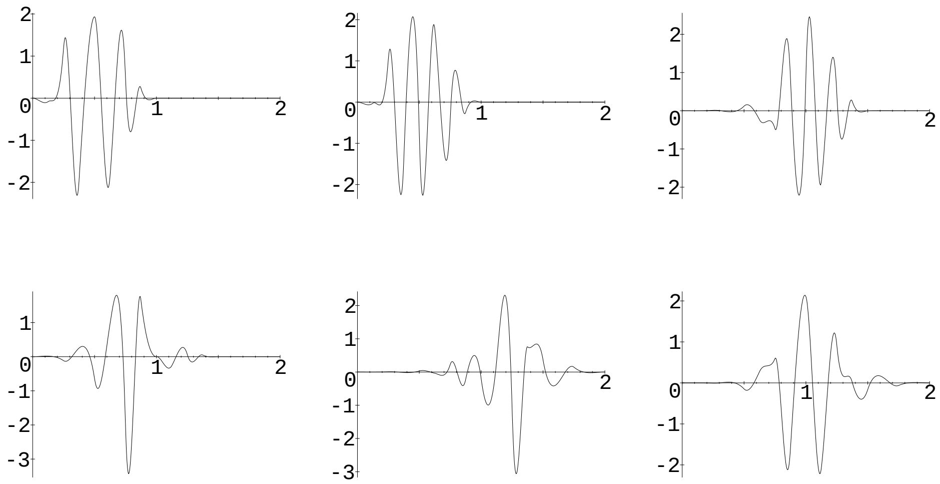
5.1 Example: $\mathcal{S}_{3,2}$—C^1 cubic scaling functions

In this case $L = 2$, $q = 0$ and V_0 is generated by the two B-splines associated with the knot sequences $(-1, -1, 0, 0, 1)$ and $(-1, 0, 0, 1, 1)$. As noted in the remark following Lemma 3.1 we will need $\mathcal{W}$ to have dimension at least 2. Now $\mathcal{A}(V_1) \ominus \mathcal{A}(V_0)$ is two dimensional which is not large enough to allow for any freedom in the choice of $\mathcal{W}$. Observe that $\mathcal{A}(V_2) \ominus \mathcal{A}(V_1)$ is $6 - 2 = 4$ dimensional.

Using Mathematica to aid in the calculations we find a 2-dimensional $\mathcal{W}$ that satisfies Equation (1). This gives a total of 6 scaling functions and 6 wavelets as shown in Figure 3.



Scaling Functions



Wavelets

Figure 3. Piecewise cubic C^1 orthogonal scaling functions (from right-to left and top to bottom $\tilde{\phi}^1(\cdot-1)$, $\tilde{\phi}^2(\cdot-1)$, $\tilde{\phi}^3$, $\tilde{\phi}^4$, $\tilde{\phi}^5$, $\tilde{\phi}^6$) and wavelets (from top-to-bottom and right-to-left $\tilde{\psi}^1$, $\tilde{\psi}^2$, $\tilde{\psi}^3(\cdot-1)$, $\tilde{\psi}^4(\cdot-1)$, $\tilde{\psi}^5(\cdot-1)$,$\tilde{\psi}^6(\cdot-1)$).

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Appendix 1: Interpolation values for scaling functions and wavelets

$\tilde{\phi}^1$	$\tilde{\phi}^2$	$\tilde{\phi}^3$	$\tilde{\psi}^1$	$\tilde{\psi}^2$	$\tilde{\psi}^3$
0	0	0	0	0	0
			$165-44\sqrt{5}$	$-3-2\sqrt{5}$	$6+4\sqrt{5}$
$3+2\sqrt{5}$	1	131	$242-352\sqrt{5}$	$-4\sqrt{5}-6$	$12+8\sqrt{5}$
			$-1265+836\sqrt{5}$	$10\sqrt{5}+15$	$-30-20\sqrt{5}$
$-18-12\sqrt{5}$	2	$150-96\sqrt{5}$	1716	$24\sqrt{5}+36$	$-72-48\sqrt{5}$
			$-1265-836\sqrt{5}$	$25+2\sqrt{5}$	$-38+4\sqrt{5}$
$5+18\sqrt{5}$	1	$-381+160\sqrt{5}$	$242+352\sqrt{5}$	$-84\sqrt{5}-82$	$92+120\sqrt{5}$
			$165+44\sqrt{5}$	$54\sqrt{5}-117$	$254-36\sqrt{5}$
132	0	0	0	264	0
				$-117-54\sqrt{5}$	$-369-125\sqrt{5}$
$5-18\sqrt{5}$				$-82+84\sqrt{5}$	$78+118\sqrt{5}$
				$25-2\sqrt{5}$	$53+17\sqrt{5}$
$-18+12\sqrt{5}$				$36-24\sqrt{5}$	$12-36\sqrt{5}$
				$-10\sqrt{5}+15$	$5-15\sqrt{5}$
$3-2\sqrt{5}$				$4\sqrt{5}-6$	$-2+6\sqrt{5}$
				$2\sqrt{5}-3$	$-1+3\sqrt{5}$
0				0	0
		Norms			
$4\sqrt{231}$	$\frac{2\sqrt{3}}{3}$	$32\sqrt{33}-10\sqrt{165}$	$44\sqrt{462}$	$8\sqrt{231}$	$28\sqrt{42}+4\sqrt{210}$

Interpolation values for piecewise linear scaling functions (at the quarter-integers) and wavelets (at the eighth-integers) of Example 3.1.

$\tilde{\phi}^1$	$\tilde{\phi}^2$	$\tilde{\phi}^3$	$\tilde{\phi}^4$	$\tilde{\psi}^1$	$\tilde{\psi}^2$	$\tilde{\psi}^3$	$\tilde{\psi}^4$
0	0	0	0	0	0	0	0
				$-93\sqrt{14}+294$	$98-31\sqrt{14}$	$-1079+288\sqrt{14}$	$127\sqrt{14}-476$
$1079-288\sqrt{14}$	1	5	$175-20\sqrt{14}$	$-58\sqrt{14}+84$	$-308+66\sqrt{14}$	$-2158+576\sqrt{14}$	$-952+254\sqrt{14}$
				$21\sqrt{14}+182$	$266-37\sqrt{14}$	$1307-304\sqrt{14}$	$588-131\sqrt{14}$
$-2386+592\sqrt{14}$	2	10	$98+24\sqrt{14}$	$228\sqrt{14}-224$	$336-12\sqrt{14}$	$4772-1184\sqrt{14}$	$2128-516\sqrt{14}$
				$101\sqrt{14}-518$	$-266+37\sqrt{14}$	$-280\sqrt{14}+965$	$-125\sqrt{14}+420$
$1421-312\sqrt{14}$	3	-1	$175-20\sqrt{14}$	$-154\sqrt{14}-308$	$-364-42\sqrt{14}$	$-2842+624\sqrt{14}$	$-1288+266\sqrt{14}$
				$-13\sqrt{14}-406$	$-98+31\sqrt{14}$	$-2105+360\sqrt{14}$	$-980+145\sqrt{14}$
$684-48\sqrt{14}$	4	-12	0	0	$672-24\sqrt{14}$	$-1368+96\sqrt{14}$	$-672+24\sqrt{14}$
				$13\sqrt{14}+406$	$-98+31\sqrt{14}$	$5737-1464\sqrt{14}$	$-387\sqrt{14}+1596$
$-2105+360\sqrt{14}$	3	-1	$-175+20\sqrt{14}$	$154\sqrt{14}+308$	$-364-42\sqrt{14}$	$-5334+1648\sqrt{14}$	$226\sqrt{14}-168$
				$-101\sqrt{14}+518$	$-266+37\sqrt{14}$	$8139-1208\sqrt{14}$	$-201\sqrt{14}+2548$
$-350-400\sqrt{14}$	2	10	$-98-24\sqrt{14}$	$-228\sqrt{14}+224$	$336-12\sqrt{14}$	$3436+608\sqrt{14}$	$396\sqrt{14}+1232$
				$-21\sqrt{14}-182$	$266-37\sqrt{14}$	$-10411+1792\sqrt{14}$	$433\sqrt{14}-2884$
$2341+48\sqrt{14}$	1	5	$-175+20\sqrt{14}$	$58\sqrt{14}-84$	$-308+66\sqrt{14}$	$-6082-1696\sqrt{14}$	$-554\sqrt{14}-2968$
				$93\sqrt{14}-294$	$98-31\sqrt{14}$	$-2553+816\sqrt{14}$	$235\sqrt{14}-3500$
$9576-672\sqrt{14}$	0	0	0	0	0	$19152-1344\sqrt{14}$	0
				Norms			
$1368\sqrt{7}-672\sqrt{2}$	$\frac{4\sqrt{3}}{3}$	$\frac{4\sqrt{21}}{3}$	$28\sqrt{21}-7\sqrt{6}$	$168\sqrt{7}-42\sqrt{2}$	$56\sqrt{21}-14\sqrt{6}$	$2736\sqrt{7}-1344\sqrt{2}$	$672\sqrt{14}-336$

Interpolation values for piecewise linear symmetric scaling functions (at the eighth-integers) and wavelets (at the sixteenth-integers) of Example 3.2. For $\tilde{\phi}^1$, $\tilde{\psi}^3$, and $\tilde{\psi}^4$ only half of the points are given. The others points may be computed using the fact that $\tilde{\phi}^1$ and $\tilde{\psi}^3$ are symmetric while $\tilde{\psi}^4$ is antisymmetric.

ϕ^1	0	0.01145260744	0.1941789118	-0.2276642461	2.188816056	-0.5189133117	0.1301883312	0.04941362556	0
$\phi^{1\prime}$	0	0.2283230026	4.395387791	14.37750265	0	1.209857886	-7.265110200	-1.032301156	0
ϕ^2	0	-0.02531744710	-0.2967038040	0.1905122953	0.6577284145	-0.7845469243	0.1871122709	0.07121519464	0
$\phi^{2\prime}$	0	-0.5135739653	-7.525014752	-24.97260290	45.02381581	2.403631242	-10.46198835	-1.487792616	0
ϕ^3	0	0.8204126541	1.640825308	0.8204126541	0	0	0	0	0
$\phi^{3\prime}$	0	4.922475925	0	-4.922475925	0	0	0	0	0
ϕ^4	0	-1.280868846	0	1.280868846	0	0	0	0	0
$\phi^{4\prime}$	0	-5.123475383	20.49390153	-5.123475383	0	0	0	0	0
ϕ^5	0	1.456996003	-1.563883833	1.456996003	0	0	0	0	0
$\phi^{5\prime}$	0	-10.35032509	-5.872827930	-8.442724283	0	0	0	0	0
ϕ^6	0	-1.025146541	-0.5127445138	1.276503733	0	0	0	0	0
$\phi^{6\prime}$	0	18.39424679	-50.39758685	-12.10638475	0	0	0	0	0

Interpolation values for piecewise cubic C^1 scaling functions of Example 4.1 and their derivatives at the quarter-integers.

Appendix 2: Matrix dilation coefficients

Suppose (V_p) is an orthogonal mulitresolution analysis generated by scaling functions $\phi^1, \ldots, \phi^r$ minimally supported on $[-1, 1]$. Let $\Phi = (\phi^1, \ldots, \phi^r)^\top$. Since $V_0 \subset V_1$ it follows that Φ must satisfy a two scale dilation equation with matrix coefficients:

$$\Phi(x) = \sqrt{2} \sum_{n=-2}^{1} H_n \Phi(2x - n) \tag{23}$$

where each H_n is an $r \times r$ real matrix. The support and orthogonality of the scaling functions imply the given summation limits. In the following tables we give H_{-2}, H_{-1}, H_0 and H_1 for the three examples given in the paper.

The wavelets $\Psi = (\psi^1, \ldots, \psi^r)^\top$ constructed from the scaling functions also satisfy a two scale dilation equation of the form;

$$\Psi(x) = \sqrt{2} \sum_{n=-2}^{1} K_n \Phi(2x - n). \tag{24}$$

These matrices are also given in the tables below.

H_{-2}	0	$-\sqrt{154}(3+2\sqrt{5})/3696$	$\sqrt{14}(2+5\sqrt{5})/1232$
	0	0	0
	0	0	0
H_{-1}	$-\sqrt{2}(3+2\sqrt{5})/44$	$\sqrt{154}(67+30\sqrt{5})/3696$	$\sqrt{14}(-10+\sqrt{5})/112$
	0	0	0
	0	0	0
H_0	$\sqrt{2}/2$	$\sqrt{154}(67-30\sqrt{5})/3696$	$\sqrt{14}(10+\sqrt{5})/112$
	0	$3\sqrt{2}/8$	$\sqrt{22}(-4+\sqrt{5})/88$
	0	$\sqrt{22}(32+7\sqrt{5})/264$	$\sqrt{2}(-5+4\sqrt{5})/88$
H_1	$\sqrt{2}(-3+2\sqrt{5})/44$	$\sqrt{154}(-3+2\sqrt{5})/3696$	$\sqrt{14}(-2+5\sqrt{5})/1232$
	$\sqrt{154}/22$	$3\sqrt{2}/8$	$\sqrt{22}(4+\sqrt{5})/88$
	$-\sqrt{70}/22$	$\sqrt{22}(-32+7\sqrt{5})/264$	$-\sqrt{2}(5+4\sqrt{5})/88$

K_{-2}	0	0	0
	0	$(3+2\sqrt{5})\sqrt{154})/3696$	$-(2+5\sqrt{5})\sqrt{14}/1232$
	0	$-\sqrt{7}(1+\sqrt{5})/336$	$\sqrt{77}(-1+3\sqrt{5})/1232$
K_{-1}	0	0	0
	$(2\sqrt{5}+3)\sqrt{2}/44$	$-\sqrt{154}(67+30\sqrt{5})/3696$	$(10-\sqrt{5})\sqrt{14}/112$
	$-\sqrt{11}(1+\sqrt{5})/44$	$\sqrt{7}(29+13\sqrt{5})/336$	$\sqrt{77}(-75+17\sqrt{5})/1232$
K_0	0	$\sqrt{77}(-2+\sqrt{5})/264$	$\sqrt{7}(13-6\sqrt{5})/88$
	$\sqrt{2}/2$	$\sqrt{154}(-67+30\sqrt{5})/3696$	$-(10+\sqrt{5})\sqrt{14}/112$
	0	$\sqrt{7}(-29+13\sqrt{5})/336$	$-\sqrt{77}(75+17\sqrt{5})/1232$
K_1	$13/22$	$-\sqrt{77}(\sqrt{5}+2)/264$	$-\sqrt{7}(13+6\sqrt{5})/88$
	$(3-2\sqrt{5})\sqrt{2}/44$	$\sqrt{154}(3-2\sqrt{5})/3696$	$(2-5\sqrt{5})\sqrt{14}/1232$
	$\sqrt{11}(1-\sqrt{5})/44$	$\sqrt{7}(1-\sqrt{5})/336$	$-\sqrt{77}(3\sqrt{5}+1)/1232$

Matrix dilation coefficients for the piecewise linear scaling functions and wavelets of Example 3.1

H_{-2}	0	$-\sqrt{6}(-2\sqrt{2}+\sqrt{7})/144$	$-\sqrt{6}(14\sqrt{14}-65)/1008$	$-\sqrt{6}/84$
	0	0	0	0
	0	0	0	0
	0	0	0	0
H_{-1}	$\sqrt{2}/28$	$\sqrt{6}\left(-2\sqrt{2}+\sqrt{7}\right)/144$	$\sqrt{6}\left(14\sqrt{14}+79\right)/1008$	$-13\sqrt{6}/84$
	0	0	0	0
	0	0	0	0
	0	0	0	0
H_0	$\sqrt{2}/2$	$\sqrt{6}\left(-2\sqrt{2}+\sqrt{7}\right)/144$	$\sqrt{6}\left(14\sqrt{14}+79\right)/1008$	$13\sqrt{6}/84$
	0	$3\sqrt{2}/8$	$3\sqrt{14}/56$	$-\sqrt{14}/14$
	0	$\sqrt{14}/8$	$-9\sqrt{2}/56$	$3\sqrt{2}/14$
	0	$\sqrt{2}\left(4\sqrt{7}+\sqrt{2}\right)/24$	$-\sqrt{2}\left(-4+\sqrt{2}\sqrt{7}\right)/24$	0
H_1	$\sqrt{2}/28$	$\sqrt{3}\left(4-\sqrt{14}\right)/144$	$\sqrt{3}\left(65\sqrt{2}-28\sqrt{7}\right)/1008$	$\sqrt{6}/84$
	$\sqrt{42}/14$	$3\sqrt{2}/8$	$3\sqrt{14}/56$	$\sqrt{14}/14$
	$-3\sqrt{6}/14$	$\sqrt{14}/8$	$-9\sqrt{2}/56$	$-3\sqrt{2}/14$
	0	$-1/12-\sqrt{14}/6$	$-(2\sqrt{2}+\sqrt{7})/12$	0

K_{-2}	0	0	0	0
	0	0	0	0
	0	$\sqrt{6}\left(-2\sqrt{2}+\sqrt{7}\right)/144$	$\sqrt{6}\left(14\sqrt{14}-65\right)/1008$	$\sqrt{6}/84$
	0	$\sqrt{6}\left(-4+\sqrt{14}\right)/144$	$\sqrt{6}\left(28\sqrt{7}-65\sqrt{2}\right)/1008$	$\sqrt{3}/42$
K_{-1}	0	0	0	0
	0	0	0	0
	$-\sqrt{2}/28$	$-\sqrt{6}\left(-2\sqrt{2}+\sqrt{7}\right)/144$	$-\sqrt{6}\left(14\sqrt{14}+79\right)/1008$	$13\sqrt{6}/84$
	$-1/14$	$-\sqrt{6}\left(-4+\sqrt{14}\right)/144$	$-\sqrt{6}\left(28\sqrt{7}+79\sqrt{2}\right)/1008$	$13\sqrt{3}/42$
K_0	0	$-\sqrt{6}\left(-2\sqrt{2}+\sqrt{7}\right)/36$	$-\sqrt{6}\left(2\sqrt{14}+1\right)/36$	$\sqrt{6}/6$
	0	0	$-2\sqrt{2}/7$	$3\sqrt{6}/14$
	$\sqrt{2}/2$	$-\sqrt{6}\left(-2\sqrt{2}+\sqrt{7}\right)/144$	$-\sqrt{6}\left(14\sqrt{14}+79\right)/1008$	$-13\sqrt{6}/84$
	0	$\sqrt{6}\left(-4+\sqrt{14}\right)/144$	$\sqrt{6}\left(28\sqrt{7}+79\sqrt{2}\right)/1008$	$13\sqrt{3}/42$
K_1	0	$\sqrt{6}\left(-2\sqrt{2}+\sqrt{7}\right)/36$	$\sqrt{6}\left(2\sqrt{14}+1\right)/36$	$\sqrt{6}/6$
	$2\sqrt{6}/7$	0	$-2\sqrt{2}/7$	$-3\sqrt{2}/14$
	$-\sqrt{2}/28$	$\sqrt{6}\left(-2\sqrt{2}+\sqrt{7}\right)/144$	$\sqrt{6}\left(14\sqrt{14}-65\right)/1008$	$-\sqrt{6}/84$
	$1/14$	$-\sqrt{6}\left(-4+\sqrt{14}\right)/144$	$-\sqrt{6}\left(28\sqrt{7}-65\sqrt{2}\right)/1008$	$\sqrt{3}/42$

Matrix dilation coefficients for the symmetric piecewise linear scaling functions and wavelets of Example 3.2.

	0	0	-0.0002208624742	-0.001924773722	-0.003288244991	-0.001947295571
	0	0	0.0005480992544	0.004856715145	0.008496591936	0.005147569977
	0	0	0	0	0	0
	0	0	0	0	0	0
H_{-2}	0	0	0	0	0	0
	0	0	0	0	0	0
	0.03702321115	0.02327794583	0.070258082635	0.3134392642	0.1926557751	0.06689140585
	-0.05789731181	-0.04178352399	-0.09405362704	-0.4199017293	-0.2020095784	-0.03215729927
	0	0	0	0	0	0
	0	0	0	0	0	0
H_{-1}	0	0	0	0	0	0
	0	0	0	0	0	0
	0.5	0	-0.01275849911	-0.1152392333	0.254509594	-0.144924649
	0.07876397049	0.25	-0.01928041017	-0.1741474187	0.3578324616	-0.1761829744
	0	0	0.3365384615	0.1901521826	0.1136121823	0.07235657604
	0	0	-0.4553644374	-0.15625	-0.07802085054	-0.04900446818
H_0	0	0	0.3801437869	-0.2233988914	-0.0859266686	-0.05492068733
	0	0	-0.1794572274	0.4142757111	0.156362129	0.09789851989
	0.04186152385	-0.03847597749	-0.001137235601	0.0103188138	-0.02303513663	0.01434079321
	0.06311599521	-0.05809141319	-0.001718550297	0.01559368852	-0.03481655981	0.02168109372
	0.3748202832	0	0.3365384615	-0.1901521826	0.1632992753	-0.1259679427
	-0.03419477048	0.1085355724	0.4553644374	-0.15625	0.1015281078	-0.06864716186
H_1	-0.3474452217	-0.03110245943	0.3428605425	0.04787803702	-0.1847362238	0.1397329126
	-0.03303819395	-0.2669053	0.2019668468	0.1235190581	-0.3007025809	0.2081776516

	0	0	0	0	0	0
	0	0	0	0	0	0
	0	0	−0.0003450330824	−0.003069463328	−0.005399552433	−0.003288096492
K_{-2}	0	0	0.003677598183	0.03314000602	0.05933009478	0.03671227884
	0	0	−0.0002078481375	−0.001876605867	−0.003368376617	−0.002089116052
	0	0	0.0001926474622	0.001660807040	0.002792288660	0.001627445103
	0	0	0	0	0	0
	0	0	0	0	0	0
	0.03130736939	0.02388558244	0.04706225017	0.2101944353	0.08540876058	0
K_{-1}	−0.1540863084	−0.1701046849	−0.07718035836	−0.3484642221	0.5489690651	0.7074414504
	0.007173437053	0.008891783345	0.0007343402982	0.003523980761	−0.04349900544	−0.04602920876
	−0.03995938295	−0.02391005881	−0.07939816359	−0.3541468171	−0.2303288536	−0.08854025638
	0	0	−0.02519015248	−0.2275254027	−0.4086088923	−0.2535449777
	0	0	−0.01623364889	−0.1466286491	−0.2633307378	−0.1634003680
	0	0.9112504122	0.00005668466137	0.0005119958472	−0.1797897710	0.2863652683
K_0	0	0.002424886694	0.00001844956122	0.0001666429418	−0.05851745969	0.09320534726
	0	−0.2035583274	−0.05337191372	−0.4820738207	0.1967490848	0.7817811565
	0.6982782210	−0.05639870192	0.02262092952	0.2043201597	−0.4452067894	0.2472916741
	0.6248862379	0	−0.001364675259	0.02429880239	−0.3319782450	0.4637528492
	0.02712562128	−0.8703879210	−0.002093878939	0.01024598177	0.1811356129	−0.3016012983
	−0.001149229936	0.0005230835228	0.000005181478463	−0.00004527100431	0.00006042145614	0
K_1	−0.0003740480676	0.0001702517267	0.000001686452768	−0.00001473467681	0.00001966580247	0
	0.1704375211	−0.1592437059	−0.004756707896	0.04316890544	−0.09656524863	0.06030056198
	−0.07418827490	0.06820633491	0.002016324222	−0.01829536575	0.04084291440	−0.02542850505

Matrix dilation coefficients for the C^1 piecewise cubic scaling functions and wavelets of Example 4.1.

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