Note Title 6/1/2015

<u>Example</u>: Find the inverse of $A = \begin{bmatrix} 1 & -1 & -2 \\ 2 & 4 & 5 \\ 6 & 0 & -3 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 6 & 0 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} \\ R_2-2R_1 \\ R_3-6R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 6 & 9 & -2 & 1 & 0 \\ 0 & 6 & 9 & -6 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} \\ R_2/6 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & -1/3 & 1/6 & 0 \\ 0 & 6 & 9 & -6 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} \\ \\ R_3-6R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & -1/3 & 1/6 & 0 \\ 0 & 0 & 0 & -4 & -1 & 1 \end{array}\right]$$

A does not have an inverse

Example: Find the inverse of $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 & 1 \end{array}\right]$$

$R_1 - R_3$
$R_2 - R_3$
$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & \boxed{1} & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_1 - R_2$
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \checkmark$$

Obs: $A \in \mathbb{R}^{n \times n}$. $Ax = 0$ has only one solution ($x = 0$) if and only if A has an inverse

Notation: A is non-singular = A has an inverse

A is singular = A does not have an inverse

Eigenvalues and eigenvectors

Def: Let $A \in \mathbb{R}^{n \times n}$. We say that v (an n-vector) is an eigenvector of A if there exists a number λ such that $Av = \lambda v$ and $v \neq 0$. λ is called an eigenvalue of A.

Example: $\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$ with eigenvalue 2

Finding eigenvalues and eigenvectors

$A v = \lambda v$

$A v = \lambda I v$ (because $I v = v$ for all vectors v)

(*) $(A - \lambda I) v = 0$. Thus, λ is an eigenvalue of A if there exists a vector $v \neq 0$ that satisfies (*). Thus, λ is an eigenvalue of A if and only if $A - \lambda I$ is singular if and only

$\boxed{\det(A-\lambda I)=0}$. In this case, the eigenvectors associated with λ are the non-zero solutions of (*).

<u>Steps to find eigenvalues and eigenvectors</u>:

1) Construct $P(\lambda)=\det(A-\lambda I)$. $P(\lambda)$ is called the characteristic polynomial of A. $P(\lambda)$ is a polynomial of degree n.

2) Find the solutions of $P(\lambda)=0$. These are the eigenvalues of A

3) For each eigenvalue λ, solve $(A-\lambda I)v=0$. These solutions (except for $v=0$) are the eigenvectors corresponding to λ.

Example: Find the eigenvalues and eigenvectors of $\begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$

$$\boxed{P(\lambda)} = \det\left(\begin{bmatrix} 1-\lambda & 2 & 1 \\ 6 & -1-\lambda & 0 \\ -1 & -2 & -1-\lambda \end{bmatrix}\right) = (1-\lambda)\det\left(\begin{bmatrix} -1-\lambda & 0 \\ -2 & -1-\lambda \end{bmatrix}\right) -$$

$$-2\det\left(\begin{bmatrix} 6 & 0 \\ -1 & -1-\lambda \end{bmatrix}\right) + \det\left(\begin{bmatrix} 6 & -1-\lambda \\ -1 & -2 \end{bmatrix}\right) = (1-\lambda)\left[(-1-\lambda)^2\right] - 2(6)(-1-\lambda)$$

$$+\left[6(-2) - (-1)(-1-\lambda)\right] = -\lambda^3 - \lambda^2 + 12\lambda = \boxed{-\lambda(\lambda-3)(\lambda+4)}$$

Eigenvalues: $\boxed{\lambda = -4,\ 0 \text{ and } 3}$

Eigenvectors: $(A-\lambda I)v=0$ $\quad A=\begin{bmatrix}1&2&1\\6&-1&0\\-1&-2&-1\end{bmatrix}$

$\boxed{\lambda=-4}$ $\quad A+4I=\begin{bmatrix}5&2&1\\6&3&0\\-1&-2&3\end{bmatrix}$

$$\begin{bmatrix}1&2/5&1/5\\0&3/5&-6/5\\0&-8/5&16/5\end{bmatrix}\quad \begin{bmatrix}\boxed{1}&0&1\\0&\boxed{1}&-2\\0&0&0\end{bmatrix}\quad v=\begin{bmatrix}-t\\2t\\t\end{bmatrix}=t\begin{bmatrix}-1\\2\\1\end{bmatrix}$$

$$\begin{bmatrix}\boxed{1}&2/5&1/5\\0&\boxed{1}&-2\\0&0&0\end{bmatrix}$$

$\boxed{\text{Eigenvector } v=\begin{bmatrix}-1\\2\\1\end{bmatrix} \text{ of eigenvalue } \lambda=-4}$

$\boxed{\lambda=0}$ $\quad \begin{bmatrix}1&2&1\\6&-1&0\\-1&-2&-1\end{bmatrix}\quad \begin{bmatrix}1&2&1\\0&-13&-6\\0&0&0\end{bmatrix}\quad \begin{bmatrix}1&0&1/13\\0&1&6/13\\0&0&0\end{bmatrix}$

$$\boxed{\text{Eigenvector } v = \begin{bmatrix} -1/13 \\ -6/13 \\ 1 \end{bmatrix} \text{ of eigenvalue } \lambda = 0}$$

$$\boxed{\lambda = 3} \quad A - 3I = \begin{bmatrix} -2 & 2 & 1 \\ 6 & -4 & 0 \\ -1 & -2 & -4 \end{bmatrix} \Bigg| \begin{bmatrix} 1 & -1 & -1/2 \\ 0 & 2 & 3 \\ 0 & -3 & -9/2 \end{bmatrix} \Bigg|$$

$$\begin{bmatrix} 1 & -1 & -1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix} \Bigg| \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boxed{\text{Eigenvector } v = \begin{bmatrix} -1 \\ -3/2 \\ 1 \end{bmatrix} \text{ of eigenvalue } \lambda = 3}$$

Example: Find the eigenvalues and eigenvectors of $\begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix}$

$$\boxed{P(\lambda) =} \det\left(\begin{bmatrix} 3-\lambda & 4 \\ -1 & 7-\lambda \end{bmatrix}\right) = (3-\lambda)(7-\lambda)+4 = \boxed{= (\lambda-5)^2}$$

Only one eigenvalue, $\boxed{\lambda = 5}$

Eigenvectors of $\lambda = 0$ $\quad A - 5I = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$

$\boxed{v = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ Eigenvector of eigenvalue } \lambda = 5}$

Example: $A = \begin{bmatrix} 9 & 1 & 1 \\ 1 & 9 & 1 \\ 1 & 1 & 9 \end{bmatrix}$

$$P(\lambda) = \det\left(\begin{bmatrix} 9-\lambda & 1 & 1 \\ 1 & 9-\lambda & 1 \\ 1 & 1 & 9-\lambda \end{bmatrix}\right) = -(\lambda-11)(\lambda-8)^2$$

Eigenvalues $\lambda_1 = 8$ and $\lambda_2 = 11$

<u>Eigenvectors $\lambda_1 = 8$</u> $\quad A - 8I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$v = \begin{bmatrix} -t_1 - t_2 \\ t_1 \\ t_2 \end{bmatrix} = t_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Eigenvectors $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ of eigenvalue $\lambda_1 = 8$

Eigenvectors of $\lambda_2 = 11$ $\quad A - 11I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & -3/2 & 3/2 \\ 0 & 3/2 & -3/2 \end{bmatrix} \Bigg| \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Bigg| \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenvector of eigenvalue $\lambda_2 = 11$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

<u>Complex numbers</u>

i is a new number

$i^2 = -1$

<u>Def</u>: a complex number is a number of the form $a+bi$, where a and b are real numbers

Example $2+3i$

Operations with complex numbers

Addition: $(a+bi)+(c+di) = (a+c)+(b+d)i$

Multiplication: $(a+bi)(c+di) = ac+adi+bic+bidi =$
$(ac-bd)+(ad+bc)i$

Examples: $(-2+3i)+(1-2i) = -1+i$

$(-2+3i)(1-2i) = -2+4i+3i+6 = 4+7i$

Def: 1) $\mathbb{C}$ denotes the set of all complex numbers

2) $z \in \mathbb{C}$, then $z = a+bi$ with a and $b \in \mathbb{R}$

a is called the real part of z and b is called the imaginary part of z

$$a = Re(z) \quad \text{and} \quad b = Im(z)$$

$$\boxed{z = Re(z) + i\, Im\, z}$$

<u>Obs</u>: $x^2+1=0$ does not have any real roots

$$x^2+1=0 \Rightarrow x^2=-1 \Rightarrow x=i \text{ or } x=-i$$

Every polynomial of degree two has at least one root if we allow complex numbers

<u>Roots of degree two polynomials</u>

$ax^2+bx+c=0 \qquad a,b,c\in\mathbb{R} \qquad a\neq 0$

$\Delta = b^2-4ac$

<u>Case 1:</u> $\Delta>0$. The roots are

$$x=\frac{-b\pm\sqrt{\Delta}}{2a}$$

<u>Case 2:</u> $\Delta=0$. Only one root $x=\dfrac{-b}{2a}$

<u>Case 3:</u> $\Delta<0$. The roots are

$$x=\frac{-b\pm i\sqrt{-\Delta}}{2a}$$