$S = \{ x \in \mathbb{R}^4 : \ x_1 + x_2 = 0 \ \text{and} \ 3x_2 - x_4 = 0 \}$

The set of solutions of
$$x_1 + x_2 = 0$$
$$3x_2 - x_4 = 0$$

Autonomous odes

These are odes where the independent variable does not appear explicitly

$$y^{(n)} = F(y^{(n-1)}, \ldots, y', y)$$

First order autonomous

$$y' = F(y)$$

Example: 1) $y' = 1 + y^2$ is 1^{st} order autonomous

2) $y' = xy$ is not autonomous

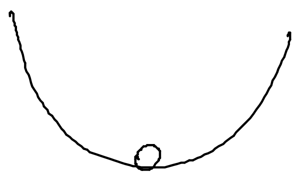
Notation: $\dot{x} = f(x)$ $\quad \dot{x} = \dfrac{dx}{dt}$ $\quad t =$ independent variable

Def: We say x_0 is a fixed point or a critical point of $\dot{x} = f(x)$ if $f(x_0) = 0$

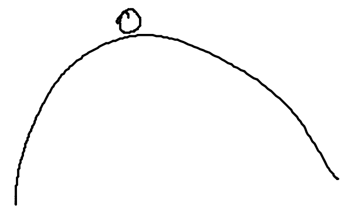
Obs: Let x_0 be a fixed point of $\dot{x} = f(x)$. Then, $x(t) = x_0$ for all t is a solution of $\dot{x} = f(x)$ because

$\frac{d}{dt} x_0 = 0$ and $f(x_0) = 0$.

Def: Fixed points are also called equilibrium solutions



stable equilibrium



unstable equilibrium

Def: Let x_0 be a fixed point of $\dot{x} = f(x)$.
We say that x_0 is stable if $x(t)$ remains close to x_0 as long as $x(0)$ was close to x_0. Otherwise, we say x_0 is unstable.

Solving first order autonomous odes

$$\frac{dx}{dt} = f(x)$$

$$\frac{dx}{f(x)} = dt$$

$$\int \frac{dx}{f(x)} = \int dt$$

then try to solve for x.

Examples: 1) $\dot{x} = kx$

$$\int \frac{dx}{x} = k \int dt$$

$$\boxed{x = A e^{kt}}$$

$A \in \mathbb{R}$

$$\int \frac{dx}{x} = k\int dt$$

$$\ln|x| = kt + C$$

$$|x| = e^{C} e^{kt}$$

---

2) $\dot{x} = x(1-x)$

$$\int \frac{dx}{x(1-x)} = \int dt$$

$$\boxed{\frac{1}{x(1-x)} = \frac{a}{x} + \frac{b}{1-x} = \frac{a(1-x)+bx}{x(1-x)} = \frac{x(b-a)+a}{x(1-x)}}$$

$a = 1$

$b = 1$

$$\int\left(\frac{1}{x} + \frac{1}{1-x}\right)dx = \int dt$$

$$\ln|x| - \ln|1-x| = t + C$$

$$\ln\left|\frac{x}{1-x}\right| = t + C$$

$$\left|\frac{x}{1-x}\right| = e^{C} e^{t}$$

$$\frac{x}{1-x} = Ae^{t} \qquad A \in \mathbb{R}$$

$$x = (1-x)Ae^{t}$$

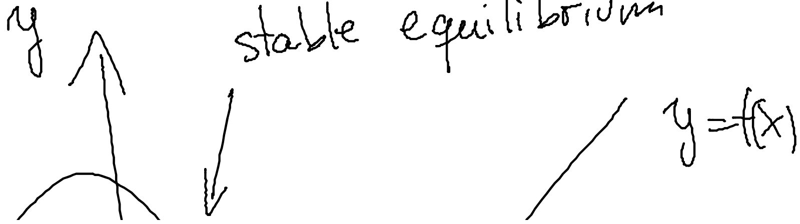
$$x(1+Ae^{t}) = Ae^{t}$$

$$\boxed{x = \frac{Ae^{t}}{1+Ae^{t}}}$$

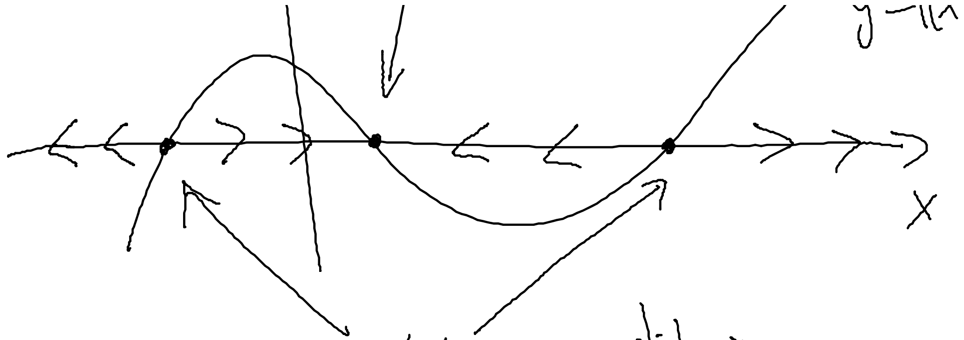
Phase line 1st order autonomous odes

$\dot{x} = f(x)$

Plot $y = f(x)$



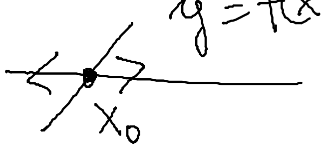
Plot $y = f(x)$



unstable equilibriums

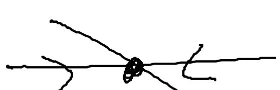
The zeros of $f(x)$ are the fixed points
the fixed points are located at the intersections of $y = f(x)$ and the x-axis.
the arrows indicate the direction of motion
this allow us to identify the stability of the equilibrium points

<u>Obs</u>: Let x_0 be a fixed point of $\dot{x} = f(x)$



If $f'(x_0) > 0 \Rightarrow x_0$ is unstable

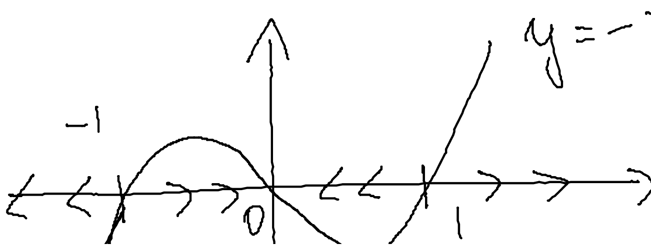
If $f'(x_0) < 0 \Rightarrow x_0$ is stable



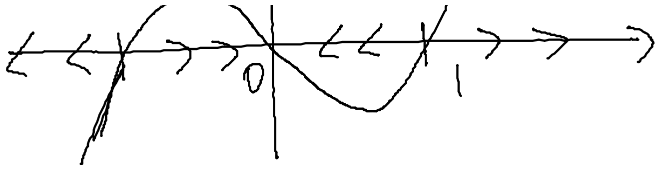
If $f'(x_0) = 0$, we do not know

<u>Example</u> $\dot{x} = -x + x^3$ $\quad f(x) = -x + x^3$

$y = -x + x^3 \quad f'(x) = -1 + 3x^2$



fixed point x_0	$f'(x_0)$	stability



point x_0	$f'(x_0)$	stability
-1	2	unstable
0	-1	stable
1	2	unstable

Separable equations

$$\frac{dy}{dx} = g(x)\, h(y)$$

$$\int \frac{dy}{h(y)} = \int g(x)\, dx \qquad \text{try to solve for } y$$

Example $\quad \dfrac{dy}{dx} = -\dfrac{x}{y} \qquad y(4) = 3$

$$\int y\, dy = -\int x\, dx \qquad \frac{y^2}{2} = -\frac{x^2}{2} + C$$

plug $x=4$ $y=3$ $\qquad \dfrac{9}{2} = -\dfrac{16}{2} + C \qquad C = \dfrac{25}{2}$

$$y^2 = 25 - x^2 \qquad \boxed{y = \sqrt{25 - x^2}}$$

Linear first order differential equations

(1) $\quad y' + P(x)\, y = f(x)$

(1) $y' + P(x)\,y = f(x)$

Integrating factor $I(x) = e^{\int P(x)dx}$

$I'(x) = P(x)\, e^{\int P(x) dx} = P(x)\, I(x)$

Multiply (1) by $I(x)$

$I y' + I P y = I f$

$I y' + I' y = I f$

$(I y)' = I f$

$I(x)\, y(x) = \int I(x) f(x)\, dx$

$$\boxed{y(x) = \frac{1}{I(x)} \int I(x) f(x)\, dx}$$

Examples 1) $y' - 3y = 6$

$P(x) = -3 \qquad I(x) = e^{\int P(x)dx} = e^{-3x}$

$e^{-3x} y' - 3 e^{-3x} y = 6 e^{-3x}$

$\left(e^{-3x} y\right)' = 6 e^{-3x}$

$e^{-3x} y = -2 e^{-3x} + C$

$$\boxed{y = -2 + C e^{3x}}$$

2) $x y' - 4y = x^6 e^x$

Devide by x

$y' - \frac{4}{x} y = x^5 e^x$

$\int -\frac{4}{x} dx \qquad e^{-4 \ln|x|}$

$y' - \frac{4}{x} y = \ldots$

$P(x) = -\frac{4}{x} \quad I = e^{\int -\frac{4}{x} dx} = e^{-4\ln|x|} = \frac{1}{|x|^4} = \frac{1}{x^4}$

$\frac{1}{x^4} y' - \frac{4}{x^5} y = x e^x$

$\left(\frac{y}{x^4}\right)' = x e^x$

$\frac{y}{x^4} = \int x e^x dx = x e^x - \int e^x dx$

$= x e^x - e^x + C$

$\boxed{y = x^5 e^x - x^4 e^x + C x^4}$

Complex valued functions

$f(x) = f_1(x) + i f_2(x)$ where $x \in \mathbb{R}$ & $f_1(x)$ & $f_2(x)$ are real valued functions.

Def: $a, b \in \mathbb{R}$ $z = a+bi$

$e^z = e^{a+bi} = e^a (\cos b + i \sin b)$

Ex: $e^{i\pi} = e^0 (\cos\pi + i \sin\pi) = -1$

Properties: z_1 & $z_2 \in \mathbb{C}$

$e^{z_1+z_2} = e^{z_1} e^{z_2}$

$e^{z_1-z_2} = \frac{e^{z_1}}{e^{z_2}} = e^{z_1} e^{-z_2}$

Ex: $\lambda = a+bi$ $a, b \in \mathbb{R}$

Ex: $\lambda = a+bi \quad a, b \in \mathbb{R}$

$$\boxed{f(x) = e^{\lambda x} = e^{ax+ibx} = e^{ax}\cos(bx) + i e^{ax}\sin bx}$$

$f_1(x) = e^{ax}\cos bx \qquad f_2(x) = e^{ax}\sin bx$

$f_1(x) = \operatorname{Re} f(x) \qquad f_2(x) = \operatorname{Im} f(x)$

Derivatives of complex valued functions

$f(x) = f_1(x) + i f_2(x) \qquad f_1 \;\&\; f_2$ real valued.

$$\boxed{f'(x) = f_1'(x) + i f_2'(x)}$$

Example $\lambda = a+bi \qquad a, b \in \mathbb{R}$

$$\boxed{\frac{d}{dx} e^{\lambda x}} = \frac{d}{dx}\left[e^{ax}\cos(bx) + i e^{ax}\sin(bx)\right] =$$

$$e^{ax}\left[a\cos(bx) - b\sin(bx)\right] + i\, e^{ax}\left[a\sin(bx) + b\cos(bx)\right] =$$

$$(a+bi)\left[e^{ax}\cos(bx) + i e^{ax}\sin(bx)\right] = \boxed{\lambda e^{\lambda x}}$$

Linear homogeneous constant coefficient odes

(1) $\quad y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$

$a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$

Obs: $f(x) = f_1(x) + i f_2(x)$ f_1 & f_2 real valued.

$f(x)$ is a solution of (1) $\Longleftrightarrow$ $f_1(x)$ & $f_2(x)$ are solutions of (1)

Looking for solutions of (1)

Set $y = e^{\lambda x}$. Plug into (1)

$$\lambda^n e^{\lambda x} + a_{n-1}\lambda^{n-1} e^{\lambda x} + \cdots + a_1 \lambda e^{\lambda x} + a_0 e^{\lambda x} = 0$$

$$P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

$P(\lambda)$ is called the characteristic polynomial.

Obs: $e^{\lambda x}$ is a solution of (1) $\Longleftrightarrow$ $P(\lambda) = 0$

Goal: Find n linearly independent solutions of (1)
real

Method: 1) Construct $P(\lambda)$.
2) Find the roots of $P(\lambda)$
3) Factorize $P(\lambda)$

$$P(\lambda) = (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_r)^{n_r}$$

$\lambda_l \neq \lambda_j$ if $l \neq j$.

$n_\ell \geq 1$ is the multiplicity of λ_ℓ as root of $r(\lambda)$

Note: $n_1 + n_2 + \cdots + n_r = n$

n linearly independent solutions (complex valued) of (1) are

$$
\begin{array}{cccc}
e^{\lambda_1 x} & e^{\lambda_2 x} & \cdots & e^{\lambda_r x} \\
x e^{\lambda_1 x} & x e^{\lambda_2 x} & & x e^{\lambda_r x} \\
\vdots & \vdots & & \vdots \\
x^{n_1 - 1} e^{\lambda_1 x} & x^{n_2 - 1} e^{\lambda_2 x} & \cdots & x^{n_r - 1} e^{\lambda_r x}
\end{array}
$$

If we have complex roots, they come in pairs of complex conjugates. Keep only one root per pair. Take the real and imaginary parts of the solutions you are left with to end up with n linearly independent real valued solutions of eq (1)