Plugging of porous media and filters: Maximum clogged porosity

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We model porous media as two-dimensional networks of channels. As suspension flows through the network, particles clog channels. Assuming no flow through clogged channels, we determine an upper bound on the number of channels that clog and leave the network impervious. More precisely, the number of channels that clog does not exceed the number of faces of the multigraph associated with the network. In terms of d, the average coordination number of the network, the number of channels that clog does not exceed $(1-2/d)$ of the total number of channels. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2883947]

A porous medium consists of interconnected pores (fluid filled spaces) within a solid matrix. The medium is pervious or permeable if fluid can flow from one side to the opposite side of the medium through its pores. The small particles suspended in the fluid within a porous medium are called fines. When a suspension (fluid with fines in suspension) flows through the medium, some of these fines may become trapped at pore throats (narrow sections of the pores) and the medium experiences loss in permeability.

Filters are pervious porous materials designed to selectively trap migrating fines. Particle removal from fluids is of critical importance in a wide range of natural processes (examples of biological filters include kidneys and lungs) and industrial applications (e.g., water treatment, refinement processes, and oil recovery).[1]

Fines migration and entrapment can be analyzed using continuum macroscale models[2] or pore-scale models.[3] Within the later group, the class of network models[4] is widely used. Our work belongs to this class of models. Works where network models were used to study particle transport and clogging in porous media include.[5] Network models have also been used to study other transport phenomena in porous media (see Ref. 6 and references therein).

We start reviewing concepts of graph theory (see details in Ref. 7). A *multigraph* G consists of a nonempty set of elements, called *vertices* or *nodes*, and a list of unordered pairs of these elements, called *edges*. It is convenient and a common practice to draw graphs on the plane. Each node is a different point and each edge a line joining its two nodes without intersecting any other node. If e is an edge joining the nodes a and b, we say that a and b are the end points of e and that e connects a and b. If $a=b$, i.e., the end points of an edge e are the same, we say that e is a loop. In a multigraph, two different edges can have the same end points. If in a given multigraph G any two different edges do not have the same pair of end points, we say that G is a *graph*.[7]

We say that two nodes a and b are connected if there exists a sequence of nodes $n_0, n_1, \ldots, n_k$ such that $a=n_0$, $b=n_k$ and for each $1 \leqslant i \leqslant k$ there exists an edge e_i that connects n_{i-1} and n_i. In this case, the alternating sequence of nodes and edges $n_0, e_1, n_1, e_2, n_2, \ldots, e_k, n_k$ forms a *walk* between a and b and $a=n_0$ and $b=n_k$ are the end points of the walk. If $n_i \neq n_j$ for all $i \neq j$, the walk is a *path*. If $n_0=n_k$ and $n_i \neq n_j$ for $i<j$ except when $(i,j)=(0,k)$, the walk is a *cycle*. We identify each walk with the trajectory it traces on the plane.

A multigraph is *connected* if there is a walk between any pair of its nodes. Every multigraph is the union of disjoint connected multigraphs called *connected components*. A multigraph is *planar* if it can be drawn on the plane in such a way that any two different edges may only intersect at one or two of their end points. Any such drawing is a *plane drawing* of the multigraph. In this letter, we only need to consider planar multigraphs. We identify each planar multigraph with one of its plane drawings. We will consider only plane multigraphs. Thus, in the rest of this letter, any multigraph that we mention or consider is a planar multigraph (accordingly, any drawing is a planar drawing).

A multigraph divides the plane into regions called *faces*, i.e., the faces are the connected components left from the plane once we remove the drawing of the multigraph from the plane. Note that the faces are open sets. Any finite multigraph has exactly one unbounded face surrounding it. The boundary of a bounded face contains a cycle. In particular, a finite connected multigraph with no cycles has only one face, its unbounded face. If G is a multigraph, we denote by n_G, e_G, f_G, and ℓ_G its number of nodes, edges, faces, and connected components, respectively. The well known *Euler formula*[7] states that $n_G+f_G=e_G+\ell_G+1$.

We now describe our modeling assumptions. We model filters as two-dimensional networks of channels, as we illustrate in Fig. 1. The voids are the interior of the channels. To each filter, we associate a multigraph in a natural way, see Fig. 1. The edges are the channels and the nodes the end

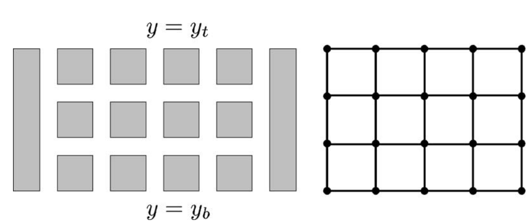

FIG. 1. Network of channels and associated multigraph G. The black circles are the nodes of the multigraph G and the solid lines its edges.

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points of the edges. The bottom and top boundaries of the filter are located at $y=y_b$ and $y=y_t$, respectively. Thus, the multigraph is included in $y_b \leq y \leq y_t$. Note that there are nodes in the bottom and top boundaries. For convenience, we also include edges in the $y=y_b$ and $y=y_b$ boundaries, as shown in Fig. 1. Precisely, there is a path of edges in $y=y_b$ connecting the left most node in the bottom boundary with the right most node in that boundary. We also include edges in $y=y_t$ analogously. We refer to the nodes and edges included in bottom or top boundaries as *exterior nodes* and *exterior edges*, respectively. The other nodes and edges are called *interior nodes* and *edges*. By construction, each exterior node is the end point of a least one interior edge. The solid matrix is rigid, i.e., the channels cannot be deformed.

In our model, edges are either open or clogged. Suspension can only flow through open edges. There is no flow through clogged edges. Within an open edge, suspension flows from the end point with higher pressure to the opposite end point. If both end points are at the same pressure, there is no flow through the edge. We assume that suspension can only flow into the filter through the bottom boundary, and can flow out of the filter only through the top boundary. Both fluid and particles are incompressible and thus, volume of suspension enters the filter through the bottom boundary at the same rate it exits the filter through the top boundary. We assume that the bottom boundary is held at constant pressure $p=p_b$ and the top boundary at $p=p_t$, where $p_b > p_t$. Note that the filter is permeable if and only if there is a path of open edges connecting the bottom boundary with the top boundary. Due to the difference in pressure between the top and bottom boundaries, there is flow through the filter if and only if it is permeable.

Initially, all the edges are open. As the suspension flows through the filter, particles are trapped causing edges to change from open to clogged. A key assumption is that different edges do not clog simultaneously. Note that an open channel can only clog if there is flow through it. Since there is no pressure difference between the end points of any exterior edge—at the upper and lower boundaries—exterior edges never clog.

We say that a sequence of edges $e_1, e_2, \ldots, e_s$ is a *feasible clogging sequence* if, for each i, there is flow through the edge e_i when $e_1, e_2, \ldots, e_{i-1}$ are clogged and all other edges of the network are open.

The above definition can be described in more mathematical terms as follows: assuming that the average velocity of the fluid within an edge is proportional to the difference between the pressure at the end points of that edge, mass conservation at a given node a leads to the equation $\Sigma k_e(p_b - p_a)=0$, where the sum is over all edges e that are open and have a as an end point, and b is the end point of e that is not a. In that equation, k_e is a constant that depends on the geometry of the channel (proportional to the conductivity of the channel). Then, $e_1, e_2, \ldots, e_s$ is a feasible clogging sequence if and only if, all these edges are different and, for each i, the pressures at the ends of e_i are different, where the pressures at the nodes are found solving the above system when only $e_1, e_2, \ldots, e_{i-1}$ are clogged and subjected to the boundary condition that the pressures at the nodes in the bottom boundary are p_b and at the nodes in top boundary are p_t (with $p_b > p_t$ any pair of fixed numbers).

According to our modeling assumptions, if $e_1, e_2, \ldots, e_s$ are the edges that actually clog and they do so in that order

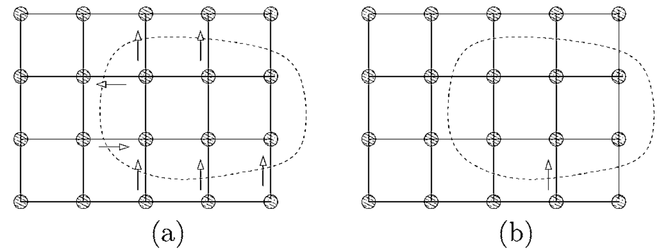

FIG. 2. $\partial\Omega$ is in dashed lines. The arrows next to the edges that intersect $\partial\Omega$ indicate the flow direction. No arrow next to an edge that intersects $\partial\Omega$ means that there is no flow through that edge. A plausible example is shown in (a). The situation of (b) is impossible since we would have a net flow into Ω.

(e_1 first, e_2 second, etc.), then $e_1, e_2, \ldots, e_s$ is a feasible clogging sequence. While there are many feasible clogging sequences that make the filter nonpermeable, only one actually realizes. The flow conditions, conductivity of the channels as well as other factors determine the feasible clogging sequence that realizes, which generally has less edges than other feasible clogging sequences. It is not our goal to find the sequence that realizes. Below, we will obtain an upper bound on the number of edges in any feasible clogging sequence and, in particular, in the number of edges that do clog (i.e., the number of edges in the sequence that does realize).

Next, we proceed with the analysis leading to our bound. The following observation will play a key role in our analysis. Here, as in the rest of this letter, G is a fixed multigraph that corresponds to one of our filters, such as the one in Fig. 1. We work at a fixed time and assume that certain edges $e_1, e_2, \ldots, e_s$ have clogged. In fact, as discussed before, we will only use the fact that $e_1, e_2, \ldots, e_s$ is a feasible clogging sequence.

Observation 1: Let Ω be an open bounded set in the plane such that (1) any edge of G intersects $\partial\Omega$, the boundary of Ω, at most at one point, (2) if an edge e intersects $\partial\Omega$, then an end point of e is in Ω and the other outside Ω, (3) Ω does not contain any of the exterior nodes of G, and (4) $\partial\Omega$ does not contain any node of G. Then, the following hold: (1) The rate at which suspension flows into Ω is equal to the rate at which suspension flows out of Ω. (2) Let e be an edge that intersects $\partial\Omega$. If all the other edges that intersect $\partial\Omega$ are clogged, then there is no flow through e. (3) If there is an edge that intersects $\partial\Omega$, then there is an edge that intersects $\partial\Omega$ that is not clogged.

This observation is illustrated in Fig. 2. Suspension enters Ω through some edges that intersect $\partial\Omega$ and leaves Ω through some other edges [see Fig. 2(a)]. Point (1) of Observation 1 is mass conservation coupled with the fact that the fluid and particles are incompressible. Point (2) is a particular case of point (1) [see Fig. 2(b)]. If all the edges that intersect $\partial\Omega$ but one are clogged, there cannot be flow through the open edge since otherwise point(1) would be violated. Point(3) is a consequence of point(2). Assume that all the edges that intersect $\partial\Omega$ clog. Let e be the last of the edges that intersect $\partial\Omega$ to clog. Once the rest of the edges are clogged, point(2) implies that there is no more flow through e. Thus, e cannot clog since one of our modeling assumptions is that an edge cannot clog if there is no flow through it.

We now construct a multigraph $C^\star$ that is associated with the set of clogged edges. This construction has its similarities with standard duality but also its differences. Note that the bounded connected components of the set $\{y_b < y$

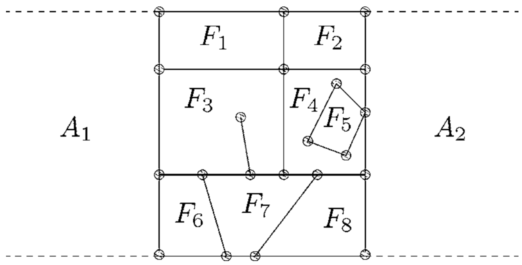

FIG. 3. The edges of G are the solid lines. The dashed lines are not part of G. F_i $(1 \leqslant i \leqslant 8)$ and A_1 and A_2 are the connected components of $\{y_b < y < y_t\} - G$.

$< y_t\} - G$ are the bounded faces of G (where $\{y_b < y < y_t\} - G$ is obtained by removing the edges and nodes of G from $\{y_b < y < y_t\}$). In addition, $\{y_b < y < y_t\} - G$ has two unbounded connected components, one to the left of G and the other to its right. An example is shown in Fig. 3.

Select a point inside each connected components of the set $\{y_b < y < y_t\} - G$. Call $\mathcal{N}^\star$ this set of points. For each edge of G that is clogged, draw exactly one edge of $C^\star$ as follows: let e be a clogged edge of G. It can be shown that e is included in the boundary of two connected components of $\{y_b < y < y_t\} - G$. Let $a^\star$ and $b^\star$ be the points of $\mathcal{N}^\star$ that are included in these components. We draw exactly one edge $e^\star$ of $C^\star$ connecting $a^\star$ and $b^\star$ in such a way that $e^\star$ intersects e in exactly one point and $e^\star$ does not intersect any other edge of G. This construction is carried out in such a way that edges of $C^\star$ may only intersect at their end points. The nodes of $C^\star$ are the end points of the edges in $C^\star$. Note that the set of nodes of $C^\star$ is a subset of $\mathcal{N}^\star$. In Fig. 4, we show an example of a set of clogged edges and the associated $C^\star$.

Observation 2: *$C^\star$ has only one face*.

Since every multigraph has exactly one unbounded face, to prove Observation 2, we need to show that $C^\star$ does not have any bounded faces. We proceed by contradiction. Assume Ω is a bounded face of $C^\star$. From the definition of $C^\star$, the edges of G that intersect $C^\star$ are clogged. Thus, all the edges of G that intersect $\partial\Omega$ are clogged. Note also that Ω satisfies the conditions in Observation 1 and the number of edges of G that intersect $\partial\Omega$ is equal to the number of edges of $C^\star$ in $\partial\Omega$, and this is a positive number. However, this is a contradiction because point 3 in Observation 1 states that there should be at least one edge that intersects $\partial\Omega$ and that is not clogged.

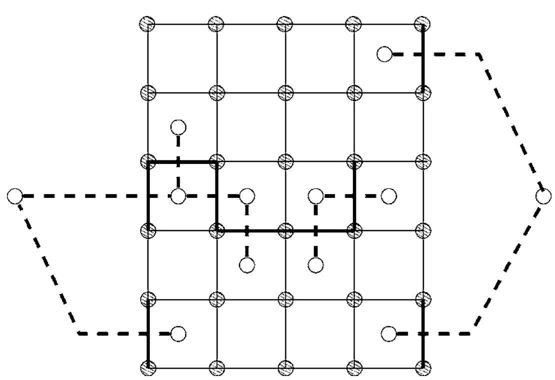

FIG. 4. The clogged edges are the thick solid lines. The open edged in thin solid lines. The thick dashed lines are the edges in $C^\star$. The white circles are the nodes of $C^\star$.

We are now ready to obtain the sought upper bound. Let $n_{C^\star}$, $e_{C^\star}$, $f_{C^\star}$, and $\ell_{C^\star}$ be the number of nodes, edges, faces, and connected components of $C^\star$. From Observation 2, we have $f_{C^\star}=1$. Since, in addition, $\ell_{C^\star} \geqslant 1$, Euler's formula, $e_{C^\star}=n_{C^\star}+f_{C^\star}-\ell_{C^\star}-1$ implies $e_{C^\star} \leqslant n_{C^\star}-1$. It is also easy to see that $e_{C^\star}$ is the number of clogged edges and $n_{C^\star} \leqslant$ number of connected components of $\{y_b < y < y_t\} - G = 1 +$ number of faces of G, and thus, we have the main theorem of this letter.

Theorem 1. Let G be the multigraph associated to a filter. Then, the number of clogged edges is less or equal to number of faces of G.

The degree d_n of a node n is the number of edges that have n as end point, where the loops are counted twice. The average degree d_G of a multigraph G is the average of the degrees d_n of the nodes of G. Since $d_G = 2e_G/n_G$, where e_G and n_G are the number of edges and nodes of G, respectively, (see Ref. 7), Euler's formula allow us to rewrite the upper bound as the number of clogged edges $\lesssim [(d_G-2)/d_G]e_G$, when $e_G \gg 1$ and $d_G > 2$.

In many situations of interest, G is a graph, i.e., no two edges have the same end points. For example, if all the edges in a multigraph are straight segments, then the multigraph is really a graph. It is a well known fact from graph theory that, if G is a planar graph, the average degree of G is bounded by 6, i.e., $d_G \leqslant 6$. Thus, we have that, if G is a graph, $e_G \gg 1$ and $d_G > 2$, then, the number of clogged edges $\lesssim (2/3)$ the number of all edges.

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