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Differential Geometry I
Fall 2006, Georgia Tech

Lecture Notes 11

Orientability

Any ordered basis $(b_1, \dots, b_n)$ of $\mathbf{R}^n$ may be viewed as a matrix $B \in GL(n)$ whose i^{th} column is b_i. Thus the set of ordered basis of $\mathbf{R}^n$ are in one-to-one correspondence with elements of $GL(n)$, and so we may partition them into two subsets: those whose corresponding matrices have positive determinant, $GL(n)^+$, and those whose corresponding matrices have negative determinant, $GL(n)^-$. By an *orientation* for $\mathbf{R}^n$ we mean the choice of $GL(n)^+$ or $GL(n)^-$ as the preferred source for picking a basis for $\mathbf{R}^n$. The *standard* orientation for $\mathbf{R}^n$ is that determined by $GL(n)^+$.

Lemma 0.0.1. *$GL(n)$ has exactly two path-connected components: $GL(n)^+$ and $GL(n)^-$.*

Proof. First note that since $\det\colon GL(n) \to \mathbf{R}^n - \{0\}$ is continuous, and $\mathbf{R} - \{0\}$ is not connected, then $GL(n)$ is not connected. So it must have at least two components $GL(n)^+ = \det^{-1}((0,\infty))$, and $GL(n)^- = \det^{-1}((-\infty,0))$. Secondly note that multiplying the first row of any element of $GL(n)^+$ yields an element of $GL(n)^-$. This implies that $GL(n)^+$ and $GL(n)^-$ are homeomorphic. So we just need to check that $GL(n)^+$ is path connected. This may be achieved in two steps: first we deform each element of $GL(n)^+$ to an element of $SO(n)$ and then show that $SO(n)$ is path connected.

Step 1: This may be achieved with the aid of the Gram-Schmidt process. In particular recall that if $(b_1, \dots, b_n)$ is any basis of $\mathbf{R}^n$ and $(b'_1, \dots, b'_n)$ is the corresponding orthonormalizaion of it, then b'_i is never antiparallel to b_i and each pair b_i and b'_i span a subspace which does not include any other of the elements of the basis. Thus $tb_i + (1-t)b'_i$ yields a continuous deformation of $(b_1, \dots, b_n)$ to $(b'_1, \dots, b'_n)$ through a family of basis.

Step 2: Let $B \in SO(n+1)$, then we may write $B = (b_1, \dots, b_n)$ where b_i are the columns of B. We may rotate B until b_n coincides with $(0, \dots, 0, 1)$. Then $b_1, \dots, b_{n-1}$ lie in $\mathbf{R}^{n-1} \times \{0\}$, and by a rotation in $\mathbf{R}^{n-1}$ we may bring b_{n-1} in coincidence with $(0, \dots, 0, 1, 0)$. Continuing this procedure we may continuously deform $(b_1, \dots, b_n)$ through a family of orthonormal basis until $b_2, \dots, b_n$ coincide with the last $n-1$ elements of the standard basis of $\mathbf{R}^n$. At that point we must either have $b_1 = (1, 0, \dots, 0)$ or $b_1 = (-1, 0, \dots, 0)$. But the latter is impossible

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because B has positive determinant, and continuous deformations of it through a family of basis preserve the sign of the determinant. So $b_1 = (1, 0, \dots, 0)$ and we are done.

$\square$

This immediately yields that

Corollary 0.0.2. *Two basis of* $\mathbf{R}^n$ *can be continuously transformed to each other, through a family of basis, if and only if they belong to the same orientation class.*

$\square$

We may also define an orientation for an abstract finite dimensional vector space V over $\mathbf{R}$: we say that a pair of ordered basis of V are equivalent provided that their image, under an isomorphism $f\colon V \to \mathbf{R}^n$ belong to the same orientation class. This is an equivalence relation which partitions the set of ordered basis of V into two subsets which we call orientation classes of V.

Exercise 0.0.3. Check that the notion of the orientation for an abstract vector space V is well defined, i.e., it does not depend on the choice of the isomorphism $f\colon V \to \mathbf{R}^n$.

We say that a smooth manifold M is *orientable* provided that we can orient each tangent space of M in *a continuos way*, i.e., we can choose an orientation class for each tangent space, which satisfies the following property: for each $p \in M$ there exists an open neighborhood U and linearly independent vector fields $X_1, \dots, X_n$ on U such that, for every $q \in U$, $(X_1(q), \dots X_n(q))$ belongs to the orientation class of T_qM. If M is an orientable manifold then by an orientation of M we mean a continuous choice of orientations for the tangent planes of M.

Let M and N be oriented manifolds. We say that a diffeomorphism $f\colon M \to N$ is orientation preserving provided df_p preserves orientation at each point p of M, i.e., whenever $(X_1, \dots X_n)$ is in the orientation class of T_pM then $(df_p(X_1), \dots df_p(X_n))$ is in the orientation class of $T_{f(p)}N$.

Lemma 0.0.4. *If* M *is orientable, then at each point* $p \in M$ *there exists a local chart* (U, ϕ) *such that* $\phi\colon U \to \mathbf{R}^n$ *is orientation preserving.*

Proof. Suppose that M is orientable, and let (U, ϕ) be a local chart of M, where we choose U sufficiently small so that there exist continuous vector fields $(X_1, \dots, X_n)$ which lie in the orientation class of M at each point of U. Then $(d\phi(X_1), \dots, d\phi(X_n))$ is a continuous basis for $\mathbf{R}^n$ and thus $(d\phi(X_1)(q), \dots, d\phi(X_n)(q))$ lies in the same orientation class of $\mathbf{R}^n$ for all $q \in U$. After replacing ϕ by a composition of ϕ with a reflection in $\mathbf{R}^n$ if necessary, we may assume that $(d\phi(X_1)(q), \dots, d\phi(X_n)(q))$ all lie in the standard orientation class of $\mathbf{R}^n$. $\square$

Corollary 0.0.5. *Every connected orientable manifold admits exactly two different orientations.*

Proof. Suppose we are given two orientations for a connected manifold M. Let $A \subset M$ be the set of points p such that these two orientations agree on T_pM. By the previous lemma there exists a local chart (U, ϕ) centered at p such that $\phi \colon U \to \mathbf{R}^n$ is orientation preserving for each of the orientations classes of M. This implies that A is open. Similarly, it can be shown that $M - A$ is open as well. So, since M is connected, we either have $A = M$ or $A = \emptyset$. So any two orientations of M must either agree on every tangent space, or be different on every tangent space. Since each tangent space can be oriented in exactly two different ways, it follows then that M also admits exactly two different orientation classes. $\square$

Proposition 0.0.6. *M is orientable if an only if it admits an atlas so that for every pair of charts (U, ϕ) (V, ψ) of M, $\phi \circ \psi^{-1}$ is orientation preserving.*

Proof. Suppose that M is orientable. Then we may cover M by a collection of orientation preserving charts. We claim that this yields the desired atlas. To see this let (U, ϕ) and (V, ψ) belong to this atlas, and observe that

$$d\phi = d(\phi \circ \psi^{-1}) \circ d\psi.$$

Now let $(X_1, \ldots, X_n)$ be a basis in the orientation class of M at $p \in U \cap V$. Then $(d\phi(X_1), \ldots, d\phi(X_n))$ and $(d\psi(X_1), \ldots, d\psi(X_n))$ belong to the same orientation class, and thus the above expression implies that the determinant of $d(\phi \circ \psi^{-1})$ must be positive.

Conversely, suppose that an atlas as in the statement of the proposition exists. Then, at each point of M we define an orientation of T_pM by pulling back the standard basis of $\mathbf{R}^n$ via a local chart centered at p. This gives a well defined orientation at each point. To see this let $p \in U \cap V$ for a pair of local charts (U, ϕ) and (V, ψ) of M. We need to check that

$$(d\phi^{-1}_{\phi(p)}(e_1), \ldots, d\phi^{-1}_{\phi(p)}(e_n)) \quad \text{and} \quad (d\psi^{-1}_{\psi(p)}(e_1), \ldots, d\psi^{-1}_{\psi(p)}(e_n))$$

belong to the same orientation class of T_pM. To this end we push forward these basis via $d\phi_p$. Then we obtain

$$(e_1, \ldots, e_n) \quad \text{and} \quad (d\phi \circ \psi^{-1}_{\psi(p)}(e_1), \ldots, d\phi \circ \psi^{-1}_{\psi(p)}(e_n))$$

which belong to the same orientation class since $\phi \circ \psi^{-1}$ is orientation preserving by assumption. So the orientation we have assigned to each T_pM is well defined. It remains only to show that this orientation is continuous. But this is immediate from our definition because $(d\phi^{-1}_{\phi(p)}(e_1), \ldots, d\phi^{-1}_{\phi(p)}(e_n))$ are continuous vector fields on U which belong to the orientation class of T_pM for each $p \in U$. $\square$

Lemma 0.0.7. *Let M and N be connected orientable manifolds and $f \colon M \to N$ be a diffeomorphism. Then either f is orientation preserving or is orientation reversing.*

Proof. Let $p \in M$, choose local charts (U, ϕ), (V, ψ) centered at p and $f(p)$ which are orientation preserving, and let $\tilde{f} := \psi \circ f \circ \phi^{-1}$. Then $\tilde{f} \colon \mathbf{R}^n \to \mathbf{R}^n$ is a diffeomorphism, so the determinant of its jacobian is either everywhere positive or everywhere negative. So either $\tilde{f}$ preserves orientation everywhere or else reverses orientation everywhere. Since ϕ and ψ both preserve orientation, this implies that either f preserves orientation throughout U or reverses orientation throughout U. So the set of points in M where f preserves orientation and the set of points where f reverses orientation are both open. So since these sets are complements of each other they are both closed as well. Thus one of these sets must be empty and the other has to coincide with all of M, since M is connected. □

Corollary 0.0.8. *If M can be covered by only two charts (U, ϕ) and (V, ψ), and $(U \cap V)$ is connected then M is orientable. In particular, $\mathbf{S}^n$ is orientable.*

Proof. Since $\psi(U \cap V)$ is connected and $\phi \circ \psi^{-1} \colon \psi(U \cap V) \to \phi(U \cap V)$ is a diffeomorphism, then either it is orientation preserving or orientation reversing everywhere by the previous lemma. If $\phi \circ \psi^{-1}$ is orientation preserving then we are done; otherwise, we may replace ϕ by its composition with a reflection. □

Next we show that the mobius strip is not orientable. To this end we first need:

Lemma 0.0.9. *Let $(b_1, \dots, b_n)$ and $(b'_1, \dots, b'_n)$ be a pair of basis in the same orientation class of $\mathbf{R}^n$. Further suppose that $(b_1, \dots, b_{n-1})$ and $(b'_1, \dots, b'_{n-1})$ belong to the same orientation class of $\mathbf{R}^{n-1}$. Then b_n and b'_n point to the same side of $\mathbf{R}^{n-1} \times \{0\}$ in $\mathbf{R}^n$.*

Proof. Since $(b_1, \dots, b_{n-1})$ and $(b'_1, \dots, b'_{n-1})$ belong to the same orientation class of $\mathbf{R}^{n-1}$, we may continuously transform $(b_1, \dots, b_{n-1}, b_n)$ to $(b'_1, \dots, b'_{n-1}, b_n)$ through a family of basis. So $(b'_1, \dots, b'_{n-1}, b_n)$ and $(b'_1, \dots, b'_{n-1}, b'_n)$ belong to the same orientation class. Next, note that we may continuously deform these basis, through a family of basis, to $(b'_1, \dots, b'_{n-1}, \overline{b_n})$ and $(b'_1, \dots, b'_{n-1}, \overline{b'_n})$, respectively, where $\overline{b_n}$ and $\overline{b'_n}$ are unit vectors orthogonal to $\mathbf{R}^{n-1} \times \{0\}$. Then $\overline{b_n} = \pm \overline{b'_n}$. If $\overline{b_n} = -\overline{b'_n}$, then $(b'_1, \dots, b'_{n-1}, \overline{b_n})$ and $(b'_1, \dots, b'_{n-1}, \overline{b'_n})$ lie in different orientation classes, which is not possible, since they are obtained by a continuous transformation, through a family of basis, from a pair of basis which where in the same orientation class. So we must have $\overline{b_n} = \overline{b'_n}$. This implies that b_n and b'_n must have been pointing into the same side of $\mathbf{R}^{n-1} \times \{0\}$. □

Proposition 0.0.10. *Let M be an orientable manifold and $N \subset M$ be an embedded submanifold. Suppose that $dim(N) = dim(M) - 1$. Then N is orientable if and only if there exists a vectorfield in M defined along N which is continuous and is never tangent to N.*

Proof. We may equip M with a Riemannian metric. Suppose that M and N are both orientable, then we may define a normal vector field along N as follows. For every $p \in N$, let $(X_1(p), \dots, X_{n-1}(p))$ be an ordered basis in the orientation class of T_pN. Then, since orthogonal complement of T_pN in T_pM is one dimensional, there exists only two vectors $Y_1(p), Y_2(p) \in T_pM$ which have unit length, and are orthogonal to T_pN. Since, $Y_1(p) = -Y_2(p)$, $(X_1(p), \dots, X_{n-1}(p), Y_1(p))$ and $(X_1(p), \dots, X_{n-1}(p), Y_2(p))$ belong to different orientation classes of T_pM. In particular, only one of these two vector fields, which we rename $Y(p)$, satisfies the condition that $(X_1(p), \dots, X_{n-1}(p), Y(p))$ belongs to the preferred orientation class of T_pM.

Next we show that Y is continuous. To see this recall that since N is orientable, by definition there exists for every p in N an open neighborhood U of p in N and continuous linearly independent vectorfields $X_1(q), \dots, X_{n-1}(q)$ in M tangent to N at each $q \in U$ such that $(X_1(q), \dots, X_{n-1}(q))$ belongs to the orientation class of T_qN. By the rank theorem, there exists an open neighborhood V of p in M and a diffeomorphism $\phi\colon V \to \mathbf{R}^n$ such that $\phi(V \cap N) = \mathbf{R}^{n-1} \times \{0\}$. Choosing U and V sufficiently small we may assume that $U \cap N = V$. Note that $(d\phi_q(X_1(q)), \dots, d\phi_q(X_{n-1}(q)))$ gives a continuous basis for $\mathbf{R}^{n-1}$. So $(d\phi_q(X_1(q)), \dots, d\phi_q(X_{n-1}(q)))$ all belong to the same orientation class of $\mathbf{R}^{n-1}$. Similarly $(d\phi_q(X_1(q)), \dots, d\phi_q(X_{n-1}(q)), d\phi_q Y(q))$ all belong to the same orientation class of $\mathbf{R}^n$. This implies, by the previous lemma, that $d\phi_q Y(q)$ always points to the same side of $\mathbf{R}^{n-1} \times \{0\}$. So $Y(q)$ must always point to the same side of N.

Conversely suppose that there exists a continuous vector field Y along N which is never tangent to N. Then we may assume that Y is normal to N, after a continuous deformation of Y. Now we pick an orientation for each T_pN as follows. Let $(X_1(p), \dots, X_{n-1}(p))$ be an ordered basis of T_pN such that $(X_1(p), \dots, X_{n-1}(p), Y(p))$ is in the orientation class of T_pM. We claim that defines an orientation on N. To see this let (U, ϕ) be an orientation preserving local chart of M centered at p such that $\phi(U \cap N) = \mathbf{R}^{n-1} \times \{0\}$. Then $d\phi(Y)$ is a continuous vector field along $\mathbf{R}^{n-1} \times \{0\}$ which is never tangent to it. So we may continuously transform $d\phi(Y)$ without ever making it tangent to $\mathbf{R}^{n-1} \times \{0\}$ until it coincides everywhere either with $(0, \dots, 0, 1)$ or $(0, \dots, 0, -1)$. We may assume, after replacing Y with $-Y$ if necessary, that $d\phi(Y)$ coincides with $(0, \dots, 0, 1)$. Now let $(b_1, \dots b_{n-1})$ be a basis for $\mathbf{R}^{n-1}$ such that $(b_1, \dots b_{n-1}, (0, \dots, 0, 1))$ lies in the standard orientation class of $\mathbf{R}^n$. Then $(d\phi^{-1}(b_1), \dots d\phi^{-1}(b_{n-1}), d\phi^{-1}((0, \dots, 0, 1)))$ lie in the orientation class of M, and consequently $(d\phi^{-1}(b_1), \dots d\phi^{-1}(b_{n-1}))$ lie in the orientation class of N because $d\phi^{-1}((0, \dots, 0, 1))$ may be continuously deformed to Y without every being tangent to N (recall that $(0, \dots, 0, 1)$ may be continuously deformed to $d\phi(Y)$). Since $(d\phi^{-1}(b_1), \dots d\phi^{-1}(b_{n-1}))$ is continuous, we conclude that the orientation defined on N is continuous. $\square$

Corollary 0.0.11. *The mobius strip is not orientable.*

Proof. The center circle in a mobius strip is an orientable submanifold, but it does not admit a continuous vectorfield which is nowhere tangent to it. Hence the mobius strip cannot be orientable. $\square$

There is another way to show that the mobius strip is not orientable as we discuss next.

Exercise 0.0.12. Let M be a two dimensional smooth manifold which admits a smooth triangulation. Show that M is orientable if and only if we may orient each triangle, i.e., order its vertices so that the orientations induced on each edge by the neighboring triangles are the opposite of each other. Check that this implies that the mobius strip is not orientable.

Exercise 0.0.13. Show that $\mathbf{RP}^2$ is not orientable in two ways: (1) show that $\mathbf{RP}^2$ contains a mobius strip; (2) use the triangulation argument of the previous exercise.

There is still a third way to show that $\mathbf{RP}^2$ is not orientable, and this argument applies to all $\mathbf{RP}^n$ when n is even:

Lemma 0.0.14. *The reflection $r\colon \mathbf{S}^n \to \mathbf{S}^n$ given by $r(x) = -x$ is orientation reversing when n is even and is orientation preserving when n is odd.*

Proof. Define a continuous normal vectorfield on $\mathbf{S}^n$ by $n(p) := p$. Then we may define an orientation on $\mathbf{S}^n$ by stating that a basis $(b_1(p), \dots, b_n(p))$ of $T_p\mathbf{S}^n$ lies in the orientation class of $T_p\mathbf{S}^n$ provided that

$$(b_1(p), \dots, b_n(p), n(p))$$

lies in the standard orientation class of $\mathbf{R}^{n+1}$. Then, to decide whether or not r is orientation preserving, we just have to check whether

$$(dr(b_1(p)), \dots, dr(b_n(p)), n(r(p)))$$

lies in the orientation class of $\mathbf{R}^{n+1}$ or not. But dr is just the restriction of $d\overline{r}$ to $\mathbf{S}^n$ where $\overline{r}\colon \mathbf{R}^n \to \mathbf{R}^n$ is the reflection through the origin of $\mathbf{R}^n$. Since $\overline{r}$ is linear $d\overline{r} = \overline{r}$. So $dr = \overline{r}$. Further note that $n(r(p)) = n(-p) = -p = -n(p) = \overline{r}(n(p))$. Thus the last displayed expression may be rewritten as

$$(\overline{r}(b_1(p)), \dots, \overline{r}(b_n(p)), \overline{r}(n(p))).$$

Since the standard orientation class of $\mathbf{R}^{n+1}$ contains $(b_1(p), \dots, b_n(p), n(p))$ by assumption, the above basis belongs to the orientation class of $\mathbf{R}^{n+1}$ if and only if $\overline{r}$ is orientation preserving, which is the case only when n is even (or $n+1$ is odd). $\square$

Theorem 0.0.15. $\mathbf{RP}^n$ *is orientable if and only if n is odd.*

Proof. Recall that we have a natural mapping $\pi\colon \mathbf{S}^n \to \mathbf{RP}^n$ which is a local diffeomorphism, and is given simply by $\pi(p) = \{p, -p\}$. Now let $r\colon \mathbf{S}^n \to \mathbf{S}^n$ be the reflection through the origin. Then

$$\pi \circ r = \pi.$$

If $\mathbf{RP}^n$ is orientable then we may assume that π preserves orientation. Then the above inequality implies that $\pi \circ r$ preserves orientation as well. This is not possible only if r preserves orientation which is the case only when n is odd. Thus $\mathbf{RP}^n$ is not orientable when n is even.

It remains to show that $\mathbf{RP}^n$ is orientable when n is odd. In this we may orient each tangent space $T_{[p]}\mathbf{RP}^n$ is as follows: take a representative from $q \in [p] = \{p, -p\}$. Choose a basis of $T_q\mathbf{S}^n$ which is in its orientation class, and let the image of this basis under $d\pi$ determine the orientation class of $T_{[p]}\mathbf{RP}^n$. This orientation is well defined because it not effected by whether $q = p$ or $q = -p$. Indeed, in $(b_1, \dots, b_n)$ is a basis in the orientation class of $T_p\mathbf{S}^n$ and $(b_1', \dots, b_n')$ is a basis in the orientation class of $T_{-p}\mathbf{S}^n$ then

$$(d\pi_p(b_1), \dots, d\pi(b_n)) \quad \text{and} \quad (d\pi_{-p}(b_1'), \dots, d\pi(b_n'))$$

belong in the same orientation class of $T_{[p]}\mathbf{RP}^n$; because,

$$d\pi_p(b_i) = d(\pi \circ r)_p(b_i) = d\pi_{r(p)} \circ dr_p(b_i) = d\pi_{-p} \circ dr_p(b_i)$$

and r preserves orientation, i.e., $(dr_p(b_1), \dots, dr_p(b_n))$ belongs in the same orientation class as $(b_1', \dots, b_n')$. □

Exercise 0.0.16. Show that $\mathbf{RP}^2$ cannot be embedded in $\mathbf{R}^3$ by using the Jordan-Brouwer separation theorem which states that if $M^n \subset \mathbf{R}^{n+1}$ is an embedded compact submanifold then $\mathbf{R}^{n+1} - M$ has exactly two components A and B and $\partial A = M = \partial B$. So we just need to check that this allows us to define a continuous normal vector field along M.

Exercise 0.0.17. Show that the tangent bundle of any manifold is orientable. (*Solution:* We show that the tangent bundle admits an atlas such that the transition functions are all orientation preserving. In particular we may use the atlas which is induced on TM by any choice of an atlas on M. Recall that if (U, ϕ) is any local chart of M, then the corresponding chart of TM is given by $(\overline{U}, \overline{\phi})$ where $\overline{U} = \cup_{p\in U} T_pM$, and $\overline{\phi}(p, v) = (\phi(p), d\phi(v))$. So if $(\overline{V}, \overline{\psi})$ is another such chart for TM then

$$\overline{\phi} \circ \overline{\psi}^{-1}(x_1, \dots, x_{2n}) = (\phi \circ \psi^{-1}(x_1, \dots, x_n), d(\phi \circ \psi^{-1})(x_{n+1}, \dots, x_{2n})).$$

This implies that

$$d(\overline{\phi} \circ \overline{\psi}^{-1})(x_1, \dots, x_{2n}) = (d(\phi \circ \psi^{-1})(x_1, \dots, x_n), d(\phi \circ \psi^{-1})(x_{n+1}, \dots, x_{2n})).$$

Consequently eigenvectors of $d(\overline{\phi} \circ \overline{\psi}^{-1})$ are $(0, v_i)$ and $(v_i, 0)$ where v_i are eigenvectors of $d(\phi \circ \psi^{-1})$. This yields that each eigenvalue of $d(\phi \circ \psi^{-1})$ occurs twice in the in the list of the eigenvalues of $d(\overline{\phi} \circ \overline{\psi}^{-1})$. Thus

$$\det(d(\overline{\phi} \circ \overline{\psi}^{-1})) = \det(d(\phi \circ \psi^{-1}))^2 > 0.)$$

Exercise 0.0.18. Show that a minfold M is orientable if and only if there does not exits a continuous family of immersions $f_t \colon B^n \to M$, $t \in [0, 1]$, such that f_0 and f_1 are embeddings, they have the same image, i.e., $f_0(B^n) = f_1(B^n)$, and $f_0 \circ f_1^{-1}$ is orientation reversing. (Recall that B^n denotes the unit ball in $\mathbf{R}^n$; we say that $f_t \colon B^n \to M$ is a continuous family of immersions, if each f_t is an immersion, an if $F \colon [0, 1] \times B^n \to M$, defined by $F(t, p) := f_t(p)$ is continuous.)

Note 0.0.19. The last exercise suggests how one may extend the definition of orientability to topological manifolds: we say that M is orientable if and only there does not exist a cotinuous family of (topological)immersions $f_t \colon B^n \to M$, $t \in [0, 1]$, such that f_0 and f_1 are embeddings, they have the same image, i.e., $f_0(B^n) = f_1(B^n)$, and $f_0 \circ f_1^{-1}$ is isotopic to the identity. A homeomorphism $h \colon B^n \to B^n$ is said to be isotopic to the identity provided that there exist a continuous family of homeomorphisms $h_t \colon B^n \to B^n$ such that $h_0 = h$ and h_1 is the identity map.