Math 497C
Curves and Surfaces
Fall 2004, PSU

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# Lecture Notes 12

## 2.6 Gauss's formulas, and Christoffel Symbols

Let $X: U \to \mathbf{R}^3$ be a proper regular patch for a surface $M$, and set $X_i := D_i X$. Then

$$\{X_1, X_2, N\}$$

may be regarded as a *moving bases* of *frame* for $\mathbf{R}^3$ similar to the Frenet Serret frames for curves. We should emphasize, however, two important differences: (i) there is no canonical choice of a moving bases for a surface or a piece of surface ($\{X_1, X_2, N\}$ depends on the choice of the chart $X$); (ii) in general it is not possible to choose a patch $X$ so that $\{X_1, X_2, N\}$ is orthonormal (unless the Gaussian curvature of $M$ vanishes everywhere).

The following equations, the first of which is known as *Gauss's formulas*, may be regarded as the analog of Frenet-Serret formulas for surfaces:

$$X_{ij} = \sum_{k=1}^{2} \Gamma_{ij}^k X_k + l_{ij} N, \qquad \text{and} \qquad N_i = -\sum_{j=1}^{2} l_i^j X_j.$$

The coefficients $\Gamma_{ij}^k$ are known as the *Christoffel symbols*, and will be determined below. Recall that $l_{ij}$ are just the coefficients of the second fundamental form. To find out what $l_i^j$ are note that

$$-l_{ik} = -\langle N, X_{ik}\rangle = \langle N_i, X_k\rangle = -\sum_{j=1}^{2} l_i^j \langle X_j, X_k\rangle = -\sum_{j=1}^{2} l_i^j g_{jk}.$$

Thus $(l_{ij}) = (l_i^j)(g_{ij})$. So if we let $(g^{ij}) := (g_{ij})^{-1}$, then $(l_i^j) = (l_{ij})(g^{ij})$, which yields

$$l_i^j = \sum_{k=1}^{2} l_{ik} g^{kj}.$$

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**Exercise 1.** What is $\det(l_i^j)$ equal to?.

**Exercise 2.** Show that $N_i = dn(X_i) = -S(X_i)$.

Next we compute the Christoffel symbols. To this end note that

$$\langle X_{ij}, X_k\rangle = \sum_{l=1}^{2} \Gamma_{ij}^l \langle X_l, X_k\rangle = \sum_{l=1}^{2} \Gamma_{ij}^l g_{lk},$$

which in matrix notation reads

$$\begin{pmatrix} \langle X_{ij}, X_1\rangle \\ \langle X_{ij}, X_2\rangle \end{pmatrix} = \begin{pmatrix} \Gamma_{ij}^1 g_{11} + \Gamma_{ij}^2 g_{21} \\ \Gamma_{ij}^1 g_{12} + \Gamma_{ij}^2 g_{22} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} \Gamma_{ij}^1 \\ \Gamma_{ij}^2 \end{pmatrix}.$$

So

$$\begin{pmatrix} \Gamma_{ij}^1 \\ \Gamma_{ij}^2 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} \langle X_{ij}, X_1\rangle \\ \langle X_{ij}, X_2\rangle \end{pmatrix} = \begin{pmatrix} g^{11} & g^{21} \\ g^{12} & g^{22} \end{pmatrix} \begin{pmatrix} \langle X_{ij}, X_1\rangle \\ \langle X_{ij}, X_2\rangle \end{pmatrix},$$

which yields

$$\Gamma_{ij}^k = \sum_{l=1}^{2} \langle X_{ij}, X_l\rangle g^{lk}.$$

In particular, $\Gamma_{ij}^k = \Gamma_{ji}^k$. Next note that

$$\begin{aligned} (g_{ij})_k &= \langle X_{ik}, X_j\rangle + \langle X_i, X_{jk}\rangle, \\ (g_{jk})_i &= \langle X_{ji}, X_k\rangle + \langle X_j, X_{ki}\rangle, \\ (g_{ki})_j &= \langle X_{kj}, X_i\rangle + \langle X_k, X_{ij}\rangle. \end{aligned}$$

Thus

$$\langle X_{ij}, X_k\rangle = \frac{1}{2}\big((g_{ki})_j + (g_{jk})_i - (g_{ij})_k\big).$$

So we conclude that

$$\Gamma_{ij}^k = \sum_{l=1}^{2} \frac{1}{2}\big((g_{li})_j + (g_{jl})_i - (g_{ij})_l\big) g^{lk}.$$

Note that the last equation shows that $\Gamma_{ij}^k$ are *intrinsic quantities*, i.e., they depend only on $g_{ij}$ (and derivatives of $g_{ij}$), and so are preserved under isometries.

**Exercise 3.** Compute the Christoffel symbols of a surface of revolution.

## 2.7 The Gauss and Codazzi-Mainardi Equations, Riemann Curvature Tensor, and a Second Proof of Gauss's Theorema Egregium

Here we shall derive some relations between $l_{ij}$ and $g_{ij}$. Our point of departure is the simple observation that if $X\colon U \to \mathbf{R}^3$ is a $C^3$ regular patch, then, since partial derivatives commute,

$$X_{ijk} = X_{ikj}.$$

Note that

$$\begin{aligned}
X_{ijk} &= \Big(\sum_{l=1}^{2} \Gamma_{ij}^{l} X_l + l_{ij} N\Big)_k \\
&= \sum_{l=1}^{2} (\Gamma_{ij}^{l})_k X_l + \sum_{l=1}^{2} \Gamma_{ij}^{l} X_{lk} + (l_{ij})_k N + l_{ij} N_k \\
&= \sum_{l=1}^{2} (\Gamma_{ij}^{l})_k X_l + \sum_{l=1}^{2} \Gamma_{ij}^{l} \Big(\sum_{m=1}^{2} \Gamma_{lk}^{m} X_m + l_{lk} N\Big) + (l_{ij})_k N - l_{ij} \sum_{l=1}^{2} l_k^l X_l \\
&= \sum_{l=1}^{2} (\Gamma_{ij}^{l})_k X_l + \sum_{l=1}^{2} \sum_{m=1}^{2} \Gamma_{ij}^{l} \Gamma_{lk}^{m} X_m + \sum_{l=1}^{2} \Gamma_{ij}^{l} l_{lk} N + (l_{ij})_k N - \sum_{l=1}^{2} l_{ij} l_k^l X_l \\
&= \sum_{l=1}^{2} \Big((\Gamma_{ij}^{l})_k + \sum_{p=1}^{2} \Gamma_{ij}^{p} \Gamma_{pk}^{l} - l_{ij} l_k^l\Big) X_l + \Big(\sum_{l=1}^{2} \Gamma_{ij}^{l} l_{lk} + (l_{ij})_k\Big) N.
\end{aligned}$$

Switching $k$ and $j$ yields,

$$X_{ikj} = \sum_{l=1}^{2} \Big((\Gamma_{ik}^{l})_j + \sum_{p=1}^{2} \Gamma_{ik}^{p} \Gamma_{pj}^{l} - l_{ik} l_j^l\Big) X_l + \Big(\sum_{l=1}^{2} \Gamma_{ik}^{l} l_{lj} + (l_{ik})_j\Big) N.$$

Setting the normal and tangential components of the last two equations equal to each other we obtain

$$\begin{gathered}
(\Gamma_{ij}^{l})_k + \sum_{p=1}^{2} \Gamma_{ij}^{p} \Gamma_{pk}^{l} - l_{ij} l_k^l = (\Gamma_{ik}^{l})_j + \sum_{p=1}^{2} \Gamma_{ik}^{p} \Gamma_{pj}^{l} - l_{ik} l_j^l, \\
\sum_{l=1}^{2} \Gamma_{ij}^{l} l_{lk} + (l_{ij})_k = \sum_{l=1}^{2} \Gamma_{ik}^{l} l_{lj} + (l_{ik})_j.
\end{gathered}$$

These equations may be rewritten as

$$(\Gamma^l_{ik})_j - (\Gamma^l_{ij})_k + \sum_{p=1}^{2} \left(\Gamma^p_{ik}\Gamma^l_{pj} - \Gamma^p_{ij}\Gamma^l_{pk}\right) = l_{ik}l^l_j - l_{ij}l^l_k, \quad \text{(Gauss)}$$

$$\sum_{l=1}^{2} \left(\Gamma^l_{ik}l_{lj} - \Gamma^l_{ij}l_{lk}\right) = (l_{ij})_k - (l_{ik})_j, \quad \text{(Codazzi-Mainardi)}$$

and are known as the *Gauss's equations* and the *Codazzi-Mainardi equations* respectively. If we define the *Riemann curvature tensor* as

$$R^l_{ijk} := (\Gamma^l_{ik})_j - (\Gamma^l_{ij})_k + \sum_{p=1}^{2} \left(\Gamma^p_{ik}\Gamma^l_{pj} - \Gamma^p_{ij}\Gamma^l_{pk}\right),$$

then Gauss's equation may be rewritten as

$$R^l_{ijk} = l_{ik}l^l_j - l_{ij}l^l_k.$$

Now note that

$$\sum_{l=1}^{2} R^l_{ijk}g_{lm} = l_{ik}\sum_{l=1}^{2} l^l_j g_{lm} - l_{ij}\sum_{l=1}^{2} l^l_k g_{lm} = l_{ik}l_{jm} - l_{ij}l_{km}.$$

In particular, if $i = k = 1$ and $j = m = 2$, then

$$\sum_{l=1}^{2} R^l_{121}g_{l2} = l_{11}l_{22} - l_{12}l_{21} = \det(l_{ij}) = K\det(g_{ij}).$$

So it follows that

$$K = \frac{R^1_{121}g_{12} + R^2_{121}g_{22}}{\det(g_{ij})},$$

which shows that $K$ is intrinsic and gives another proof of Gauss's Theorema Egregium.

**Exercise 4.** Show that if $M = \mathbf{R}^2$, hen $R^l_{ijk} = 0$ for all $1 \leq i, l, j, k \leq 2$ both intrinsically and extrinsically.

**Exercise 5.** Show that (i) $R^l_{ijk} = -R^l_{ikj}$, hence $R^l_{ijj} = 0$, and (ii) $R^l_{ijk} + R^l_{jki} + R^l_{kij} \equiv 0$.

**Exercise 6.** Compute the Riemann curvature tensor for $\mathbf{S}^2$ both intrinsically and extrinsically.

## 2.8 Fundamental Theorem of Surfaces

In the previous section we showed that if $g_{ij}$ and $l_{ij}$ are the coefficients of the first and second fundamental form of a patch $X\colon U \to M$, then they must satisfy the Gauss and Codazzi-Maindardi equations. These conditions turn out to be not only necessary but also sufficient in the following sense.

**Theorem 7** (Fundamental Theorem of Surfaces)**.** *Let $U \subset \mathbf{R}^2$ be an open neighborhood of the origin $(0,0)$, and $g_{ij}\colon U \to \mathbf{R}$, $l_{ij}\colon U \to \mathbf{R}$ be differentiable functions for $i$, $j = 1, 2$. Suppose that $g_{ij} = g_{ji}$, $l_{ij} = l_{ji}$, $g_{11} > 0$, $g_{22} > 0$ and $\det(g_{ij}) > 0$. Further suppose that $g_{ij}$ and $l_{ij}$ satisfy the Gauss and Codazzi-Mainardi equations. Then there exists and open set $V \subset U$, with $(0,0) \in V$ and a regular patch $X\colon V \to \mathbf{R}$ with $g_{ij}$ and $l_{ij}$ as its first and second fundamental forms respectively. Further, if $Y\colon V \to \mathbf{R}^3$ is another regular patch with first and second fundamental forms $g_{ij}$ and $l_{ij}$, then $Y$ differs from $X$ by a rigid motion.*