Math 528 Jan 11, 2005[1]
Geometry and Topology II
Fall 2005, PSU

Lecture Notes 1

1 Topological Manifolds

The basic objects of study in this class are manifolds. Roughly speaking, these are objects which locally resemble a Euclidean space. In this section we develop the formal definition of manifolds and construct many examples.

1.1 The Euclidean space

By $\mathbf{R}$ we shall always mean the set of real numbers, which is a well ordered field with the following property.

Completeness Axiom. *Every nonempty subset of* $\mathbf{R}$ *which is bounded above has a least upper bound.*

The set of all n-tuples of real numbers $\mathbf{R}^n := \{(p_1, \dots, p_n) \mid p_i \in \mathbf{R}\}$ is called the Euclidean n-space. So we have

$$p \in \mathbf{R}^n \iff p = (p_1, \dots, p_n), \quad p^i \in \mathbf{R}.$$

Let p and q be a pair of points (or vectors) in $\mathbf{R}^n$. We define $p + q := (p^1 + q^1, \dots, p^n + q^n)$. Further, for any scalar $r \in \mathbf{R}$, we define $rp := (rp^1, \dots, rp^n)$. It is easy to show that the operations of addition and scalar multiplication that we have defined turn $\mathbf{R}^n$ into a vector space over the field of real numbers. Next we define the standard inner product on $\mathbf{R}^n$ by

$$\langle p, q\rangle = p_1q_1 + \ldots + p_nq_n.$$

Note that the mapping $\langle\cdot,\cdot\rangle\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is linear in each variable and is symmetric. The standard inner product induces a norm on $\mathbf{R}^n$ defined by

$$\|p\| := \langle p, p\rangle^{1/2}.$$

If $p \in \mathbf{R}$, we usually write $|p|$ instead of $\|p\|$.

Theorem 1.1.1. (The Cauchy-Schwartz inequality) *For all* p *and* q *in* $\mathbf{R}^n$

$$|\langle p, q\rangle| \leqslant \|p\|\,\|q\|.$$

The equality holds if and only if $p = \lambda q$ *for some* $\lambda \in \mathbf{R}$.

[1] Last revised: October 2, 2006

Proof I. If $p = \lambda q$ it is clear that equality holds. Otherwise, let $f(\lambda) := \langle p - \lambda q, p - \lambda q\rangle$. Then $f(\lambda) > 0$. Further, note that $f(\lambda)$ may be written as a quadratic equation in λ:

$$f(\lambda) = \|p\|^2 - 2\lambda\langle p, q\rangle + \lambda^2\|q\|^2.$$

Hence its discriminant must be negative:

$$4\langle p, q\rangle^2 - 4\|p\|^2\|q\|^2 < 0$$

which completes the proof. $\square$

Proof II. Again suppose that $p \neq \lambda q$. Then

$$\langle p, q\rangle = \|p\|\|q\| \left\langle \frac{p}{\|p\|}, \frac{q}{\|q\|} \right\rangle.$$

Thus it suffices to prove that for all unit vectors $\overline{p}$ and $\overline{q}$ we have

$$|\langle \overline{p}, \overline{q}\rangle| \leq 1,$$

and equality holds if and only if $p = \pm q$. This may be proved by using the method of lagrangne multipliers to find the maximum of the function $\langle x, y\rangle$ subject to the constraints $\|x\| = 1$ and $\|y\| = 1$. More explicitly we need to find the critical points of

$$\begin{aligned} f(x, y, \lambda_1, \lambda_2) &:= \langle x, y\rangle + \lambda_1(\|x\|^2 - 1) + \lambda_2(\|y\|^2 - 1) \\ &= \sum_{i=1}^{n}(x_i y_i + \lambda_1 x_i^2 + \lambda_2 y_i^2) - \lambda_1 - \lambda_2. \end{aligned}$$

At a critical point we must have $0 = \partial f/\partial x_i = y_i + 2\lambda_1 x_i$, which yields that $y = \pm x$. $\square$

The standard Euclidean distance in $\mathbf{R}^n$ is given by

$$\mathrm{dist}(p, q) := \|p - q\|.$$

The proof of the following fact is left as an exercise.

Corollary 1.1.2. (The triangle inequality) *For all p, q, r in $\mathbf{R}^n$.*

$$\mathrm{dist}(p, q) + \mathrm{dist}(q, r) \geqslant \mathrm{dist}(p, r)$$

$\square$

By a *metric* on a set X we mean a mapping $d\colon X \times X \to \mathbf{R}$ such that

1. $d(p, q) \geqslant 0$, with equality if and only if $p = q$.

2. $d(p, q) = d(q, p)$.

3. $d(p, q) + d(q, r) \geqslant d(p, r)$.

These properties are called, respectively, positive-definiteness, symmetry, and the triangle inequality. The pair (X, d) is called a *metric space*. Using the above exercise, one immediately checks that $(\mathbf{R}^n, \mathrm{dist})$ is a metric space. Geometry, in its broadest definition, is the study of metric spaces.

Finally, we define the *angle* between a pair of vectors in $\mathbf{R}^n$ by

$$\mathrm{angle}(p, q) := \cos^{-1} \frac{\langle p, q \rangle}{\|p\| \, \|q\|}.$$

Note that the above is well defined by the Cauchy-Schwartz inequality.

Exercise 1.1.3. (The Pythagorean theorem) Show that in a right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the length of the sides.

Exercise 1.1.4. Show that the sum of angles in a triangle is π.

1.2 Topological spaces and continuous maps

By a *topological space* we mean a set X together with a collection T of subsets of X which satisfy the following properties:

1. $X \in T$, and $\emptyset \in T$.

2. If U_1, $U_2 \in T$, then $U_1 \cap U_2 \in T$.

3. If $U_i \in T$, $i \in I$, then $\cup_i U \in T$.

The elements of T are called *open* sets. Note that property 2 implies that any <u>finite</u> intersection of open sets is open, and property 3 states that the union of any collection of open sets is open. Any collection of subsets of X satisfying the above properties is called a *topology* on X.

Exercise 1.2.1 (Metric Topology). Let (X, d) be a metric space. For any $p \in X$, and $r > 0$ define the open ball of radius r centered at p as

$$B_r(p) := \{x \in X \mid d(x, p) < r\}.$$

We say $U \subset X$ is open if for each point p of U there is an $r > 0$ such that $B_r(p) \subset U$. Show that this defines a topology on X. In particular, $(\mathbf{R}^n, \mathrm{dist})$ is a topological space.

We say a subset of a topological space is *closed* if its complement is open. Note that a subset of a toplogical space maybe both open and closed or neither. A *limit point* of a subset $A \subset X$ is a point $p \in X$ such that every open neighborhood of p intersects A in a point other than p.

Proposition 1.2.2. *A subset A of a topological space X is closed if and only if it contains all of its limit points.*

Proof. Suppose that A is closed and p is a limit point of A. Since $X - A$ is open and disjoint from X, p may not belong to $X - A$. So $p \in X$.

Conversely, suppose that X contains all of its limit points. Then every point of $X - A$ must belong to an open set which is disjoint from X. Then union of all these open sets is disjoint from X and cotains $X - A$. So it must be $X - A$. But the union of open sets is open. Thus $X - A$ is open. □

A mapping $f\colon X \to Y$ between topological spaces is *continuous* if for every open set $U \subset X$, $f^{-1}(U)$ is open in Y. When X and Y are metric spaces, this condition is equivalent to the requirement that f map nearby points to nearby points:

Proposition 1.2.3. *Let X and Y be metric spaces. Then $f\colon X \to Y$ is continuous if and only if whenever a sequence of points $x_i \in X$ has a limit point x, the sequnce $f(x_i)$ has a limit point $f(x)$.*

Proof. Suppose that f is continuous and x is a limit point of x_i. Now if there exists an open neighborhood U of $f(x)$ which is disjoint from $f(x_i)$, then $f^{-1}(U)$ is an open neighborhood of x which is disjoint form x_i, which is a contradiction.

Conversely, suppose that whenever x is a limit point of x_i then $f(x)$ is a limit point of $f(x_i)$. Let $U \subset Y$ be open. We need to show that $f^{-1}(U)$ is open, which is equivalent to $X - f^{-1}(U)$ being closed. To prove the latter, let x be a limit point of $X - f^{-1}(U)$. Then there exists a sequence of points x_i of $X - f^{-1}(U)$ such that x is a limit point of x_i (we may construct x_i be letting $x_i \in B_{1/i}(x) \cap (X - f^{-1}(U))$). Then $f(x)$ is a limti point of $f(x_i)$. But $f(x_i) \in Y - U$ which is closed. Thus $f(x) \in Y - U$, which implies $x \in X - f^{-1}(U)$ as desired. □

Corollary 1.2.4. *If X is a metric space then its distance function $d\colon X \times X \to \mathbf{R}$ is continuous.*

Proof. Let (x_i, y_i) be a sequence of points of $X \times X$ converging to (x, y). Note that by the triangle inequality

$$d(x_i, y_i) \leq d(x_i, x) + d(x, y) + d(y, y_i)$$

which implies that

$$\lim_{i\to\infty} d(x_i, y_i) \leq d(x, y).$$

Further note that the triangle inequality also implies that

$$d(x, y) \leq d(x, x_i) + d(x_i, y_i) + d(y_i, y).$$

So

$$d(x, y) \leq \lim_{i \to \infty} d(x_i, y_i).$$

We conclude then that $\lim_{i\to\infty} d(x_i, y_i) = d(x, y)$ $\square$

Another characterization of continuous maps between metric spaces is as follows.

Proposition 1.2.5. *Let X, Y be metric spaces. Then $f : X \to Y$ is continuous if and only if for every $p \in X$ and $\epsilon > 0$, there exists a $\delta > 0$ such that whenever* $\operatorname{dist}(x, p) < \delta$, *then* $\operatorname{dist}(f(x), f(p)) < \epsilon$.

Proof. Suppose f is continuous. Then $f^{-1}(B_\epsilon f(p))$ is open. So there exists $\delta > 0$ such that $B_\delta(p) \subset f^{-1}(B_\epsilon f(p))$. Then $f(B_\delta(p)) \subset B_\epsilon(f(p))$ as desired.

Conversely, suppose that the ϵ-δ property holds. Let $U \subset Y$ be open, and $p \in f^{-1}(U)$. Then $f(p) \in U$ and there exists $\epsilon > 0$ such that $B_\epsilon(f(p)) \subset U$. There exists $\delta > 0$ such that $f(B_\delta(p)) \subset B_\epsilon(f(p)) \subset U$, which means that $B_\delta(p) \subset f^{-1}(U)$. So $f^{-1}(U)$ is open. $\square$

Let o denote the *origin* of $\mathbf{R}^n$, that is

$$o := (0, \dots, 0).$$

The n-dimensional Euclidean sphere is defined as

$$\mathbf{S}^n := \{\, x \in \mathbf{R}^{n+1} \mid \operatorname{dist}(x, o) = 1 \,\}.$$

The next exercise shows how we may define a topology on $\mathbf{S}^n$.

Exercise 1.2.6 (Subspace Topology). Let X be a topological space and suppose $Y \subset X$. Then we say that a subset V of Y is open if there exists an open subset U of X such that $V = U \cap Y$. Show that with this collection of open sets, Y is a topological space.

The n-dimensional torus T^n is defined as the cartesian product of n copies of $\mathbf{S}^1$,

$$T^n := \mathbf{S}^1 \times \cdots \times \mathbf{S}^1.$$

The next exercise shows that T^n admits a natural topology:

Exercise 1.2.7 (The Product Topology). Let X_1 and X_2 be topological spaces, and $X_1 \times X_2$ be their Cartesian product, that is

$$X_1 \times X_2 := \{\, (x_1, x_2) \mid x_1 \in X_1 \text{ and } x_2 \in X_2 \,\}.$$

We say that $U \subset X_1 \times X_2$ is open if for each $p \in U$ there exist open sets $U_1 \subset X_1$ and $U_2 \subset X_2$ such that $p \in U_1 \times U_2 \subset U$. Show that this defines a topology on $X_1 \times X_2$.

A *partition* P of a set X is defined as a collection P_i, $i \in I$, of subsets of X such that $X \subset \cup_i P_i$ and $P_i \cap P_j = \emptyset$ whenever $i \neq j$. For any $x \in X$, the element of P which contains x is called the equivalence class of x and is denoted by $[x]$. Thus we get a mapping $\pi\colon X \to P$ given by $\pi(x) := [x]$. Suppose that X is a topological space. Then we say that a subset U of P is open if $\pi^{-1}(U)$ is open in X.

Exercise 1.2.8 (Quotient Topology). Let X be a topological space and P be a partition of X. Show that P with the collection of open sets defined above is a topological space.

Exercise 1.2.9. Let P be a partition of $[0,1] \times [0,1]$ consisting of the following sets: (i) all sets of the form $\{(x,y)\}$ where $(x,y) \in (0,1) \times (0,1)$; (ii) all sets of the form $\{(x,1),(x,0)\}$ where $x \in (0,1)$; (iii) all sets of the form $\{(1,y),(0,y)\}$ where $y \in (0,1)$; and (iv) the set $\{(0,0),(0,1),(1,0),(1,1)\}$. Sketch the various kinds of open sets in P under its quotient topology.

1.3 Homeomorphisms, compactness and connectedness

We say that two topological spaces X and Y are *homeomorphic* if there exists a bijection $f\colon X \to Y$ which is continuous and has a continuous inverse. The main problem in topology is deciding when two topological spaces are homeomorphic.

Exercise 1.3.1. Show that $\mathbf{S}^n - \{(0,0,\dots,1)\}$ is homeomorphic to $\mathbf{R}^n$.

Exercise 1.3.2. Let $X := [1,0] \times [1,0]$, T_1 be the subspace topology on X induced by $\mathbf{R}^2$ (see Exercise 1.2.6), T_2 be the product topology (see Exercise 1.2.7), and T_3 be the quotient topology of Exercise 1.2.9. Show that (X,T_1) is homeomorphic to (X,T_2), but (X,T_3) is not homeomorphic to either of these spaces.

Exercise 1.3.3. Show that $B_1^n(o)$ is homeomorphic to $\mathbf{R}^n$ (*Hint:* Consider the mapping $f\colon B_1^n(o) \to \mathbf{R}^n$ given by $f(x) = \tan(\pi\|x\|/2)x$).

For any a, $b \in \mathbf{R}$, we set

$$[a,b] := \{x \in \mathbf{R} \mid a \leq x \leq b\},$$

and

$$(a,b) := \{x \in \mathbf{R} \mid a < x < b\}.$$

Exercise 1.3.4. Let P be a partition of $[0,1]$ consisting of all sets $\{x\}$ where $x \in (0,1)$ and the set $\{0,1\}$. Show that P, with respect to its quotient topology, is homeomorphic to $\mathbf{S}^1$ (Hint: consider the mapping $f\colon [0,1] \to \mathbf{S}^1$ given by $f(t) = e^{2\pi i t}$).

Exercise 1.3.5. Let P be the partition of $[0,1] \times [0,1]$ described in Exercise 1.2.9. Show that P, with its quotient topology, is homeomorphic to T^2.

Let P be the partition of $\mathbf{S}^n$ consisting of all sets of the form $\{p, -p\}$ where $p \in \mathbf{S}^n$. Then P with its quotient topology is called the real projective space of dimension n and is denoted by $\mathbf{RP}^n$.

Exercise 1.3.6. Let P be a partition of $B_1^2(o)$ consisting of all sets $\{x\}$ where $x \in U_1^2(o)$, and all sets $\{x, -x\}$ where $x \in \mathbf{S}^1$. Show that P, with its quotient topology, is homeomorphic to $\mathbf{RP}^2$.

Next we show that $\mathbf{S}^n$ is not homeomorphic to $\mathbf{R}^m$. This requires us to recall the notion of compactness. We say that a collection of subsets of X *cover* X, if X lies in the union of these subsets. Any subset of a cover which is again a cover is called a *subcover*. A topological space X is *compact* if every open cover of X has a finite subcover.

Exercise 1.3.7. Show that (i) if X is compact and Y is homeomorphic to X, then Y is compact as well and (ii) if X is compact and $f\colon X \to Y$ is continuous, then $f(X)$ is compact.

Exercise 1.3.8. Show that every closed subset of a compact space is compact.

We say that a sequence of points $p_i \in X$ converges provided there exists a point $p \in X$ such that for every open neighborhood U of p there exists a number N such that $p_i \in U$ for all $i \geq N$.

Lemma 1.3.9. *If a metric space X is compact, then every infinite sequence of points of X has a convergent subsequence.*

Proof. Suppose that X is compact and let p_i be a suequence of points of X. First we show that p_i must have a limit point. Otherwise for every $x \in X$ we may find an open neighborhood U_x which contains at most one point of p_i. Since X is compact the Collection U_x must have a finite subcover. This would imply that the sequence p_i form a finite subset of X. So some point of X must conicide with the elemnets of p_i infinitely often, which yields our convergent subsequence. $\square$

Lemma 1.3.10 (Lebeague Number lemma). *Let X be a compact metric space, and suppose $\mathcal{U}$ is any open conver of X. Then there exists a number $\delta > 0$ such for every point $p \in X$ there is an open set $U \in \mathcal{U}$ with $B_\delta(p) \subset U$.*

Proof. Suppose that there exists no such δ, i.e., for every $\delta > 0$, there exists a point $p \in X$ such that $B_\delta(p)$ does not lie in any element of $\mathcal{U}$. For each natural number n let p_n be a point such that $B_{1/n}(p_n)$ does not lie in any element of $\mathcal{U}$. We need to show that p_n has no convergent subsequence. Suppose that that there exists a convergent subsequence p_{n_i} converging to p. Note that there exists $\epsilon > 0$ such that $B_\epsilon(p)$ lies in an element of $\mathcal{U}$. Choose n large enough so that $\operatorname{dist}(p_{n_i}, p) < \epsilon/2$ and $1/n_i < \epsilon/2$. Then $B_{1/n_i}(p_{n_i}) \subset B_\epsilon(p)$, because, by the triangle inequality, for every $x \in B_{1/n_i}(p_{n_i})$,

$$\operatorname{dist}(x, p) \leq \operatorname{dist}(x, p_{n_i}) + \operatorname{dist}(p_{n_i}, p) < \epsilon/2 + \epsilon/2.$$

So $B_{1/n_i}(p_{n_i})$ belongs to an element of $\mathcal{U}$ which is a contradiction. $\square$

Theorem 1.3.11. *A metric space space X is comact if and only if every infinite sequence of points of X has a convergent subsequence.*

Proof. The first half of the theorem was proved in a previous lemma. To prove the second half suppose that every infinite sequence of points of X has a convergent subsequence. We need to show that then X is compact. This is proved in two steps:

Step 1. We claim that for every $\epsilon > 0$ there exists a finite covering of X by ϵ-balls. Suoppose that that this is not true. Let x_1 be any poin of X, and let x_2 be a point of $X - B_\epsilon(x_1)$. Mor genrally, let x_{n+1} be point of X which does not belong to $B_\epsilon(x_i)$ for $i \leq n$. Then x_i does not contain a covergent subsequence which is a contradiction.

Step 2. Let $\mathcal{U}$ be an open cover of X. Then $\mathcal{U}$ has a Lebesqgue number δ. By the previous step there exists a finite covering of X with δ-balls. Foe each of these balls pick an element of $\mathcal{U}$ which contains it. That yields our finite subcover. $\square$

Corollary 1.3.12. *Any finite product of comapct metric spaces is compact.*

Proof. Let $X = X_1 \times \cdots \times X_n$ be a product of compact metric spacex, and $p_i = (p_i^1, \ldots, p_i^n)$ be an infinite sequence in X. We show that p_i has a convergent subsequence. Too see this first note that each component of p_i has a convergent subsequence. Further, anu subsequence of a convergent sequence converges. Thus by taking sunccessive refinements of p_i we obtain a convergent subsequence. $\square$

The n-dimensional Euclidean *ball* of radius r centered at p is defined by

$$B_r^n(p) := \{x \in \mathbf{R}^n \mid \operatorname{dist}(x,p) < r\}.$$

A subset A of $\mathbf{R}^n$ is *bounded* if $A \subset B_r^n(o)$ for some $r \in \mathbf{R}$.

Lemma 1.3.13. *Any closed interval $[a,b] \subset \mathbf{R}^n$ is compact.*

Proof. We just need to check that evey infinite sequence $x_i \in [a,b]$ has convergent subsequence. This is certainly the case if x_i has a limit point, say x, for then we may choose a convergent subsequence by picking elements from a nested sequence of balls centered at x. So suppose that x_i has no limit points. Further, we may suppose that $\{x_i\}$ is an infinite subset of $[a,b]$, i.e., no point of $[a,b]$ coincides with an element of x_i more than a finite number of times, for then we would have a constant subsequence which is of course convergent. So, after passing to a subsequence of x_i, we may assume that x_i are all distinct points of $[a,b]$. Now let y_1 be the greatest lower bound of x_i. Note that y_1 must coincide with an element of x_i because otherwise it would be a limit point of x_i, which we have assumed not to exist. let A_1 be the set $\{x_i\} - \{y_1\}$ and let y_2 be the greatest lower bound of A_1. Then $y_2 \in A_1$ because A_1, since it is a subset of $\{x_i\}$, has no limit points. Next we let A_2 be the collection

$A_1 - \{y_2\}$ and let y_3 be the greatest lower bound of A_2. Continuiung this procedure we obtain a sequence y_i which is increasing. Since y_i is bounded above (by b), then it must have a greatest lower bound, say y, which must necessarily be a limit point of y_i. But any limit point of y_i is a limit point of x_i which is a contradiction. □

Theorem 1.3.14 (Heine-Borel). *A subset of* $\mathbf{R}^n$ *is compact if and only if it is closed and bounded.*

Proof. Suppose $X \subset \mathbf{R}^n$ is compact. Then every sequnce of points of X must have a convergent subsequence (which converges to a point of X). This easily implies that X must be closed and bounded.

Conversely, suppose that X is closed and bounded. Since X is bounded it lies insice a cube $C = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Since every closed interval is compact, and the product of compact metric spaces is compact, C is compact. But a closed subset of a compact set is compact. So X is compact. □

Corollary 1.3.15. $\mathbf{S}^n$ *is not homeomorphic to* $\mathbf{R}^m$*.*

Proof. Since $\mathbf{R}^m$ is not bounded it is not compact. On the other hand $\mathbf{S}^n$ is clearly bounded. Furthermore $\mathbf{S}^n$ is closed becuase the function $f\colon \mathbf{R}^n \to \mathbf{R}$, given by $f(x) := \|x\| = \mathrm{dist}(x, o)$ is continuous, and therefore $\mathbf{S}^n = f^{-1}(1)$ is closed. So $\mathbf{S}^n$ is compact. □

Next, we show that $\mathbf{R}^2$ is not homeomorphic to $\mathbf{R}^1$. This can be done by using the notion of connectedness. We say that a topological space X is *connected* if and only if the only subsets of X which are both open and closed are $\emptyset$ and X.

Exercise 1.3.16. Show that (i) if X is connected and Y is homeomorphic to X then Y is connected, and (ii) if X is connected and $f : X \to Y$ is continuous, then $f(X)$ is connected.

We also have the following fundamental result:

Theorem 1.3.17. $\mathbf{R}$ *and all of its intervals* $[a, b]$*,* (a, b) *are connected.*

We say that X is path connected if for every x_0, $x_1 \in X$, there is a continuous mapping $f\colon [0, 1] \to X$ such that $f(0) = x_0$ and $f(1) = x_1$.

Exercise 1.3.18. Show that if X is path connected and Y is homeomorphic to X then Y is path connected.

Exercise 1.3.19. Show that if X is path connected, then it its connected.

Exercise 1.3.20. Show that $\mathbf{R}^2$ is not homeomorphic to $\mathbf{R}^1$. (Hint: Suppose that there is a homeomorphism $f\colon \mathbf{R}^2 \to \mathbf{R}$. Then for a point $p \in \mathbf{R}^2$, f is a homeomorphism between $\mathbf{R}^2 - p$ and $\mathbf{R} - f(p)$.)

The technique hinted in Exercise 1.3.20 can also be used in the following:

Exercise 1.3.21. Show that the figure "8", with respect to its subspace topology, is not homeomorphic to $\mathbf{S}^1$.

Finally, we show that $\mathbf{R}^n$ is not homeomorphic to $\mathbf{R}^m$ if $m \neq n$. This may be done by using the following fundamental result:

Theorem 1.3.22 (Invariance of domain). *Let U, $V \subset \mathbf{R}^n$. Suppose that U is open and there exists a homeomorphism $f : U \to V$. Then V is also open.*

Corollary 1.3.23. $\mathbf{R}^n$ *is not homeomorphic to* $\mathbf{R}^m$ *unles* $m = n$.

Proof. Suppose that $m < n$. Then $\mathbf{R}^m$ is homeomprphic to the set of points in $\mathbf{R}^m$ whose last $n - m$ coordinates are zero. This set is not open in $\mathbf{R}^n$ (because its complement is). Therefore it may not be homeomorphic to $\mathbf{R}^n$ (which is an open subset of itself). □

Invariance of domain is usually proved by using homology theory, or more precisely, one uses homology to prove the Jordan Brouwer seperation theorem and then uses the Jordan-Brouwer theorem to prove the invriance of domain. But this would take up too much and would require us to go deeply into the realm of algebraic topology. We sketch here another proof which is taken from Alexandrov's book *Convex Polyhedra*.

Sketch of the proof of the Theorem 1.3.22. A set $T \subset \mathbf{R}^n$ is said to be a *simplex* if it is the convex hull of $n + 1$ points which are affinely independent. Note that for every point $p \in U$ we may find a simplex $T \subset A$ such that p is an interior point of T. Thus to prove the theorem it suffices to show that whenever p is an interior point of a simplex T then $f(p)$ is an interior point of $\widetilde{T} := f(T)$, i.e., there exists $\epsilon > 0$ such that $B_\epsilon(f(p)) \subset \widetilde{T}$. To this end, it is enough to show that the property of being an interior point of a homeomorphic image $\widetilde{T}$ of a simplex T may be characterized in terms which are invariant under homeomorphisms. What we propose as this characterization is the following covering property: a point p is an interior point of $\widetilde{T}$ provided that there is a covering of $\widetilde{T}$ by $n+1$ compact sets $X_i \subset \widetilde{T}$ such that (i) p is the only point which belongs to all X_i, and (ii) any covering of $\widetilde{T}$ by a set of $n+1$ compact sets $X'_i \subset \widetilde{T}$ which coincide with X_i outside an open neighborhood of p has a point p' which belongs to all X'_i. Since homeomorphisms preserve compactness, it is clear that this covering property is invariant under homeomoprhisms. So it remains to show that (step 1) if a point of $\widetilde{T}$ satisfies this covering property, then it is an interior point of $\widetilde{T}$ and (step 2) every interior point of $\widetilde{T}$ satisfies this covering property. □