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Geometry and Topology II
Spring 2005, PSU

Lecture Notes 3

1.8 Immersions and Embeddings

Let X and Y be topological spaces and $f\colon X \to Y$ be a continuous map. We say that f is an *immersion* provided that it is locally one-to-one, and f is an *embedding* provided that f is a homeomorphism between X and $f(X)$ with respect to the subspace topology.

Exercise 1. Show that if X is compact, Y is Hausdorf and $f\colon X \to Y$ is a one-to-one continuous map, then f is an embedding. Demonstrate, by means of a counterexample, that the previous sentence may not be true if X is not compact.

Exercise 2. Show that if $f\colon M \to N$ is an immersion between manifolds of the same dimension, then f is open. In particular, if M is compact, and N is connected but not compact, then there exists no immersion $f\colon M \to N$. (*Hint:* Use the invariance domain: if two subsets of $\mathbf{R}^n$ are homeomorphic and one of them is open then the other is open as well.)

Exercise 3. Let $C \subset M$ be a compact set, and $f\colon M \to X$ be an immersion which is one-to-one on C. Show that f is one-to-one on a neighborhood of C.

Exercise 4. Show that $\mathbf{RP}^2$ may be embedded in $\mathbf{R}^4$. (*Hint:* Consider the restriction to the sphere of the mapping $f\colon \mathbf{R}^3 \to \mathbf{R}^4$ given by $f(x, y, z) = (xy, yz, xz, x^2 + 2y^2 + 3z^2)$.)

Theorem 5. *Any compact manifold admits an embedding into a Euclidean space.*

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Proof. By definition, for every $p \in M$ there exists an open neighborhood U of p in M and a homeomorphism $\phi\colon U \to \mathbf{R}^n$. Let $V := \phi^{-1}(\operatorname{int} B^n(1))$, where $B^n(1)$ is the unit ball in $\mathbf{R}^n$. Since ϕ is continuous, V is open, and thus we obtain an open covering of M. But M is compact, so there exists a finite cover U_i, $1 \leq i \leq m$, of M and homeomorphisms $\phi_i\colon U_i \to \mathbf{R}^n$, such that $V_i := \phi_i^{-1}(\operatorname{int} B^n(1))$ also cover M.

Now let $\lambda\colon \mathbf{R}^n \to \mathbf{R}$ be a continuous map such that $\lambda \neq 0$ on $B^n(1)$ and $\lambda = 0$ on $\mathbf{R}^n - B^n(2)$. Define $\lambda_i\colon M \to \mathbf{R}$ by $\lambda_i(p) := \lambda(\phi_i(p))$ if $p \in U_i$ and $\lambda_i(p) = 0$ otherwise. We claim that, since M is hausorf, λ_i is continuous. To see this, let $K_i := \phi_i^{-1}(B^n(2))$. Then K_i is compact. So K_i is closed, since M is hausdorf. This yields that $M - K_i$ is open. In particular, since $K_i \subset U_i$, $\{U_i, M - K_i\}$ is an open cover M. Further, λ_i is continuous on U_i, since there it is the composition of two continuous functions, and λ_i is continuous on $M - K_i$, since there it is 0. This completes the argument that λ_i is continuous (if the resitriction of a function to each element of an open conver of a topological space is continuous, then that function is continuous over the entire space).

Next define $f_i\colon M \to \mathbf{R}^n$ by $f_i(p) = \lambda_i(p)\phi_i(p)$ if $p \in U_i$, and $f_i(p) = 0$ otherwise. Then f_i is continuous, since, similar to the argument we gave for λ_i above, f_i is continuous on U_i and $M - K_i$. Finally, define $f\colon M \to \mathbf{R}^{m(n+1)}$ by

$$f(p) = \big(\lambda_1(p), \ldots, \lambda_m(p), f_1(p), \ldots, f_m(p)\big).$$

Since each component function of f is continuous, f is continuous. We claim that f is the desired embedding. To this end, since f is continuous, and M is compact, it suffices to check that f is one-to-one.

Suppose that $f(p) = f(q)$. Then $f_i(p) = f_i(q)$, and $\lambda_i(p) = \lambda_i(q)$. Since V_i cover M, $p \in V_j$ for some fixed j. Consequently

$$\lambda_j(q) = \lambda_j(p) \neq 0,$$

which yields that $q \in U_j$. Since $p, q \in U_j$, it follows, from definition of f_i, that

$$\lambda_j(p)\phi_j(p) = f_j(p) = f_j(q) = \lambda_j(q)\phi_j(q).$$

So we conclude that $\phi_j(p) = \phi_j(q)$, which yields that $p = q$. □

Note that the above proof shows that if M can be covered by m open balls, then it may be embedded in $\mathbf{R}^{m(n+1)}$.

Exercise 6. Show that every compact manifold is metrizable.

Exercise 7. Show that there exists a compact topological space which is locally homeomorphic to a Euclidean space but is not Hausdorf. Conclude then that the previous theorem is not true if M is not Hausdorf.