Math 497C
Curves and Surfaces
Fall 2004, PSU

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Lecture Notes 7

1.17 The Frenet-Serret Frame and Torsion

Recall that if $\alpha\colon I \to \mathbf{R}^n$ is a unit speed curve, then the unit tangent vector is defined as

$$T(t) := \alpha'(t).$$

Further, if $\kappa(t) = \|T'(t)\| \neq 0$, we may define the principal normal as

$$N(t) := \frac{T'(t)}{\kappa(t)}.$$

As we saw earlier, in $\mathbf{R}^2$, $\{T, N\}$ form a moving frame whose derivatives may be expressed in terms of $\{T, N\}$ itself. In $\mathbf{R}^3$, however, we need a third vector to form a frame. This is achieved by defining the *binormal* as

$$B(t) := T(t) \times N(t).$$

Then similar to the computations we did in finding the derivatives of $\{T, N\}$, it is easily shown that

$$\begin{pmatrix} T(t) \\ N(t) \\ B(t) \end{pmatrix}' = \begin{pmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{pmatrix} \begin{pmatrix} T(t) \\ N(t) \\ B(t) \end{pmatrix},$$

where τ is the *torsion* which is defined as

$$\tau(t) := -\langle B', N \rangle.$$

Note that torsion is well defined only when $\kappa \neq 0$, so that N is defined. Torsion is a measure of how much a space curve deviates from lying in a plane:

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Exercise 1. Show that if the torsion of a curve $\alpha\colon I \to \mathbf{R}^3$ is zero everywhere then it lies in a plane. (*Hint*: We need to check that there exist a point p and a (fixed) vector v in $\mathbf{R}^3$ such that $\langle \alpha(t) - p, v\rangle = 0$. Let $v = B$, and p be any point of the curve.)

Exercise 2. Computer the curvature and torsion of the circular helix

$$(r\cos t, r\sin t, ht)$$

where r and h are constants. How does changing the values of r and h effect the curvature and torsion.

1.18 Curves of Constant Curvature and Torsion

The above exercise shows that the curvature and torsion of a circular helix are constant. The converse is also true

Theorem 3. *The only curve $\alpha\colon I \to \mathbf{R}^3$ whose curvature and torsion are nonzero constants is the circular helix.*

The rest of this section develops a number of exercises which lead to the proof of the above theorem

Exercise 4. Show that $\alpha\colon I \to \mathbf{R}^3$ is a circular helix (up to rigid motion) provided that there exists a vector v in $\mathbf{R}^3$ such that

$$\langle T, v\rangle = const,$$

and the projection of α into a plane orthogonal to v is a circle.

So first we need to show that when κ and τ are constants, v of the above exercise can be found. We do this with the aid of the Frenet-Serret frame. Since $\{T, N, B\}$ is an orthonormal frame, we may write

$$v = a(t)T(t) + b(t)N(t) + c(t)B(t).$$

Next we need to find a, b and c subject to the condtions that v is a constant vector, i.e., $v' = 0$, and that $\langle T, v\rangle = const$. The latter implies that

$$a = const$$

because $\langle T, v\rangle = a$. In particular, we may set $a = 1$.

Exercise 5. By setting $v' = 0$ show that

$$v = T + \frac{\kappa}{\tau} B,$$

and check that v is the desired vector, i.e. $\langle T, v\rangle = const$ and $v' = 0$.

So to complete the proof of the theorem, only the following remains:

Exercise 6. Show that the projection of α into a plane orthogonal to v, i.e.,

$$\overline{\alpha}(t) := \alpha(t) - \langle \alpha(t), v\rangle \frac{v}{\|v\|^2}$$

is a circle. (*Hint*: Compute the curvature of $\overline{\alpha}$.)

1.19 Intrinsic Characterization of Spherical Curves

In this section we derive a characterization in terms of curvature and torsion for unit speed curves which lie on a shphere. Suppose $\alpha \colon I \to \mathbf{R}^3$ lies on a sphere of radius r. Then there exists a point p in $\mathbf{R}^3$ (the center of the sphere) such that

$$\|\alpha(t) - p\| = r.$$

Thus differentiation yields

$$\langle T(t), \alpha(t) - p\rangle = 0.$$

Differentiating again we obtain:

$$\langle T'(t), \alpha(t) - p\rangle + 1 = 0.$$

The above expression shows that $\kappa(t) \neq 0$. Consequently N is well defined, and we may rewrite the above expression as

$$\kappa(t)\langle N(t), \alpha(t) - p\rangle + 1 = 0.$$

Differentiating for the third time yields

$$\kappa'(t)\langle N(t), \alpha(t) - p\rangle + \kappa(t)\langle -\kappa(t)T(t) + \tau(t)\langle B(t), \alpha(t) - p\rangle = 0,$$

which using the previous expressions above we may rewrite as

$$-\frac{\kappa'(t)}{\kappa(t)} + \kappa(t)\tau(t)\langle B(t), \alpha(t) - p\rangle = 0.$$

Next note that, since $\{T, N, B\}$ is orthonormal,

$$\begin{aligned} r^2 &= \|\alpha(t) - p\|^2 \\ &= \langle \alpha(t) - p, T(t)\rangle^2 + \langle \alpha(t) - p, N(t)\rangle^2 + \langle \alpha(t) - p, B(t)\rangle^2 \\ &= 0 + \frac{1}{\kappa^2(t)} + \langle \alpha(t) - p, B(t)\rangle^2. \end{aligned}$$

Thus, combining the previous two calculations, we obtain:

$$\left(\frac{\kappa'(t)}{\kappa^2(t)}\right)^2 = \tau^2(t)\left(r^2 - \frac{1}{\kappa^2(t)}\right).$$

Exercise 7. Check the converse, that is supposing that the curvature and torsion of some curve satisfies the above expression, verify whether the curve has to lie on a sphere of radius r.

To do the above exercise, we need to first find out where the center p of the sphere could lie. To this end we start by writing

$$p = \alpha(t) + a(t)T(t) + b(t)N(t) + c(t)B(t),$$

and try to find $a(t)$, $b(t)$ and $c(t)$ so that $p' = (0, 0, 0)$, and $\|\alpha(t) - p\| = r$. To make things easier, we may note that $a(t) = 0$ (why?). Then we just need to find $b(t)$ and $c(t)$ subject to the two constraints mentioned above. We need to verify whether this is possible, when κ and τ satisfy the above expression.

1.20 The Local Canonical form

In this section we show that all C^3 curve in $\mathbf{R}^3$ essentially look the same in the neighborhood of any point which has nonvanishing curvature and a given sign for torsion.

Let $\alpha\colon (-\epsilon, \epsilon) \to \mathbf{R}^3$ be a C^3 curve. By Taylor's theorem

$$\alpha(t) = \alpha(0) + \alpha'(0)t + \frac{1}{2}\alpha''(0)t^2 + \frac{1}{6}\alpha'''(0)t^3 + R(t)$$

where $\lim_{t\to 0} |R(t)|/t^3 = 0$, i.e., for t small, the remainder term $R(t)$ is negligible. Now suppose that α has unit speed. Then

$$\begin{gathered} \alpha' = T \\ \alpha'' = T' = \kappa N \\ \alpha''' = (\kappa N)' = \kappa' N + \kappa(-\kappa T + \tau B) = -\kappa^2 T + \kappa' N + \kappa\tau B. \end{gathered}$$

So we have

$$\begin{aligned}
\alpha(t) &= \alpha(0) + T_0 t + \frac{\kappa_0 N_0 t^2}{2} + \frac{(-\kappa_0^2 T_0 + \kappa_0' N_0 + \kappa\tau_0 B_0)t^3}{6} + R(t) \\
&= \alpha(0) + (t - \frac{\kappa_0^2}{6}t^3)T_0 + (\frac{\kappa_0}{2}t^2 + \frac{\kappa_0'}{6}t^3)N_0 + (\frac{\kappa_0\tau_0}{6}t^3)B_0 + R(t)
\end{aligned}$$

Now if, after a rigid motion, we suppose that $\alpha(0) = (0,0,0)$, $T = (1,0,0)$, $N = (0,1,0)$, and $B = (0,0,1)$, then we have

$$\alpha(t) = \left(t - \frac{\kappa_0^2}{6}t^3 + R_x,\ \frac{\kappa_0}{2}t^2 + \frac{\kappa_0'}{6}t^3 + R_y,\ \frac{\kappa_0\tau_0}{6}t^3 + R_z\right),$$

where $(R_x, R_y, R_z) = R$. It follows then that when t is small

$$\alpha(t) \approx \left(t,\ \frac{\kappa_0}{2}t^2,\ \frac{\kappa_0\tau_0}{6}t^3\right).$$

Thus, up to third order of differentiation, any curve with nonvanishing curvature in space may be approximated by a cubic curve. Also note that the above approximtion shows that any planar curve with nonvanishing curvature locally looks like a parabola.

Exercise 8. Show that the curvature of a space curve α at a point t_0 with nonvanishing curvature is the same as the curvature of the projection of α into the the osculating plane at time t_0. (The osculating plane is the plane generated by T and N).