THE RELATIVE ISOPERIMETRIC INEQUALITY OUTSIDE A CONVEX DOMAIN IN $\mathbf{R}^n$

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Abstract. We prove that the area of a hypersurface Σ which traps a given volume outside a convex domain C in Euclidean space $\mathbf{R}^n$ is bigger than or equal to the area of a hemisphere which traps the same volume on one side of a hyperplane. Further, when C has smooth boundary ∂C, we show that equality holds if and only if Σ is a hemisphere which meets ∂C orthogonally.

1. Introduction

Let $\mathbf{H}^n := \{(x_1, ..., x_n) \in \mathbf{R}^n : x_n \geqslant 0\}$ be the closed upper half of Euclidean space $\mathbf{R}^n$. Given $D \subset \mathbf{H}^n$, reflection across the boundary $\partial\mathbf{H}^n$ and the classical isoperimetric inequality in $\mathbf{R}^n$ imply that

$$\big(\operatorname{area}(\partial D \sim \partial\mathbf{H}^n)\big)^n \geqslant \frac{1}{2}\, n^n \omega_n\, (\operatorname{vol} D)^{n-1},$$

with equality if and only if D is a half ball and $\partial D \sim \partial\mathbf{H}^n$ is a hemisphere. Here *area* and *vol* (*volume*) denote, respectively, the $(n-1)$ and n dimensional Hausdorff measures, ω_n is the volume of the unit ball in $\mathbf{R}^n$, and $\sim$ is the set exclusion operator. In this paper we prove that the above inequality holds outside any convex set $C \subset \mathbf{R}^n$ with interior points, i.e.,

$$\big(\operatorname{area}(\partial D \sim \partial C)\big)^n \geqslant \frac{1}{2}\, n^n \omega_n (\operatorname{vol} D)^{n-1}, \tag{1.1}$$

for any $D \subset \mathbf{R}^n \sim C$. Further we show that when ∂C is smooth, equality holds if and only if D is a half ball and $\partial D \sim \partial C$ is a hemisphere.

We call (1.1) the *relative isoperimetric inequality of* D with supporting set C. The proof of this inequality for $n = 2$ is easy once one reflects the convex hull of D about its linear boundary. For $n \geqslant 3$ some partial results were known: I. Kim [10] proved (1.1) for $C = U \times \mathbf{R}$, where U is the epigraph of a $\mathcal{C}^2$

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convex function, and J. Choe [2] proved (1.1) when $\partial D \cap \partial C$ is a graph which is symmetric about $(n-1)$ hyperplanes of $\mathbf{R}^n$. More recently, J. Choe and M. Ritoré [4] have shown that (1.1) holds outside convex sets in 3-dimensional Cartan-Hadamard manifolds, with equality if and only if D is a flat half ball and $\Sigma := \partial D \sim \partial C$ is a hemisphere. The main ingredients of the proof in [4] are the estimate $(\sup_\Sigma H^2)\operatorname{area}\Sigma \geqslant 2\pi$, and the analysis of the equality case, where H is the mean curvature of Σ; however, the methods used in [4], which were inspired by the work of P. Li and S.-T. Yau [11], are valid only when $n = 3$.

We obtain inequality (1.1), and the characterization of its equality case presented below, from the estimate $(\sup_\Sigma H^{n-1})\operatorname{area}\Sigma \geqslant \mathbf{c}_{n-1}/2$, where $\mathbf{c}_{n-1}$ is the area of the unit sphere $\mathbf{S}^{n-1} \subset \mathbf{R}^n$. This inequality follows from the arithmetic-geometric mean inequality between H (the average of the principal curvatures) and the Gauss-Kronecker curvature GK (the product of principal curvatures) of Σ, once we show that the total Gauss-Kronecker curvature of the set of regular points of $\partial D \sim \partial C$ with positive principal curvatures is larger than or equal to $\mathbf{c}_{n-1}/2$. We proved the latter inequality in [3] assuming more regularity than is warranted in the present case; however, as we verify below, that proof essentially works here as well.

2. Preliminaries: Existence and Regularity

Throughout this paper $C \subset \mathbf{R}^n$ denotes a *proper convex domain*, i.e., a closed convex set with interior points and nonempty boundary ∂C. Further, unless noted otherwise we assume that ∂C is $\mathcal{C}^2$. For any $A \subset \mathbf{R}^n$, let $A_C := A \sim \overline{C}$, where $\overline{C}$ denotes, as usual, the closure of C. The *relative isoperimetric profile* of $\mathbf{R}^n_C$ is the function $I_C : \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$I_C(v) := \inf_D \big\{ \operatorname{area}(\partial D)_C : D \Subset \mathbf{R}^n_C,\ \operatorname{vol} D = v \big\},$$

where $D \Subset \mathbf{R}^n_C$ means that D is relatively compact in $\mathbf{R}^n_C$. Note that

$$I_{\mathbf{H}^n}(v) = n\,(\omega_n/2)^{1/n} v^{(n-1)/n}.$$

So the relative isoperimetric inequality (1.1) is equivalent to

$$I_C(\operatorname{vol} D) \geqslant I_{\mathbf{H}^n}(\operatorname{vol} D). \tag{2.1}$$

An *isoperimetric region* $D \subset \mathbf{R}^n_C$ is one for which the equality $\operatorname{area}(\partial D)_C = I_C(\operatorname{vol} D)$ holds. An isoperimetric region need not exist for a given volume.

Denote by $\mathcal{C}^1_0(T\mathbf{R}^n_C)$ the set of $\mathcal{C}^1$ vector fields with compact support in $\mathbf{R}^n_C$. Let E be the closure of a bounded open set in $\mathbf{R}^n_C$, and for any $D \subset E$ define the *perimeter* of D relative to C as

$$\mathcal{P}_C(D) := \sup\left\{ \int_D \operatorname{div} X : X \in \mathcal{C}_0^1(T\mathbf{R}^n_C),\ |X| \leqslant 1 \right\},$$

where $|X|$ is the supremum norm. Stokes theorem implies that the perimeter of a set and the area of its boundary coincide for sets whose boundary is a piecewise $\mathcal{C}^1$ hypersurface; see Giusti's book [5] for background on finite perimeter sets. In order to prove (1.1), we need to minimize $\mathcal{P}_C$ subject to a volume constraint, i.e., given $v \in (0, \operatorname{vol} E)$, we wish to find $\Omega_0 \subset E$, with $\operatorname{vol}\Omega_0 = v$, such that

$$\mathcal{P}_C(\Omega_0) \leqslant \mathcal{P}_C(\Omega),$$

for any $\Omega \subset E$ with $\operatorname{vol}\Omega = v$. The existence of Ω_0 is guaranteed by the boundedness of E, see [5], and the regularity properties of Ω_0 which we need may be summarized as follows

Lemma 2.1. *Let E be the closure of a bounded domain with smooth boundary in $\mathbf{R}^n_C$. Then, for any $v \in (0, \operatorname{vol} E)$, there is a set $\Omega_0 \subset E$ of volume v minimizing $\mathcal{P}_C$. Moreover*

(i) ([6]) $\partial\Omega_0$ *has constant mean curvature and is smooth in the interior of E except for a singular set of Hausdorff dimension less than or equal to $(n-8)$.*
(ii) ([8, p. 263]) $\overline{(\partial\Omega_0)_C}$ *meets ∂C orthogonally except for a singular set of Hausdorff dimension less than or equal to $(n-8)$. In fact $\overline{(\partial\Omega_0)_C}$ is smooth at every point of $\overline{(\partial\Omega_0)_C} \cap \partial C$ away from this singular set.*
(iii) ([14, Thm. 3.6]) *If $(\partial E)_C$ is strictly convex then $(\partial\Omega_0)_C$ meets $(\partial E)_C$ tangentially and it is $\mathcal{C}^{1,1}$ in a neighborhood of $(\partial E)_C$.*
(iv) *At every point $x_0 \in (\partial\Omega_0)_C$ there is a tangent cone obtained by blowing up the set Ω_0 from x_0. If this tangent cone is contained in a half space of $\mathbf{R}^n$, then it is the half space and $(\partial\Omega_0)_C$ is regular at x_0* [5]. *At points in $\overline{(\partial\Omega_0)_C} \cap \partial C \sim \overline{(\partial E)_C}$ we have the same result, as described in* [8].

The $\mathcal{C}^{1,1}$ regularity of $(\partial\Omega_0)_C$ near $(\partial E)_C$ will be enough for our purposes here since, by Rademacher's Theorem, a $\mathcal{C}^{1,1}$ hypersurface has principal curvatures defined almost everywhere, and thus we will be able to apply the integral curvature estimates obtained in the next two sections.

3. The Estimate for Total Positive Curvature

First we give a general definition for total curvature τ^+. Let $\Sigma = (\Sigma \sim \Sigma_0) \cup \Sigma_0$ be the compact union of a $\mathcal{C}^{1,1}$ hypersurface $\Sigma \sim \Sigma_0$ with boundary and a singular set Σ_0 of Hausdorff dimension less than or equal to $(n-8)$ so that $\Sigma_0 \subset \overline{\Sigma \sim \Sigma_0}$. The points in $\Sigma \sim \Sigma_0$ will be called *regular points* of Σ. A hyperplane $\Pi \subset \mathbf{R}^n$ is called a *restricted support hyperplane* of Σ at a point p, if $p \in \Pi \cap \Sigma$, Σ lies on one side of Π, and Π is tangent to Σ when $p \in \partial\Sigma \sim \Sigma_0$. An *outward normal* of Π is a normal vector to Π which points towards a side of Π not containing Σ. If Π is a restricted support hyperplane for an open neighborhood U_p of p in Σ, then Π is called a *restricted local support hyperplane*; furthermore, p is a *locally strictly convex point* of Σ, or $p \in \Sigma^+$, provided that $\Pi \cap U_p = \{p\}$. The *total positive curvature* τ^+ of Σ is defined as the algebraic area of the unit normals to restricted local support hyperplanes of Σ at points of Σ^+, where by *area* we mean the $(n-1)$ dimensional Hausdorff measure. As we are assuming that $\Sigma \sim \Sigma_0$ is a $\mathcal{C}^{1,1}$ hypersurface, the principal curvatures are defined for almost every point of $\Sigma \sim \Sigma_0$ and so the Gauss-Kronecker curvature GK, the product of all principal curvatures, may be integrated on $\Sigma \sim \Sigma_0$. Moreover, in case there are no restricted local support hyperplanes of Σ at points of Σ_0, we have

$$\tau^+(\Sigma) = \int_{\Sigma^+ \sim \Sigma_0} GK.$$

As remarked in the introduction, the main ingredient in the proof of the relative isoperimetric inequality is the following estimate. We state below the version for convex sets with smooth boundary we shall need. The proof of this result is a slight modification of the appendix of [3].

Lemma 3.1. *Let $\Sigma = (\Sigma \sim \Sigma_0) \cup \Sigma_0 \subset \mathbf{R}^n$ be the union of a $\mathcal{C}^{1,1}$ immersed hypersurface $\Sigma \sim \Sigma_0$ and a singular set Σ_0 such that $\partial\Sigma \sim \Sigma_0$ is a $\mathcal{C}^2$ submanifold that lies on the boundary of a convex set $C \subset \mathbf{R}^n$ with $\mathcal{C}^2$ boundary ∂C. Suppose that there are no restricted local support hyperplanes of Σ at points of Σ_0, and that, at each point $p \in \partial\Sigma \sim \Sigma_0$, the inward conormal $\sigma(p)$ of $\partial\Sigma$ is the outward unit normal to C at p. Then*

$$\tau^+(\Sigma) \geqslant \frac{\mathbf{c}_{n-1}}{2}, \tag{3.1}$$

and equality holds if and only if $\partial\Sigma$ lies in a hyperplane.

Proof. Let $\partial\Sigma_r := \partial\Sigma \sim \Sigma_0$ be the regular part of the boundary of Σ,

$$U\partial\Sigma_r := \left\{ (p,u) \mid p \in \partial\Sigma_r, u \in \mathbf{S}^{n-1}, u \perp T_p\partial\Sigma_r \right\}$$

be the *unit normal bundle* of $\partial\Sigma_r$, and $\nu\colon U\partial\Sigma_r \to \mathbf{S}^{n-1}$, given by

$$\nu(p,u) := u,$$

be its *Gauss map*. Define $I \subset J \subset U\partial\Sigma_r$ by

$$I := \big\{ (p,u) \in U\partial\Sigma_r \mid \langle x-p, u\rangle \leqslant 0, \quad \forall x \in \Sigma \big\},$$
$$J := \big\{ (p,u) \in U\partial\Sigma_r \mid \langle x-p, u\rangle \leqslant 0, \quad \forall x \in \partial\Sigma \big\}.$$

Note that if $(p,u) \in J \sim I$, then the height function $x \mapsto \langle x-p, u\rangle$ achieves its maximum in the interior of Σ, and thus Σ has a restricted support hyperplane with outward normal u. By hypothesis, the point p must then lie in the regular part $\Sigma \sim \Sigma_0$ of Σ. Hence

$$\tau^+(\Sigma) \geqslant \operatorname{area} \nu(J \sim I),$$

since almost every support hyperplane of Σ intersects Σ at a single point [13, Thm. 2.2.9]. So to prove (3.1) it suffices to show that

$$\operatorname{area} \nu(J \sim I) \geqslant \frac{\mathbf{c}_{n-1}}{2}. \tag{3.2}$$

To this end note that, since, again by [13, Thm. 2.2.9], almost every element of $\nu(I-J)$ has multiplicity one,

$$\operatorname{area} \nu(J \sim I) = \int_{J\sim I} \operatorname{Jac}\nu = \int_J \operatorname{Jac}\nu - \int_I \operatorname{Jac}\nu,$$

where $\operatorname{Jac}\nu$ denotes the Jacobian of ν, which may be defined as the pull back via ν of the volume element of $\mathbf{S}^{n-1}$. Further note that, since every unit vector $u \in \mathbf{S}^{n-1}$ is the outward normal to some support hyperplane of $\partial\Sigma$,

$$\int_J \operatorname{Jac}\nu = \operatorname{area}\nu(J) = \mathbf{c}_{n-1}.$$

Thus to establish (3.2) it suffices to show that

$$\int_I \operatorname{Jac}\nu \leqslant \frac{1}{2}\int_J \operatorname{Jac}\nu.$$

In particular, if I_p and J_p denote the fibers of I and J respectively, then, by Fubini's theorem, it suffices to show that

$$\int_{I_p} \operatorname{Jac}\nu \leqslant \frac{1}{2}\int_{J_p} \operatorname{Jac}\nu, \tag{3.3}$$

for all $p \in \partial\Sigma_r$.

The above inequality is trivially satisfied whenever $I_p = \emptyset$ or $\nu(I_p)$ consists only of a pair of antipodal points of $\mathbf{S}^{n-1}$. Thus, by [3, Lemma 3.1], we may assume that I_p is nonempty and connected, which in turn yields that J_p is nonempty and connected as well.

For every $u \in \mathbf{S}^{n-1}$, let $h_u \colon \partial\Sigma \to \mathbf{S}^{n-1}$ be the height function given

$$h_u(p) := \langle p, u\rangle.$$

Then we have the following well-known identity

$$\operatorname{Jac}\nu_{(p,u)} = |\det(\operatorname{Hess} h_u)_p|,$$

where $(\operatorname{Hess} h_u)_p \colon T_p\partial\Sigma \times T_p\partial\Sigma \to \mathbf{R}$ denotes the Hessian of h_u at p. (To see this one may note that $U\partial\Sigma_r$ can be identified with a hypersurface $\overline{U\partial\Sigma_r}$ of $\mathbf{R}^n$ via the endpoint map $(p,u) \mapsto p+u$. Then the Gauss map $\overline{\nu}$ of $\overline{U\partial\Sigma_r}$, is given by $\overline{\nu}(p+u) := u = \nu(p,u)$. Consequently $\operatorname{Jac}\nu_{(p,u)} = \operatorname{Jac}\overline{\nu}_{p+u} = |\det(\mathrm{II}_{p+u})|$, where II_{p+u} is the second fundamental form of $\overline{U\partial\Sigma_r}$ at $p+u$. But $\mathrm{II}_{p+u} = (\operatorname{Hess}\overline{h}_u)_{p+u}$, where $\overline{h}_u \colon \overline{U\partial\Sigma_r} \to \mathbf{R}$ is the height function $\overline{h}_u(p+u) := \langle p+u, u\rangle$. In particular, $\overline{h}_u(p+u) = h_u(p)+1$, which yields that $\det(\operatorname{Hess}\overline{h}_u)_{p+u} = \det(\operatorname{Hess} h_u)_p$.)

Next let $\sigma^\perp(p)$ be a unit normal vector of $\partial\Sigma_r$ at p which is orthogonal to $\sigma(p)$, and is chosen so that the function $\langle x-p, \sigma^\perp(p)\rangle$ is positive for some $x \in \Sigma$ or vanishes for all $x \in \Sigma$. For $\theta \in [\pi, -\pi]$ define

$$u(\theta) := \cos\theta\,\sigma(p) + \sin\theta\,\sigma^\perp(p), \quad \text{and} \quad H_\theta := (\operatorname{Hess} h_{u(\theta)})_p.$$

Then, since $\|\partial u/\partial\theta\| = 1$, the change of variables formula allows us to rewrite (3.3) as

$$\int_{\theta_0}^{\theta_1} |\det(H_\theta)|\,d\theta \leqslant \frac{1}{2}\int_{\phi_0}^{\phi_1} |\det(H_\theta)|\,d\theta,$$

where $[\theta_0,\theta_1] \subset [\phi_0,\phi_1] \subset [-\pi,\pi]$, and $u([\theta_0,\theta_1]) = \nu(I_p)$, $u([\phi_0,\phi_1]) = \nu(J_p)$.

Note that if $u \in \nu(I_p)$, then $\langle u, \sigma(p)\rangle$ and $\langle u, \sigma^\perp(p)\rangle$ must both be non-positive. Thus $[\theta_0,\theta_1] \subset [-\pi, -\frac{\pi}{2}]$. Further, since $0 \in [\phi_0,\phi_1]$, it follows that $[\theta_0, 0] \subset [\phi_0,\phi_1]$. Hence to prove the above inequality it is enough to show that

$$\int_{\theta_0}^{-\pi/2} |\det(H_\theta)|\,d\theta \leqslant \int_{-\pi/2}^{0} |\det(H_\theta)|\,d\theta. \tag{3.4}$$

To this end note that for any tangent vectors $X_p, Y_p \in T_p\partial\Sigma_r$, with local extensions X, Y,

$$\begin{aligned} H_\theta(X_p,Y_p) &= X_p\big(Y h_{u(\theta)}\big) = \big\langle D_{X_p}Y, u(\theta)\big\rangle \\ &= \cos\theta\big\langle D_{X_p}Y, u(0)\big\rangle + \sin\theta\left\langle D_{X_p}Y, u\left(\frac{\pi}{2}\right)\right\rangle \\ &= \cos\theta\, H_0(X_p,Y_p) + \sin\theta\, H_{\frac{\pi}{2}}(X_p,Y_p), \end{aligned}$$

where D denotes the standard covariant derivative, or Levi-Civitá connection on $\mathbf{R}^n$.

Also note that H_0 is negative semidefinite because by assumption $u(0) = \sigma(p) \in \nu(J_p)$. Further, since $\theta_0 \in [-\pi, -\frac{\pi}{2}] \cap [\phi_0,\phi_1]$, and $0 \in [\phi_0,\phi_1]$, it follows that $-\frac{\pi}{2} \in [\phi_0, \phi_1]$. So $u(-\frac{\pi}{2}) \in \nu(J_p)$, which yields that $H_{\frac{\pi}{2}}$ is positive semidefinite. For any $\theta \in [-\pi, -\frac{\pi}{2}]$, let $\theta' := -\pi - \theta \in [-\frac{\pi}{2}, 0]$. Then $\cos\theta' = -\cos\theta < 0$, and $\sin\theta' = \sin\theta < 0$. Thus

$$-H_{\theta'}(X_p, X_p) \geqslant -H_\theta(X_p, X_p).$$

Hence the eigenvalues of $-H_{\theta'}$ are bigger than or equal to those of $-H_\theta$. But for all $\theta \in [\theta_0, -\frac{\pi}{2}]$, H_θ and $H_{\theta'}$ are both negative semidefinite, because $u(\theta)$, $u(\theta') \in \nu(I_p)$. So $-H_\theta$ and $-H_{\theta'}$ are positive semidefinite. Consequently

$$|\det(H_{\theta'})| = \det(-H_{\theta'}) \geqslant \det(-H_\theta) = |\det(H_\theta)|,$$

which yields that

$$\int_{\theta_0}^{-\pi/2} |\det(H_\theta)|\, d\theta \leqslant \int_{-\pi/2}^{\theta_0'} |\det(H_\theta)|\, d\theta. \tag{3.5}$$

Since $\theta_0' \leqslant 0$, this yields (3.4), which in turn completes the proof of (3.1).

Now suppose that equality holds in (3.1), then equality holds in the above inequalities. In particular, equalities hold in (3.4) and (3.5), which yields

$$\int_{-\pi/2}^{\theta_0'} |\det(H_\theta)|\, d\theta = \int_{-\pi/2}^{0} |\det(H_\theta)|\, d\theta.$$

So we conclude

$$\int_{\theta_0'}^{0} |\det(H_\theta)|\, d\theta = 0.$$

This implies that $(\operatorname{Hess} h_{u(\theta)})_p \equiv 0$ for all $\theta_0'(p) \leqslant \theta \leqslant 0$, as p ranges over $\partial\Sigma_r$. But it is a well-known consequence of Sard's theorem that h_u is a Morse function [1], i.e., it has nondegenerate Hessian, for almost all $u \in \mathbf{S}^{n-1}$. So we must have $\theta_0' = 0$, which yields that $\theta_0 = -\pi$, for some p. So $u(-\pi) \in \nu(J_p)$. But $-u(-\pi) = u(0) = \sigma(p) \in \nu(J_p)$ as well. Hence $\partial\Sigma_r$ and therefore $\partial\Sigma$ lie in a hyperplane. □

4. The Mean Curvature Estimate

As we will see in the next section, in order to prove (1.1), we need to construct a bounded region E outside C in $\mathbf{R}^n$, and minimize the perimeter $\mathcal{P}_C$ under a volume constraint inside E. We shall see in our next result that the boundary of any isoperimetric region so obtained satisfies the hypotheses of Lemma 3.1. In particular, the lower curvature bound (3.1) holds for such regions, which in turn yields the following estimate for mean curvature.

Proposition 4.1. *Let $p \in \partial C$, $E := \overline{B(p,r)_C}$, $\Omega \subset E_C$ be a set minimizing the perimeter $\mathcal{P}_C$ under a volume constraint, and H_Σ be the (constant) mean curvature of the regular part of $\Sigma := \overline{(\partial\Omega)_C}$. Then*

$$H_\Sigma^{n-1}\,\mathcal{P}_C(\Omega) \geqslant \frac{\mathbf{c}_{n-1}}{2}. \tag{4.1}$$

Equivalently, if $H_0(a)$ denotes the mean curvature of a hemisphere of area a, then

$$H_\Sigma \geqslant H_0\big(\mathcal{P}_C(\Omega)\big).$$

Equality holds in these inequalities if and only if Ω is a half ball and Σ meets ∂C orthogonally.

Proof. It is enough to show that, if Π is a support hyperplane of Ω at $p \in \Sigma$, then p is a regular point of Σ.

If $p \in \Sigma \cap \operatorname{int}(E)$ then the minimal tangent cone of Ω at p is contained in a half space. By [5, Thm. 15.5], it must be a half space and so Σ is regular at p.

If $p \in \partial\Sigma \sim \overline{(\partial E)_C}$ then we consider the integer multiplicity rectifiable current $\partial[\Omega]$. Reflecting it with respect to ∂C and blowing up from p we get an area-minimizing oriented tangent cone T [7], [8]. Let H be the tangent hyperplane of ∂C at p, and H^+ the closed half space determined by H whose interior does not meet C. Assuming there is a support hyperplane Π of Ω at p, we get that the support of T, $\operatorname{supp}(T)$, is contained in a region of $\mathbf{R}^n$ bounded by $H_1 \cup H_2$, where $H_1 = \Pi \cap H^+$ and H_2 is the reflection of H_1 with respect to H. Let $S = H_1 \cap H = H_1 \cap H_2$. We have that S is an $(n-2)$-dimensional linear submanifold of $\mathbf{R}^n$ which is contained in H.

Rotating H_1, H_2 with respect to S until they first touch $\operatorname{supp}(T) \sim S$, using the maximum principle, and a connectedness argument, we get that $\operatorname{supp}(T) = H_1 \cup H_2$, which is not area-minimizing unless $H_1 \cup H_2$ is a hyperplane orthogonal to H. Hence Σ is regular at p.

Observe that $\partial\Sigma \cap \partial C \cap \overline{(\partial E)_C} = \emptyset$: if $x_0 \in \partial\Sigma \cap \partial C \cap \overline{(\partial E)_C}$, then the outer normal ν to ∂C and the outer normal $\tilde{\nu}$ to $\partial B(p,r)$ satisfy $\langle \nu, \tilde{\nu}\rangle(x_0) > 0$. Reasoning as in the two previous paragraphs, reflecting and blowing up from x_0 we get a cone which minimizes area in a wedge of angle less than π, thus getting a contradiction.

So we can apply Lemma 3.1 to conclude that

$$\int_{\Sigma^+ \sim \Sigma_0} GK = \tau^+(\Sigma) \geqslant \frac{\mathbf{c}_{n-1}}{2}.$$

By [14, Thm. 3.7], H_Σ, the constant mean curvature of the regular part $\Sigma \sim \Sigma_0$ of Σ in the interior of E, is an upper bound for the mean curvature of Σ. So we have

$$H_\Sigma^{n-1}\,\mathcal{P}_C(\Omega) \geqslant \int_{\Sigma^+\sim\Sigma_0} H_\Sigma^{n-1} \geqslant \int_{\Sigma^+\sim\Sigma_0} GK \geqslant \frac{\mathbf{c}_{n-1}}{2},$$

which establishes the first desired inequality. To obtain the second inequality note that, if r is the radius of a hemisphere of area $\mathcal{P}_C(\Omega)$, then

$$\big(H_0(\mathcal{P}_C(\Omega))\big)^{n-1}\mathcal{P}_C(\Omega) = \left(\frac{1}{r}\right)^{n-1}\frac{\mathbf{c}_{n-1}r^{n-1}}{2} = \frac{\mathbf{c}_{n-1}}{2} \leqslant H_\Sigma^{n-1}\,\mathcal{P}_C(\Omega).$$

If equality holds then $\Sigma \sim \Sigma_0 = \Sigma^+$, and $H_\Sigma^{n-1} = GK$, which implies that $\Sigma \sim \Sigma_0$ is totally umbilical and so Σ_0 is empty. Further $\partial\Sigma$ lies in a hyperplane by Lemma 3.1, and so Ω is a half ball and Σ intersects ∂C orthogonally. □

5. Proof of the Relative Isoperimetric Inequality in $\mathbf{R}^n_C$

With the aid of Proposition 4.1 we are now in a position to prove the main result of this paper.

Theorem 5.1. *Let $C \subset \mathbf{R}^n$ be a proper convex domain with $\mathcal{C}^2$ boundary. For any bounded set $D \subset \mathbf{R}^n_C$ with finite perimeter,*

$$\big(\operatorname{area}(\partial D)_C\big)^n \geqslant \frac{1}{2}\,n^n\omega_n(\operatorname{vol} D)^{n-1}, \tag{5.1}$$

with equality if and only if D is a half ball and $(\partial D)_C$ is a hemisphere.

Remark 5.2. If C is bounded then, from the results in [12], it can be proved that any perimeter minimizing sequence of sets in $\mathbf{R}^n_C$ of given volume has a subsequence converging to an isoperimetric region. In this case the proof of Theorem 5.1 can be slightly simplified. However, when C is unbounded, we have to deal with the possibility of nonexistence of minimizers in $\mathbf{R}^n_C$.

Proof of Theorem 5.1. First we construct an exhaustion of $\mathbf{R}^n_C$. Fix $p_0 \in \partial C$, and let $\{r_m\}_{m\in\mathbb{N}}$ be a diverging sequence of positive increasing numbers. In case C is bounded we require that $C \subset B(p_0, r_m)$. We define $E_m := \overline{B(p_0, r_m)_C}$.

Since E_m is bounded, isoperimetric regions exist in E_m for any given volume $v \in (0, \operatorname{vol} E_m)$. Let $\Omega \subset E_m$ be an isoperimetric region minimizing $\mathcal{P}_C$ in E_m under a volume constraint, and let $\Sigma := \overline{(\partial\Omega)_C}$.

By Proposition 4.1, for every component Ω' of Ω touching the boundary of C, with $\Sigma' := \overline{(\partial\Omega')_C}$, we have

$$H_{\Sigma'}^{n-1}\,\mathcal{P}_C(\Sigma') \geqslant \mathbf{c}_{n-1}/2,$$

with equality if and only if Ω' is an open half ball and Σ' is an open hemisphere. Observe that, for a component Ω'' of Ω not touching the boundary of C, with $\Sigma'' := \overline{(\partial\Omega'')_C}$, one easily obtains

$$H_{\Sigma''}^{n-1}\,\mathcal{P}_C(\Sigma'') \geqslant \mathbf{c}_{n-1},$$

with equality if and only if Ω'' is a ball and Σ'' a round sphere.

Breaking Ω into components touching ∂C and components in the interior of $\mathbf{R}^n_C$ we get

$$H_{\Sigma}^{n-1}\,\mathcal{P}_C(\Omega) \geqslant \mathbf{c}_{n-1}/2,$$

and equality holds if and only if Ω consists of one connected component which is a half ball, and Σ an open hemisphere.

Let I_m be the isoperimetric profile of E_m. From standard arguments, see [9, p. 170–172,], it follows that (i) I_m is continuous and increasing, (ii) if I_m is smooth at v_0, then $I_m'(v_0) = (n-1)H$, where H is the constant mean curvature in the interior of E_m of *any* isoperimetric region of volume v_0, and (iii) left and right derivatives of I_m exist everywhere. When (i), (ii) and (iii) hold it is then known that I_m is an absolutely continuous function. For a proof of (i), (ii) and (iii) we refer the reader to [4].

Let J_m be the restriction of the isoperimetric profile of a half space of $\mathbf{R}^n$ to the interval $(0, \operatorname{vol} E_m)$, and $f(a)$, $g(a)$ be the inverse functions of I_m, J_m, respectively. We know that

$$g'(a) = J_m'(a)^{-1} = \frac{1}{(n-1)H_0(a)},$$

where $H_0(a)$ is the mean curvature of the hall ball of area a. We also know that, when f' exists,

$$f'(a) = I_m'(a)^{-1} = \frac{1}{(n-1)H},$$

where H is the mean curvature in the interior of E_m of any isoperimetric region of volume $f(a)$. From Proposition 4.1 we obtain that $g'(a) \geqslant f'(a)$ a. e. As f, g are absolutely continuous then $g(a) \geqslant f(a)$. Since J_m is increasing it easily follows that $I_m \geqslant J_m$.

If equality holds for some v_0, then for $a_0 = J_k(v_0) = I_k(v_0)$ we have $g(a_0) = f(a_0)$. Since $g' \geqslant f'$ we obtain that $f \equiv g$ in the interval $(0, a_0)$ and so $H_0(a_0)^{-1} = H(a_0)^{-1}$. If Ω_0 is any isoperimetric region of volume v_0 then Proposition 4.1 implies that Ω_0 is isometric to a half ball in $\mathbf{R}^n$ of volume v_0.

Finally let $\Omega \subset \mathbf{R}^n_C$ be relatively compact with smooth boundary. Then $\Omega \subset E_m$, for some m, and

$$\mathcal{P}_C(\Omega) \geqslant I_m(\operatorname{vol}\Omega) \geqslant I_{\mathbf{H}^n}(\operatorname{vol}\Omega).$$

If equality holds then Ω is an isoperimetric region in E_m and $I_m(\operatorname{vol}\Omega) = I_{\mathbf{H}^n}(\operatorname{vol}\Omega)$. By the discussion in the above paragraph, Ω is isometric to a half ball in $\mathbf{R}^n$ of volume $\operatorname{vol}\Omega$. □

Finally we show that the relative isoperimetric inequality (1.1) also holds outside any convex domain in $\mathbf{R}^n$, with no additional assumptions on the regularity of its boundary

Theorem 5.3. *If $C \subset \mathbf{R}^n$ is any closed convex set with interior points and $D \subset \mathbf{R}^n_C$ is a bounded set with finite perimeter, then*

$$\big(\operatorname{area}(\partial D)_C\big)^n \geqslant \frac{1}{2}\, n^n \omega_n (\operatorname{vol} D)^{n-1}.$$

Proof. Using standard results on the Hausdorff metric, we can find a sequence of convex sets with smooth boundary $C_m \subset \mathbf{R}^n$, and with $C \subset C_m$ for all $m \in \mathbb{N}$, converging locally in the Hausdorff distance to C. Let $D \subset \mathbf{R}^n_C$ be a bounded set with $(\partial D)_C$ smooth. Define $D_m := D \cap (\mathbf{R}^n)_{C_m}$. Then $\lim_{m\to\infty} \operatorname{vol} D_m = \operatorname{vol} D$ and $\mathcal{P}_C(D) \geqslant \mathcal{P}_{C_m}(D_m)$. Since, by Theorem 5.1, the relative isoperimetric inequality (1.1) is satisfied in $(\mathbf{R}^n)_{C_m}$, we have

$$\big(\operatorname{area}(\partial D)_C\big)^n \geqslant \big(\operatorname{area}(\partial D_m)_{C_m}\big)^n \geqslant \frac{1}{2}\, n^n \omega_n (\operatorname{vol} D_m)^{n-1}.$$

Taking limits when $m \to \infty$, we get (1.1). □

Remark 5.4. Reasoning as in [4], one can easily see that equality is never attained if C is strictly convex. The analysis of equality in the isoperimetric inequality for a general convex set cannot be treated with the tools used in this paper.

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