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THE CONVEX HULL PROPERTY AND TOPOLOGY OF HYPERSURFACES WITH NONNEGATIVE CURVATURE

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ABSTRACT. We prove that, in Euclidean space, any nonnegatively curved, compact, smoothly immersed hypersurface lies outside the convex hull of its boundary, provided the boundary satisfies certain required conditions. This gives a convex hull property, dual to the classical one for surfaces with nonpositive curvature. A version of this result in the nonsmooth category is obtained as well. We show that our boundary conditions determine the topology of the surface up to at most two choices. The proof is based on uniform estimates for radii of convexity of these surfaces under a clipping procedure, a noncollapsing convergence theorem, and a gluing procedure.

CONTENTS

1. INTRODUCTION

The main objects of study in this paper are locally convex hypersurfaces with boundary immersed in Euclidean space, and their infinitesimal counterparts, smooth hypersurfaces of nonnegative sectional curvature. In contrast to the complete case *without boundary*, which has been well studied [17, 29, 18, 19], the behavior of these objects is in general quite varied and complex [12]. Thus it is desirable to find natural boundary conditions which induce nice global behavior. Our primary motivation in this study is the classical convex hull property [26] which states that in Euclidean

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space a compact nonpositively curved hypersurface lies within the convex hull of its boundary. We show that this phenomenon has a dual, and in the process develop some basic tools for studying locally convex immersions. Our principal result in the smooth category is as follows:

Theorem 1.1. *Let M be a compact, connected, and smooth n-manifold, $n \geqslant 2$, $f\colon M \to \mathbf{R}^{n+1}$ be a smooth immersion with nonnegative sectional curvature, and $C := \operatorname{conv} f(\partial M)$ denote the convex hull of the image of the boundary of M (where we assume $\partial M \neq \emptyset$). Suppose that:*

1. *$f(\partial M) \subset \partial C$,*
2. *f has positive curvature on ∂M,*
3. *f is an embedding on each component of ∂M.*

Then the image of the interior of M lies completely outside the convex hull of the image of its boundary:

$$f(\operatorname{int} M) \cap C = \emptyset.$$

Further, f is an embedding on ∂M, and M is homeomorphic to the closure of a component of $\partial C - f(\partial M)$ with boundary $f(\partial M)$. (If C is degenerate, i.e. $\operatorname{int} C = \emptyset$, then we take ∂C to be two copies of C glued along their relative boundaries.)

It is obvious that, as far as our convex hull property (CHP) is concerned, condition (1) in Theorem 1.1 is necessary. Conditions (2) and (3) cannot be omitted, as demonstrated in Figures 1(a) and 1(b) respectively. Also note that even though Theorem 1.1 implies that $f(\partial M)$ is embedded, in general $f(M)$ is not embedded, see Figure 1(c). Theorem 1.1 implies that the topology of M is uniquely determined

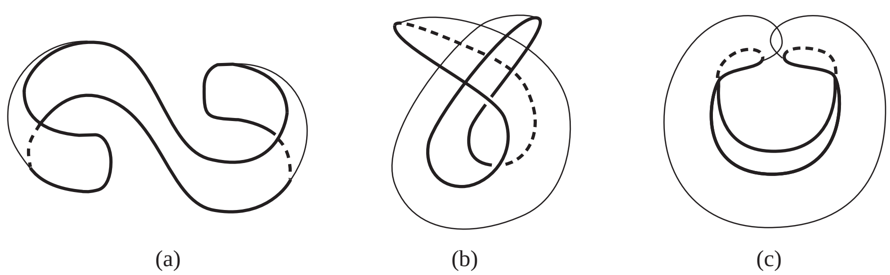

FIGURE 1

by the image of ∂M if ∂M has multiple components or is homeomorphic to $\mathbf{S}^{n-1}$. But if ∂M is a torus and $f\colon \partial M \to \partial C$ is a knotted embedding, then the components of $\partial C - f(\partial M)$ are not homeomorphic [13], and so in general the topology of M is determined only up to two choices. This degree of topological rigidity is significant, since it fails if condition 1 of Theorem 1.1 is violated: Figure 2 illustrates how to construct a curve which, for any given k, bounds k topologically distinct embedded surfaces of positive curvature.

The first step of the proof of Theorem 1.1 (Section 3) is to show that f is *locally convex* and *one-sided*, as defined below. This involves extending certain aspects of previous work of Sacksteder [29], and Greene and Wu [14], and is the only place

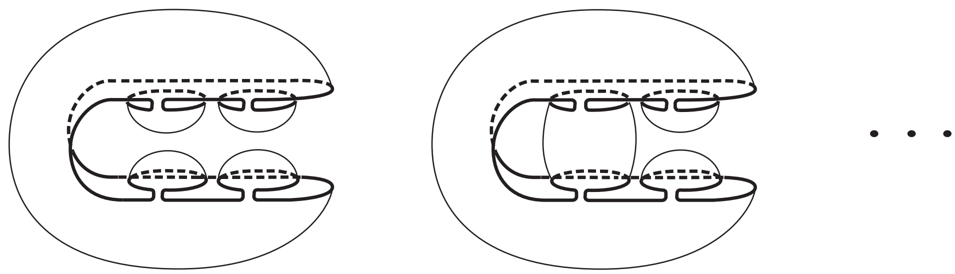

FIGURE 2

where smoothness is used in the proof of the Theorem 1.1. This theorem then follows from its counterpart in the nonsmooth setting, Theorem 1.2.

We say the immersion $f\colon M \to \mathbf{R}^{n+1}$ is *locally convex* if f has an extension $\widetilde{f}$ to a manifold without boundary $\widetilde{M}$, where every point $p \in M$ has a neighborhood U_p in $\widetilde{M}$ that is embedded by $\widetilde{f}$ into the boundary of a convex body $K_p \subset \mathbf{R}^{n+1}$. (The assumption that f extends to a collaring manifold $\widetilde{M}$ is a compatibility condition on the K_p for boundary points p; see the discussion below.) f is *one-sided* if U_p and K_p may be chosen so that the side of $f(U_p)$ on which K_p lies varies continuously with p. (Figure 1(a) illustrates an immersion which is locally convex but not one-sided.) If, for all $p \in \partial M$, U_p may be chosen so that $f(U_p \cap M)$ contains no line segments, we say f is *locally strictly convex* on a neighborhood of ∂M.

Theorem 1.2. *Let M be a compact connected topological n-manifold, $n \geq 2$, $f\colon M \to \mathbf{R}^{n+1}$ be a locally convex immersion, and $C := \operatorname{conv} f(\partial M) \neq \emptyset$. Suppose that:*

(1) *$f(\partial M) \subset \partial C$,*
(2) *f is locally strictly convex on a neighborhood in M of ∂M,*
(3) *f is an embedding on each component of ∂M.*

Then f satisfies the strong CHP:

$$f(\operatorname{int} M) \cap C = \emptyset.$$

Further, f is an embedding on ∂M, and M is homeomorphic to the closure of a component of $\partial C - f(\partial M)$ with boundary $f(\partial M)$. If condition (2) is replaced by

(2′) *f is one-sided,*

then the same conclusions hold, but with the strong CHP *replaced by the* CHP:

$$f(M) \cap \operatorname{int} C = \emptyset.$$

There is a nonintuitive distinction between our notion of local convexity and what may be referred to as *weak local convexity*. By the latter we mean that for each point $p \in M$ the restriction of f to *some neighborhood of* p in M extends to an embedding of a manifold without boundary into the boundary of a convex body K_p. Thus weak local convexity does not require that the local extensions across the boundary may be made compatible. The two notions of local convexity are equivalent in the setting of Theorem 1.1 (see Lemma 3.2). On the other hand, Figure 3 illustrates a weakly locally convex immersion (a box with a hole and a folded-down tab that pierces the box) for which the CHP fails (since the box clearly intersects the convex hull of its boundary). This example satisfies (1) and (3) of Theorem 1.2, and is one-sided in the sense that it is one-sided on the interior of M and locally has extensions across the boundary that are convex on the same side. In short, Theorem 1.2 is false for one-sided *weakly* locally convex immersions.

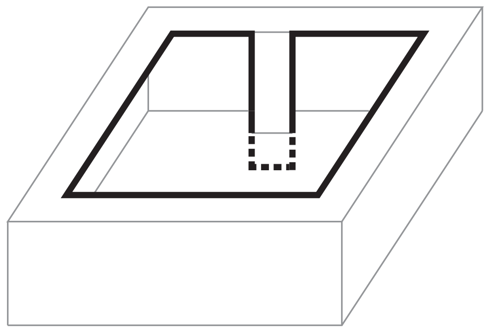

FIGURE 3

A special case of Theorem 1.2 where stronger conclusions are possible occurs when each component of ∂M lies in some hyperplane. We give a comprehensive treatment of this case in Section 5, where our result complements work of Rodríguez [28, Thm 4] and Kühnel [21] on tight immersions of manifolds with boundary.

The basic strategy for proving Theorem 1.2 is highly intuitive, and roughly may be described as follows. For simplicity, we suppress the immersion f. We wish to show that if ∂M lies on the boundary of its own convex hull C, then all points of M lie outside $\operatorname{int} C$. If points in M lie outside C, we "clip" them off with planes in such a way that C remains intact; replacing the clipped-off part of M with a topological disk in the clipping plane, we obtain a new locally convex surface with the same topology and boundary (Section 4). We continue to clip and modify M until in the limit, there are no points outside C. We then have a locally convex surface M' with the same boundary as M and all points inside C. We extend this surface to a closed locally convex surface without boundary by gluing on portions of ∂C (Section 8). By a theorem of van Heijenoort (Lemma 4.1), the resulting surface must lie on the boundary of a convex body. That convex body obviously contains C and, hence, M' lies in ∂C. Since we didn't clip any points in M that were also in $\operatorname{int} C$, there must not have been any.

One of the principal challenges in this proof is controlling the limit of the clippings. To this end, we find uniform bounds for *radii of convexity* (Section 6). These bounds may be considered as generalized a priori curvature estimates, and lead to an analogue of Blaschke's selection principle for surfaces with boundary (Section 7). The remaining key ingredient is the somewhat subtle gluing argument (Section 8).

A version of the convex hull properties proved in this paper was originally conjectured in the second author's PhD thesis [9, Conjecture E.0.5]. A proof of a version of Theorem 1.1, assuming local convexity, was obtained subsequently to the present work by Guan and Spruck [16], using their work on Monge-Ampére equations. Labourie has investigated the singularities in the convergence of smooth locally convex hypersurfaces in Euclidean space under uniform upper bounds on sectional curvature [22]. For inequalities relating intrinsic and extrinsic invariants of locally convex hypersurfaces, see [23] and Gromov [15, p. 280]. Ends of complete, smooth locally convex hypersurfaces with compact boundary were studied by Currier and the first author [1]. For an application of our main results, see [10, Appendix A], where it is proved that there exists a smooth simple closed curve without inflection points which lies on the boundary of a convex body, but bounds no surfaces of positive curvature. This result, which is related to a question of S.-T. Yau [31, Problem 26] on characterizing curves which bound surfaces of positive curvature, was one of the original motivations for developing the convex hull properties proved in this paper.

2. Basic Definitions and Notation

By M, we always mean a connected n-dimensional manifold with boundary, where $n \geq 2$. The boundary and interior of M are denoted respectively by ∂M and $\operatorname{int} M := M - \partial M$. By smooth we mean infinitely differentiable (C^∞). But with the exception of Section 3, and unless stated otherwise, the manifolds and mappings in this paper are not a priori assumed to have any degree of regularity. By an *immersion* we mean a locally one-to-one continuous map. An *embedding* is a mapping which is a homeomorphism onto its image. We say $f\colon M \to \mathbf{R}^{n+1}$ is a *smooth immersion* if M and f are smooth, and f has nonvanishing Jacobian.

The standard Euclidean innerproduct is denoted by $\langle \cdot, \cdot \rangle$, and $\|\cdot\| := \langle \cdot, \cdot \rangle^{1/2}$ is the corresponding norm. $\mathbf{S}^n$ and B^{n+1} denote respectively the unit sphere and closed unit ball in $\mathbf{R}^{n+1}$. For any pair of subsets X, $Y \subset \mathbf{R}^{n+1}$ we have the *Euclidean distance*:

$$\operatorname{dist}(X, Y) := \inf\{\, \|x - y\| \mid x \in X, \text{ and } y \in Y \,\},$$

and the *Hausdorff distance*:

$$\operatorname{dist}_H(X, Y) := \inf\{\, r \geq 0 \mid X \subset Y + rB^{n+1}, \text{ and } Y \subset X + rB^{n+1} \,\}.$$

By a *convex body* in $\mathbf{R}^{n+1}$ we mean a compact convex set with nonempty interior. For any subset $X \subset \mathbf{R}^{n+1}$, $\operatorname{int} X$ and $\operatorname{conv} X$ denote, respectively, the interior and *convex hull* of X. An intersection of finitely many closed halfspaces is called a convex *polyhedron*.

Recall that by saying $f\colon M \to \mathbf{R}^{n+1}$ is *locally convex*, we mean f has an extension $\widetilde{f}$ to a manifold without boundary $\widetilde{M}$, where every point $p \in M$ has a neighborhood U_p in $\widetilde{M}$ that is embedded by $\widetilde{f}$ into the boundary of a convex body K_p. If, for all $p \in \partial M$, U_p may be chosen so that $f(U_p \cap M)$ contains no line segments, we say f is *locally strictly convex* on a neighborhood of ∂M. (It is not ruled out that $\widetilde{f}(U_p)$ may contain line segments.) If we may choose $U_p = \widetilde{M}$, we say f is a *convex embedding.* If the side of $f(U_p)$ on which K_p lies, which we call the *positive side*, varies continuously with p, we say that f is *one-sidedly locally convex*, or merely *one-sided.*

If f is locally convex, then each neighborhood U_p may be chosen so that $\widetilde{f}(U_p)$ is supported by a hyperplane H_p at $f(p)$, and is the graph of a convex, Lipschitz height function over an open disk in H_p [5]. Since M is connected, it easily follows that any two points of $\widetilde{M}$ can be joined by an arc whose image under $\widetilde{f}$ has finite length. Therefore $\widetilde{f}$ induces an intrinsic distance on $\widetilde{M}$, which we denote by $\mathrm{dist}_{\widetilde{f}}(p,q)$. This distance is locally bi-Lipschitz equivalent to the Euclidean distance [5, p. 6 and p. 79]. Since each neighborhood U_p is bi-Lipschitz equivalent to a Euclidean disk, we obtain an atlas on $\widetilde{M}$ for which the transition functions are bi-Lipschitz. When we refer to a Lipschitz structure on $\widetilde{M}$, this is the one we mean.

Note that we have made no regularity assumptions on ∂M, which might, for example, exhibit fractal behavior. Thus there may be no arc in M, from a given boundary point to another point of M, whose image has finite length. Therefore the intrinsic distance dist_f, while always finite on pairs of points in $\mathrm{int}\, M$, may take value ∞ on pairs that include a point of ∂M. In any case, *completeness* of M under dist_f is defined as usual, namely, Cauchy sequences in M converge in M.

3. Local Convexity and One-sidedness

In this section we show that if an immersion satisfies the conditions of Theorem 1.1, then it is one-sidedly locally convex. We also examine one-sidedness in nonsmooth locally convex immersions.

First we give a condition guaranteeing that any component of the set of *flat points* of a smooth nonnegatively curved hypersurface is a convex set. By flat points we mean the set where the second fundamental form vanishes. Our Lemma 3.1 generalizes both Theorem 1 of [29], where M is assumed to be complete and without boundary, and a proposition in [14], where M is assumed to be homeomorphic to the 2-sphere with finitely many points deleted. Both our proof and that of [14] use the method of [29], to which we refer for certain arguments; however, our proof is shorter and more direct than the other two.

Below, by the *completion* of M, we mean its completion in the intrinsic metric dist_f. The *Cauchy boundary* consists of the added points.

Lemma 3.1. *Let $f\colon M \to \mathbf{R}^{n+1}$ be a smooth immersion with nonnegative curvature, where M is a manifold without boundary. Suppose f has strictly positive curvature on $M \cap U$ where U is a neighborhood of the Cauchy boundary of M in the completion of M. Then the restriction of f to each component of the set of flat points of f is an embedding onto a convex set.*

Proof. Let X be a component of the set of flat points of f. Since the Gauss map has rank 0 on X, Sard's theorem [8, 3.4.3] implies its image has vanishing one-dimensional Hausdorff measure, hence is totally disconnected (here we are using the fact that f is C^∞). Therefore the Gauss map is constant on X, say n_0. Let $h\colon M \to \mathbf{R}$ be the *height function* given by $h(p) := \langle f(p), n_0\rangle$. Then the differential of h vanishes on X. Consequently, by a theorem of Morse [24], $f(X)$ lies in a hyperplane H orthogonal to n_0.

Choose a point p in X, and let S be the maximal subset of X that is *starshaped about* p, that is, S consists of all points that can be joined to p by an arc γ in X that is mapped homeomorphically by f onto a line segment γ^* in H (we call γ an H*-segment*). It suffices to show $S = X$. Indeed, since p is arbitrary, it follows from this that $f(X)$ is a convex subset of H and the restriction of f to X is an isometry.

By assumption, all Cauchy sequences in X, with respect to the metric dist_f on M, converge in X. Therefore the closed dist_f-balls in X are compact. It follows then by an Arzela-Ascoli argument [4, (5.16)] that S is closed in X. Thus to complete the proof it remains to show that S is open in X.

Suppose S is not open in X. Then there is a point $q \in S$ and a sequence $q_i \in X - S$ converging to q. If γ is an H-segment in X ¿from p to q, then γ has a tubular neighborhood W in M that is homeomorphic to an open disk. Since f is tangent to H along γ, we may choose W so small that $f(W)$ is the homeomorphic image of W and is the graph of a smooth function $g\colon W^* \to \mathbf{R}$, where W^* is the homeomorphic image of $f(W)$ under orthogonal projection to H. Let γ_i^* be the segment in W^* from $f(p)$ to $f(q_i)$, and γ_i be the arc in W whose f-image projects orthogonally to γ_i^*. Since $q_i \notin S$, then $\gamma_i \not\subset X$, and so γ_i intersects some component A of the set of nonflat points of W. Equivalently, γ_i^* intersects the orthogonal projection A^* of $f(A)$ to H. Note that A^* is the nonvanishing set of the hessian of g.

Now we obtain a contradiction by showing that γ_i^* cannot intersect the orthogonal projection A^*. In a simply connected manifold, any component of the complement of an open connected set has connected boundary [29, Corollary 1]. Therefore the boundary in W of each component U of $W - A$ is connected. The boundary of U in W, being a connected set of flat points, lies in a hyperplane (by the argument above), and the normal to the hyperplane agrees with the normal to f on the boundary of U. But then there is a unique C^1 function g_A on W^* that agrees with g on A^* and agrees with some linear function on each component of $W^* - A^*$. In fact, since the hessian of g vanishes on the boundary of A^* in W^*, it follows that g and g_A agree up to third order on the boundary of A^*, and hence g_A is C^2. Since the nonvanishing set A^* of the hessian of g_A is connected, we may assume that the hessian of g_A is positive semidefinite, and hence that g_A is convex on W^*. Since g_A, its differential and its hessian all vanish at $f(p)$ and $f(q_i)$, the convexity of g_A implies that they vanish at every point of γ_i^* (this claim is easily verified geometrically, or by calculation as in [29, Pg. 616] or [14, Pg. 462]). Since g_A and g agree on the open set A^*, and the hessian of g does not vanish on A^*, it follows that γ_i^* cannot intersect A^*. This contradiction completes the proof. □

Lemma 3.2. *Let $f\colon M \to \mathbf{R}^{n+1}$ be a smooth immersion which is locally convex on* $\operatorname{int} M$*, and has positive curvature on ∂M. Then f is locally convex on M.*

Proof. Using the notion of a double of a manifold [25] it can be shown that there exists a manifold without boundary $\widetilde{M} \supset M$ and a smooth immersion $\widetilde{f}\colon \widetilde{M} \to \mathbf{R}^{n+1}$ such that $\widetilde{f} = f$ on M. Since $\widetilde{f}$ has positive curvature on ∂M, there is a connected open neighborhood U of ∂M in $\widetilde{M}$ on which $\widetilde{f}$ has positive curvature. Since the second fundamental form of $\widetilde{f}$ on U is definite, there is a continuous choice of normal on U with respect to which it is positive definite. Therefore $\widetilde{f}$ is locally convex on U. □

Proposition 3.3. *Let M be compact, and $f\colon M \to \mathbf{R}^{n+1}$ be a smooth immersion whose curvature is nonnegative everywhere and positive on ∂M. Then f is one-sidedly locally convex.*

Proof. First we apply Lemma 3.1 to the interior of M: since each component X of the set of flat points of f is compact, then X is embedded by f onto a compact convex set.

Next let $\Pi\colon M^* \to M$ be the universal cover of M, define $F\colon M^* \to \mathbf{R}^{n+1}$ by $F\colon = f \circ \Pi$, and note that the restriction of F to the interior of M^* also satisfies the hypotheses of Lemma 3.1. Therefore F embeds each component Y of the set of flat points of F onto a convex set. In particular, Y is embedded onto $\Pi(Y)$. Further, since $\Pi(Y)$ lies in a component X of the set of flat points of f, and X is path-connected, it easily follows that $\Pi(Y) = X$. So Y is isometric to a compact convex set.

But then the set of nonflat points of F is connected. Indeed, a simply connected manifold is separated by a closed subset only if it is separated by a component of that subset [29, Corollary 2]. And a component Y of the flat set of F, being isometric to a compact convex set and lying in the interior of M^*, does not separate M^*.

Since simply connected manifolds are orientable, we may choose a continuous unit normal vector field $n\colon M^* \to \mathbf{S}^n$ corresponding to the immersion F. Since the set of nonflat points of F is connected, we may assume that n points in the positive direction at every nonflat point. That is, the second fundamental form of F with respect to n is positive semidefinite in a neighborhood of each point. Consequently, the restriction of F to the interior of M^* is one-sidedly locally convex. Hence so is the restriction of f to the interior of M.

Finally, since f is smooth and is one-sidedly locally convex in the interior of M, then by Lemma 3.2, f is locally convex on M. Clearly f is one-sidedly locally convex. □

Now we turn to nonsmooth locally convex immersions. The following lemma is related to Lemma 3.1. Note, however, that Lemma 3.1 did not assume local convexity (but rather was used to prove local convexity). Hence the proof of the following is not as subtle.

Lemma 3.4. *Let $f\colon M \to \mathbf{R}^{n+1}$ be a locally convex immersion, where M is complete. Let H be a local support hyperplane for f at $p \in \operatorname{int} M$, and X be the component of $f^{-1}(H)$ containing p. Suppose that $X \subset \operatorname{int} M$. Then the restriction of f to X is an embedding onto a closed convex subset of H.*

Proof. By the definition of local convexity, each $q \in M$ has a neighborhood U_q in $\widetilde{M}$ such that $\widetilde{f}$ embeds U_q into the boundary of a convex body $K_q \subset \mathbf{R}^{n+1}$. Let $A \subset f^{-1}(H)$ be the set of points in M at which H locally supports $\widetilde{f}$. By the continuity of local support planes to a convex body [5, (1.6)], A is open and closed in $f^{-1}(H)$. Therefore H locally supports f at every point of X. Thus we may choose U_q and K_q so that $f(U_q \cap X) = K_q \cap H$, for all $q \in X$. Therefore every point q of X has a neighborhood in X which is mapped homeomorphically by f onto a convex subset of H.

But then f maps all of X homeomorphically onto a convex subset of H. This claim, which completes our proof, is closely related to a classical theorem of Tietze [20, 6] on local characterization of convex sets. The proof, which is quite similar to that of Tietze's theorem, is outlined in essentially the form we need in [29, Lemma 1], also see [19, Proposition 1]. Specifically, one shows that the set of points that can be joined by a finitely broken H-segment to a given point $p \in X$ is open and closed. Secondly, if p and q, and q and r, respectively, lie on H-segments, then it can be shown that so do p and r. (By an H-segment, we again mean a path in M which is mapped homeomorphically by f onto a line segment in H.) Completeness of M under dist_f is used to show that if an H- segment varies with one endpoint fixed and the image of the other endpoint moving along a half-open line segment in H, then a limit H-segment exists. □

The following may be considered as the analogue of Proposition 3.3 in the non-smooth category.

Proposition 3.5. *If M is compact and $f\colon M \to \mathbf{R}^{n+1}$ is a locally convex immersion that is locally strictly convex on a neighborhood in M of ∂M, then f is one-sidedly locally convex.*

Proof. Let Y be the subset of M each of whose points has an open neighborhood that is mapped by f into a hyperplane. Then the restriction of f to $M - Y$ has a uniquely determined and continuously varying positive side.

Let Y_1 be a component of Y. Then Y_1 lies in a component X of $f^{-1}(H)$ for some hyperplane H. By the strict convexity assumption at the boundary, and since X has nonempty interior, we may conclude that $X \subset \text{int}\, M$. So, by Lemma 3.4, the restriction of f to X is a homeomorphism onto a compact convex set.

Then $\text{int}\, f(X) = f(\text{int}\, X)$ is open and convex in H. It follows from the definition of Y_1 that $Y_1 = \text{int}\, X$. In particular $\partial Y_1 = \partial X$. Since $f(X)$ is a compact convex set, ∂Y_1 is connected. Since in addition, $\partial Y_1 \subset M - Y$, it follows that the positive side of f at each point of ∂Y_1 corresponds to a fixed side of H. Thus we may extend the choice of positive side of f continuously to Y_1, and hence to all of M. Therefore f is one-sidedly locally convex. □

Note 3.6. In Proposition 3.5, if "locally strictly convex" is replaced by "one-sided", then the proposition is no longer true; see Figure 4. Indeed, Figure 4 is locally convex (the required extension across the boundary may be obtained by collaring, in each of the three planes of the figure, the portion of the boundary in that plane) and one-sided on a neighborhood of the boundary, but not one-sided.

Recall that in Lemma 3.4 we assumed that $X \cap \partial M = \emptyset$, as was indeed necessary for the desired conclusion. We shall also require the following lemma, which allows us to address the case in which X intersects ∂M.

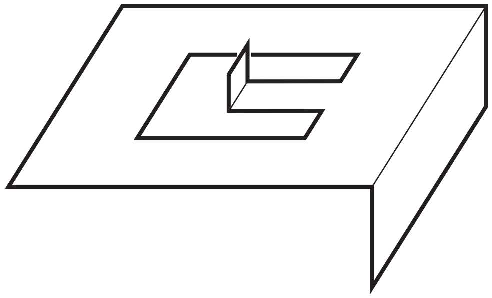

FIGURE 4

Lemma 3.7. *Let $f\colon M \to \mathbf{R}^{n+1}$ be a one-sidedly locally convex immersion, where M is complete and the restriction of f to each component of ∂M is an embedding. Let H be a local support hyperplane for $\widetilde{f}$ at $p \in M$, and X be the component of $f^{-1}(H)$ containing p. Suppose X contains every component ∂M_α of ∂M that intersects X (it is possible that there are none). Then the restriction of f to X is an embedding onto a closed convex subset K_H of H with open subsets D_α of $\operatorname{int} K_H$ removed, where D_α is a component of $H - f(\partial M_\alpha)$.*

Proof. Recall that, by definition, there is a neighborhood U_p of p in $\widetilde{M}$ such that $\widetilde{f}(U_p) \subset \partial K_p$ where K_p is a convex body lying on the positive side of $\widetilde{f}$ at p. By the continuity of local support planes to a convex body, the points of $f^{-1}(H)$ at which H supports such a convex body K_p form an open and closed subset of $f^{-1}(H)$. Therefore H locally supports the positive side of $\widetilde{f}$ on a fixed side of H, at every point of X.

Let ∂M_α denote a component of ∂M that lies in X, so $f(\partial M_\alpha)$ lies in H. By the generalized Jordan-Brouwer separation theorem [2, p. 353–355], $H - f(\partial M_\alpha)$ has exactly two components and $f(\partial M_\alpha)$ is the boundary of each. We may cover ∂M_α by open sets $U_i := U_{p_i}$ chosen as above, for which $\partial M_\alpha \cap U_i$ is connected. Set $K_i := K_{p_i}$. Choose $x \in \operatorname{int} K_i$, and consider the projection map π_{ix} to H through x. Specifically, let $\pi_{ix} \circ \widetilde{f}$ be a homeomorphism of a neighborhood U_{ix} of $\partial M_\alpha \cap U_i$ in U_i onto its image in H, where $\operatorname{int} M \cap U_{ix}$ is connected. Define D_α to be the component of $H - f(\partial M_\alpha)$ that does not contain the image of $\operatorname{int} M \cap U_{ix}$. This choice of D_α is independent of $x \in \operatorname{int} K_i \cap \operatorname{int} C$. Moreover, if $U_i \cap U_j \neq \emptyset$, then one-sidedness implies as above that $\operatorname{int} K_i \cap \operatorname{int} K_j \neq \emptyset$. For $x \in \operatorname{int} K_i \cap \operatorname{int} K_j$, the choices of D_α respectively determined by π_{ix} and π_{jx} agree. Since ∂M_α is connected, it follows that the choice of D_α is independent of both x and i.

Let $\overline{M}$ be the result of gluing the closure of D_α, for each α, to M along ∂M_α. Further, let $\overline{f} : \overline{M} \to R^{n+1}$ be the natural extension of f to $\overline{M}$. Our projection maps and the choice of D_α make clear that $\overline{M}$ is a manifold and $\overline{f}$ is an immersion. Let $\overline{X}$ be the component containing X of $\overline{f}^{-1}(H)$. Then $\overline{X}$ is the result of gluing the D_α to X. By construction, $\overline{f}$ has one-sided local supports at every point, and so $\overline{f}$ is a locally convex immersion of $\overline{M}$.

Now $\overline{X} \subset \operatorname{int} \overline{M}$ by construction. Clearly $\overline{M}$ is complete. Therefore by Lemma 3.4, the restriction of $\overline{f}$ to $\overline{X}$ is an embedding onto a closed convex subset of H. □

4. Convex Caps and Clippings

Here we describe the clipping procedure we mentioned in the introduction. We begin with two lemmas that are basic tools in this paper.

Lemma 4.1 (van Heijenoort's Theorem [18])**.** *Let M be a manifold without boundary, and $f\colon M \to \mathbf{R}^{n+1}$ be a complete locally convex immersion. Suppose that f has a strict local support hyperplane at some point of M. Then f is an embedding and $f(M)$ bounds a convex body in $\mathbf{R}^{n+1}$.*

Let $K \subset \mathbf{R}^{n+1}$ be a convex body, and $H \subset \mathbf{R}^{n+1}$ be a hyperplane. Suppose that $K \cap H$ has interior points in H, and let $\operatorname{int} H^-$ be one of the open half-spaces determined by H such that $K \cap \operatorname{int} H^- \neq \emptyset$. By a *convex cap*, cut from ∂K by H, we mean the closure of $\partial K \cap \operatorname{int} H^-$. The following lemma is due to Labourie.

Lemma 4.2 ([22])**.** *Let M be compact and $f\colon M \to \mathbf{R}^{n+1}$ be a locally convex immersion. Let $H \subset \mathbf{R}^{n+1}$ be a hyperplane which determines closed half-spaces H^+ and H^-, and set $M^- := f^{-1}(\operatorname{int} H^-)$. Suppose that $f(\partial M) \subset \operatorname{int} H^+$. Then f maps the closure of each component of M^- homeomorphically onto a convex cap.*

Lemmas 4.1 and 4.2 are proved by hyperplane slicing arguments. The second may be deduced from the first by a projective transformation that sends H to the hyperplane at infinity. We also need an addendum to Lemma 4.2:

Lemma 4.3. *Under the hypotheses of the preceding lemma, $f^{-1}(H^+)$ is connected.*

Proof. Let H_* be a hyperplane obtained by moving H a small distance parallel to itself inside H^+. Let H_*^+ denote the half-space containing $f(\partial M)$, and Y_* be the closure of the component of $f^{-1}(\operatorname{int} H_*^-)$ containing Y. By Lemma 4.2, the closures of the components of M^- and M_*^- map homeomorphically to convex caps that are pairwise nested. It follows that the closures of the components of M^- lie in $\operatorname{int} M$, are pairwise disjoint, and are finite in number. Moreover, each is homeomorphic to a closed n-disk. Therefore $M - M^-$ is connected. □

Lemma 4.2 allows us to "clip"convex hypersurfaces by convex polyhedra, as in the following proposition:

Proposition 4.4. *Let $f\colon M \to \mathbf{R}^{n+1}$ be a locally convex immersion of a compact manifold M, and $P \subset \mathbf{R}^{n+1}$ be a convex polyhedron. Suppose that $f(\partial M) \subset \operatorname{int} P$. Then there exists a locally convex immersion $f_P\colon M \to \mathbf{R}^{n+1}$ such that $f_P(M) \subset P$ and $f_P = f$ on $f^{-1}(P)$. If f is one-sidedly locally convex, then so is f_P. Moreover, there is a distance-nonincreasing map from M in its f-induced metric onto M in its f_P-induced metric. We say that f_P is a* clipping *of f by P.*

Proof. We shall assume P is a closed half-space H^+ determined by some hyperplane H; then finitely many iterations will yield the claim. Let H_* be a hyperplane parallel to H in H^+, and sufficiently close to H as to satisfy $f(\partial M) \subset H_*^+$. By

Lemma 4.2, the closures Y_β of the components of $M^- := f^{-1}(\operatorname{int} H^-)$ are mapped homeomorphically by f onto convex caps B_β with boundary in H. Moreover, each Y_β lies in the closure of a component of $M_*^- := f^{-1}(\operatorname{int} H_*^-)$ which is mapped homeomorphically by f onto a larger convex cap $B_{*\beta}$ with boundary in H_*. By the *base* of B_β, we mean the convex body D_β in H that is bounded by ∂B_β. Then there is a 1-1 correspondence from the collection of boundary components of $f^{-1}(H^+)$ that intersect H onto the collection of ∂D_β, where f maps each of these boundary components of $f^{-1}(H^+)$ homeomorphically onto the corresponding ∂D_β. Let M_H be the result of gluing the D_β to $f^{-1}(H^+)$ via these homeomorphisms. Define $g_H : M_H \to \mathbf{R}^{n+1}$ by setting $g_H = f$ on $f^{-1}(H^+)$ and g_H equal to the inclusion map on each D_β.

Since the convex caps B_β lie in the larger convex caps $B_{*\beta}$, it follows that g_H is a locally convex immersion of M_H. Moreover, if f is one-sidedly locally convex, then the insides of the convex cap correspond to the positive side of f, and so g_H is also one-sidedly locally convex. Since there is a homeomorphism from each convex cap B_β onto its base D_β that fixes $\partial B_\beta = \partial D_\beta$, there is a homeomorphism h of M onto M_H. Thus, for $P = H^+$, the locally convex immersion $f_P = g_H \circ h \colon M \to \mathbf{R}^{n+1}$ satisfies all the claims of the proposition except possibly the last, which remains to be verified.

The nearest-point projection from a Euclidean space to a convex subset is distance-nonincreasing [30, Thm. 1.2.2]. Therefore there is a distance nonincreasing map from each convex cap B_β (in its Euclidean metric, and hence in its f-induced metric) to its base D_β. It is easily checked that this procedure, successively applied to the finitely many convex caps B_β, yields a distance nonincreasing map from M in its f-induced metric onto M_H in its g_H-induced metric. Since the latter is isometric to M in its f_P-induced metric, the proposition follows. □

Note 4.5. Lemma 4.2 and Proposition 4.4 remain valid under the weaker hypotheses that $f(\partial M) \subset H^+$ and $f(\partial M) \subset P$, respectively (as opposed to $f(\partial M) \subset \operatorname{int} H^+$ and $f(\partial M) \subset \operatorname{int} P$). However, these require more proof, and we do not need these stronger statements in this paper.

5. Flat Boundary Components

In this section we analyze the case in which the boundary components of a locally convex hypersurface lie in hyperplanes. The conclusion of the following theorem is stronger than the CHP, since in this case f is shown to be a convex embedding. Theorem 5.1 is related to a result of Rodríguez [28, Thm 4], where f is assumed to be smooth and its restriction to each boundary component a convex embedding; but, instead of being locally convex, the surface is only required to have nonnegative curvature; see also the paper by Kühnel [21]. Both of these papers are based on the theory of tight immersions and the two-piece-property of Banchoff. Our proof, however, uses van Heijenoort's Theorem (Lemma 4.1), and assumes no smoothness. There is also a version of the following theorem for smooth surfaces with nonvanishing curvature [11].

Theorem 5.1. *Let M^n be compact, and $f\colon M \to \mathbf{R}^{n+1}$ be a locally convex immersion that is locally strictly convex on a neighborhood in M of ∂M. Let ∂M_α, $i = 1, \ldots k$, be the components of ∂M, and suppose that each $f(\partial M_\alpha)$ lies in a hyperplane H_α. Furthermore, suppose that either $n > 2$, or else all nonembedded boundary components of M lie in the same hyperplane. Then f is a convex embedding. In particular, $f(M - \partial M_\alpha) \cap H_\alpha = \emptyset$, and M is homeomorphic to $\mathbf{S}^n$ with k convex open subsets removed.*

The requirement that the nonembedded boundary components lie in the same hyperplane when $n = 2$ is not superfluous, see Figure 5(a). Also note that the above does not remain valid if the surface is not locally *strictly* convex on a neighborhood of the boundary, see Figure 5(b).

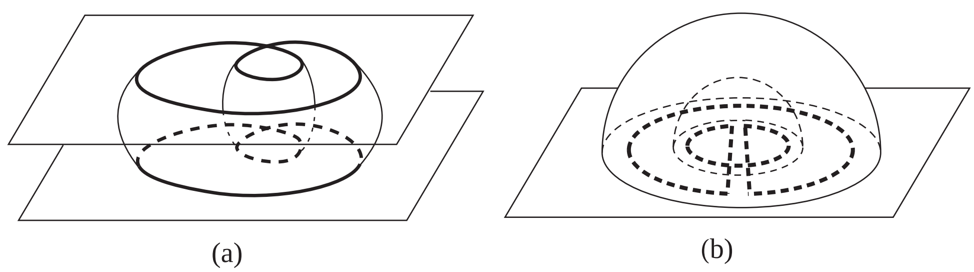

FIGURE 5

Proof(Theorem 5.1). Let $\widetilde{f}\colon \widetilde{M} \to \mathbf{R}^{n+1}$ be a locally convex extension of f.

(Part 1) We show that the restriction of f to each boundary component is a locally strictly convex immersion into the corresponding hyperplane. Note that it is only necessary to prove local convexity, since by hypothesis, the restriction of f to ∂M contains no line segments.

For a given $p \in \partial M_\alpha$, there exists an open neighborhood U in $\widetilde{M}$ and a convex body $K \subset \mathbf{R}^{n+1}$ such that $\widetilde{f}$ embeds U into ∂K. Furthermore, $\widetilde{f}(U \cap M)$ contains no line segments. Set $K^0 := K \cap H_\alpha$, and $U^0 := U \cap \partial M_\alpha$. Then K^0 is a convex body in H_α. Indeed, since K^0 is clearly compact and convex, we need only show that K^0 has interior points in H_α. Suppose not. Then K^0 lies in an affine subspace L of H_α, where the dimension of L is less than n. Since $f(U^0) \subset K^0 \subset L$ and f is an embedding on U^0, then by invariance of domain, the dimension of L is $n-1$ and $f(U^0)$ is open in L. In particular, a segment of any line in L passing through $f(p)$ must be contained in $f(U^0)$, contradicting the strict convexity assumption.

(1.1) Suppose that K is not supported by H_α. Then $\partial K \cap H_\alpha = \partial K^0$. Therefore $f(U^0)$ lies in the $n-1$ dimensional manifold ∂K^0, and $f|U^0$ is an imbedding onto an open subset of ∂K^0. Therefore $f|U^0$ is a locally convex immersion into H_α, as required.

(1.2) Suppose K is supported by H_α. Then $f(U^0) \subset K^0 = \partial K \cap H_\alpha$. We again claim $f(U^0) \subset \partial K^0$, from which it follows that $f|U^0$ is a locally convex immersion into H_α. Suppose, to the contrary, that $f(q)$ lies in the interior of K^0

for some $q \in U^0$. Since a neighborhood in ∂K of $f(q)$ lies in H_α, then f embeds a neighborhood of q in M into H_α, in violation of the strict convexity assumption.

(Part 2) We construct an extension of M and a locally convex extension of f:

(2.1) First we show that ∂M_α has a neighborhood V_α in M such that $f(V_\alpha - \partial M_\alpha)$ lies in an open halfspace determined by H_α, say $\operatorname{int} H_\alpha^-$. Indeed, in both cases (1.1) and (1.2) above, $\widetilde{f}$ maps a neighborhood of p in $\widetilde{M}$ onto an open subset of ∂K, and maps a neighborhood of p in U^0 to ∂K^0. It follows, directly in case (1.1) and ¿from the strict convexity assumption in case (1.2), that p has a neighborhood V in M such that $f(V - \partial M_\alpha)$ lies in an open halfspace determined by H_α. Covering ∂M_α by finitely many such V yields the claim.

(2.2) Let ∂M_β, $\beta = 1 \dots k'$, denote those components of ∂M such that restriction of f to ∂M_β, say ∂f_β, is an embedding. Then ∂f_β is a convex embedding by Part 1. Let D_β be the convex body in H_β bounded by $\partial f_\beta(\partial M_\beta)$. Since $\partial f_\beta \colon \partial M_\beta \to \partial D_\beta$ is a homeomorphism, we may glue D_β to M along ∂M_β via ∂f_β, to obtain a compact, connected manifold $\overline{M}$. Let $\overline{f} \colon \overline{M} \to \mathbf{R}^{n+1}$ be the natural extension of f.

(2.3) We claim that $\overline{f}$ is a locally convex immersion. Indeed, by (2.1), ∂M_β has a neighborhood V_β in M such that $f(V_\beta - \partial M_\beta)$ lies in an open halfspace $\operatorname{int} H_\beta^-$. It follows that $\overline{f}$ is locally one-to-one. Since $\overline{f}$ is continuous, it is an immersion. Local convexity needs to be checked only at the points where the gluing occurs. Consider $p \in \partial M_\beta$, and let K be a convex body such that $\widetilde{f}$ embeds a neighborhood of p in $\widetilde{M}$ into ∂K. Then we are in either case (1.1) or (1.2). In the latter case, $\overline{f}$ embeds a neighborhood of p in $\overline{M}$ into ∂K. In the former, $\overline{f}$ embeds a neighborhood of p in $\overline{M}$ into the boundary of $K \cap H_\beta^-$. Thus in both cases, we have local convexity at p.

(Part 3) Now we prove that $\overline{M}$ is a closed manifold (i.e., $\partial M = \emptyset$). Suppose not. Then, by definition of $\overline{M}$, ∂M must have some nonembedded boundary components. Consequently $n = 2$, because if $n > 2$, then by Part 2 and van Heijenoort's theorem (4.1), each boundary component of M is embedded. By assumption, all the nonembedded boundary components of M lie in the same hyperplane H. Since the nonembedded boundary components of M coincide with the boundary components of $\overline{M}$ under the inclusion map, all boundary components of $\overline{M}$ lie in H.

Let ∂M_1 be such a component. By (2.1), ∂M_1 has a neighborhood V_1 in M such that $f(V_1 - \partial M_1)$ lies in an open halfspace $\operatorname{int} H^-$. We may take $V_1 - \partial M_1$ connected. Let X be the closure of the component of $\overline{f}^{-1}(\operatorname{int} H^-)$ containing $V_1 - \partial M_1$.

By combining van Heijenoort's theorem with a projective transformation that sends H to the hyperplane at infinity, we find that $\overline{f}$ embeds $\operatorname{int} X$ into the boundary in $\operatorname{int} H^-$ of an open convex set in $\operatorname{int} H^-$. Let K be the closure in $\mathbf{R}^{n+1}$ of this convex body. Then $f(\partial X)$ lies in the convex set $K \cap H$. Since $f(\partial M_1) \subset \partial X$ by definition of X, and the restriction of f to ∂M_1 is strictly locally convex by Part 1, then $K \cap H$ is a convex body in H. Thus the restriction of f to ∂X embeds ∂X onto the boundary in H of $K \cap H$. Now ∂M_1 is a compact connected $(n-1)$-manifold immersed in the connected $(n-1)$-manifold ∂X. Therefore $f(\partial M_1) = \partial X$, and ∂M_1 is embedded, contrary to assumption.

(Part 4) Since $\overline{M}$ is a closed manifold and $\overline{f}$ is a locally convex immersion, then by van Heijenoort's theorem it follows that $\overline{f}$ is a homeomorphism onto the boundary of a convex body $K \subset \mathbf{R}^{n+1}$. Therefore f is a homeomorphism onto the complement in ∂K of $\operatorname{int} D_\alpha$, $\alpha = 1 \ldots k$, where D_α is a convex body in H_α.

By (2.1), a given D_α has a neighborhood W in $\overline{M}$ such that $\overline{f}(W - D_\alpha)$ lies in an open halfspace $\operatorname{int} H_\alpha^-$. By global convexity, it follows that $\overline{f}(\overline{M} - D_\alpha)$ lies in $\operatorname{int} H_\alpha^-$. Therefore $f(M - \partial M_\alpha) \cap H_\alpha = \emptyset$. □

6. Radii of Convexity: Uniform Estimates

In this section we start by establishing some basic tools for studying locally convex hypersurfaces, namely *radius of convexity* and *inradius of convexity*. They will be used in this section to show that the clipping procedure developed in Section 4 satisfies certain uniform bounds, and in the next section to formulate a convergence and finiteness theorem.

Throughout this section, we let $f\colon M \to \mathbf{R}^{n+1}$ be a one-sidedly locally convex immersion, with locally convex extension $\widetilde{f}\colon \widetilde{M} \to \mathbf{R}^{n+1}$. For $p \in M$ and $R > 0$, we denote by $U_{p,R} = U_{p,R}(f)$, the component containing p of $\widetilde{f}^{-1}(\operatorname{int} B_R(f(p))$. For R sufficiently small, f embeds $U_{p,R}$ into the boundary of a convex body which, by intersection, we may take to lie in $B_R(f(p))$. Indeed, for R sufficiently small, there is a convex body $K_{p,R}$, lying in $B_R(f(p))$ and on the positive side of f, such that $\widetilde{f}$ embeds $U_{p,R}$ homeomorphically onto $\partial K_{p,R} \cap \operatorname{int} B_R(f(p))$. These conditions uniquely determine $K_{p,R} = K_{p,R}(f)$, which we call the *convex piece for* $\widetilde{f}$ centered at p of (Euclidean) radius R. Note that we do not assume $U_{p,R}$ is simply connected.

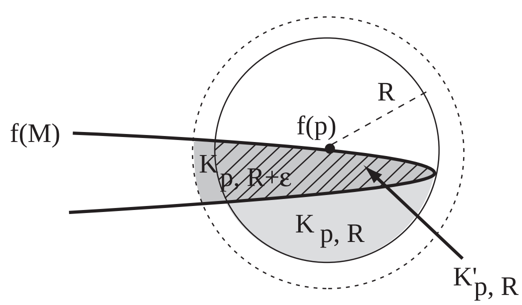

Figure 6

The radii of convex pieces centered at p form an interval, as follows ¿from Lemma 6.1.1 below, by taking $p = q$. We denote the supremum of radii of convex pieces at p by $R_p(f)$, the *radius of convexity of* f *at* p. Furthermore,we set

$$R(f) := \inf_{p \in M} R_p(f),$$

the *radius of convexity of* f.

Corresponding to each $R < R(f)$, we define the *inradius of convexity* $r_{p,R}(f)$ *of* f *at* p to be the inradius of the convex piece $K_{p,R}$. It follows by homothety that

$r_{p,R}(f)/R$ is nonincreasing in R for $R \in (0, R(f))$. (See Lemma 6.1.3 for a more general inequality, which yields this one when $p = q$.) The *inradius of convexity* $r_R(f)$ *of* f is defined by

$$r_R(f) := \inf_{p \in M} r_{p,R}(f).$$

Note that these notions depend on the choice of extension $\widetilde{f}$, which we fix in our arguments.

Lemma 6.1. *Let R_2 be the radius of a convex piece centered at q. Suppose $B_{R_1}(f(p)) \subset B_{R_2}(f(q))$, and $p \in U_{q,R_2}$. Then*

1. *R_1 is the radius of a convex piece centered at p,*
2. *$K_{q,R_2} \cap B_{R_1}(f(p)) \subset K_{p,R_1}$,*
3. *$r_{p,R_1}(f) \geqslant \frac{R_1}{R_2 + d(f(p), f(q))} r_{q,R_2}(f)$.*

Proof. Let K be the convex body $K_{q,R_2} \cap B_{R_1}(f(p))$. Then $\widetilde{f}(U_{p,R_1})$ is a component of $\widetilde{f}(U_{q,R_2}) \cap \operatorname{int} B_{R_1}(f(p))$. Let x be an interior point of K, and $\pi_x : \partial K \to \partial B_{R_1}(f(p))$ be the homeomorphism given by projection from x. Then if a sequence in $\widetilde{f}(U_{p,R_1})$ converges to a boundary point of $\widetilde{f}(U_{p,R_1})$ in ∂K, its image sequence under π_x converges to the same point. It follows that $\partial B_{R_1}(f(p))$ is homeomorphic to the union of $\widetilde{f}(U_{p,R_1})$ and the complement of its π_x-image in $\partial B_{R_1}(f(p))$. The body bounded by this union is locally supported everywhere, hence convex, and satisfies the defining conditions of the convex piece K_{p,R_1}. Therefore 1 holds. By construction, K_{p,R_1} satisfies 2.

For any ball of radius δ and center x contained in the convex body K_{q,R_2}, the cone from $f(p)$ over this ball also lies in K_{q,R_2}. The extreme distance d^* from $f(p)$ in this cone satisfies

$$d^* = d(f(p), x) + \delta \leq d(f(p), f(q)) + d(f(q), x) + \delta \leq d(f(p), f(q)) + R_2.$$

It follows by homothety with center $f(p)$ that a ball of radius $\delta R_1/(d(f(p), f(q)) + R_2)$ lies in $K_{q,R_2} \cap B_{R_1}(f(p))$. Therefore 2 implies 3. ☐

Next we introduce convex bodies $K'_{p,R}$ that vary lower semicontinuously in p. The $K_{p,R}$ lack this property, as may be seen in Figure 6.

Proposition 6.2. *For $R < R_p(f)$, there is a convex body $K'_{p,R} \subset K_{p,R}$ defined by*

$$K'_{p,R} = \lim_{\epsilon \to 0} K_{p,R+\epsilon}.$$

If $R(f) > 0$, then for any $R < R(f)$, the convex bodies $K'_{p,R}$ vary lower semicontinuously with p. That is, for any convergent sequence $K'_{p_i,R}$ where $p_i \to q$, we have

$$K'_{q,R} \subset \lim_{i \to \infty} K'_{p_i,R}.$$

Proof. By Lemma 6.1.2, the convex bodies $K_{p,R+\epsilon} \cap B_R(f(p))$ increase monotonically as ϵ decreases, which proves the first claim of our proposition.

For the second claim, fix a sequence $\epsilon_i \to 0$. Since $K'_{p_i,R} = \lim_{\epsilon \to 0} K_{p_i,R+\epsilon}$, then we may choose a sequence $\delta_i \to 0$ such that $\lim_{i \to \infty} K'_{p_i,R} = \lim_{i \to \infty} K_{p_i,R+\delta_i}$.

By passing to a subsequence of the p_i, we may also assume that $B_{R+\delta_i}(f(p_i)) \subset B_{R+\epsilon_i}(f(q))$. Then

$$\begin{aligned} K'_{q,R} &= \lim_{i\to\infty} K_{q,R+\epsilon_i} = \lim_{i\to\infty} K_{q,R+\epsilon_i} \cap B_{R+\delta_i}(f(p_i)) \\ &\subset \lim_{i\to\infty} K_{p_i,R+\delta_i} = \lim_{i\to\infty} K'_{p_i,R}. \end{aligned}$$

Here, the inclusion is by Lemma 6.1.2. The second equality is by the following property of compact convex bodies: if $C_i \to C \subset B$ and $B_i \to B$, then $C_i \cap B_i \to C$. ☐

Proposition 6.3. *Suppose that M is compact. Let $P \subset \mathbf{R}^{n+1}$ be a convex polyhedron such that $f(\partial M) \subset \operatorname{int} P$. Let f_P be the corresponding* clipping *of f, as defined in Proposition 4.4. Then $R(f_P) \geqslant R(f)/2$.*

Proof. Let $p \in M$, and $U = U_{p,R/2}$ be the component of $f_P^{-1}(B_{R/2}(f_P(p)))$ which contains p. Further, let $M^+ := f^{-1}(P)$. There are two cases to consider: either (i) $U \cap M^+ = \emptyset$, or (ii) $U \cap M^+ \neq \emptyset$.

(Case i) Suppose $U \cap M^+ = \emptyset$. The convex embedding of P in $\mathbf{R}^{n+1}$ has infinite radius of convexity, hence has a convex piece of radius $R/2$ at every point by Lemma 6.1.1. By assumption, f_P has the same convex piece of radius $R/2$ centered at p.

(Case ii) Suppose $U \cap M^+ \neq \emptyset$. Let $q \in U \cap M^+$. Then clipping by P does not reduce the radius of convexity at q: $R_q(f_P) \geqslant R_q(f) = R$. Thus R is the radius of a convex piece for f_P centered at q. We have $B_{R/2}(f(p)) \subset B_R(f(q))$. Since q lies in $U_{p,R/2}$, then p lies in $U_{q,R}$ (where $U_{q,R}$ and $U_{p,R/2}$ are both defined relative to the map f_P). Now the claim follows from Lemma 6.1.1, applied to f_P. ☐

Proposition 6.4. *Let $P \subset \mathbf{R}^{n+1}$ be a convex polyhedron such that $f(\partial M) \subset \operatorname{int} P$. Suppose M is compact, f is one-sided, and* $\operatorname{int} C \neq \emptyset$, *where $C := \operatorname{conv} f(\partial M)$. Then there exist $R > 0, \lambda > 0$ and $\delta > 0$, all independent of P, such that if* $\operatorname{dist}_{\mathrm{H}}(P, C) \leqslant \lambda$, *then $r_R(f_P) \geqslant \delta$.*

To prove this we use the following lemma.

Lemma 6.5. *Let M be compact and $f\colon M \to \mathbf{R}^{n+1}$ be a one-sidedly locally convex immersion whose restriction to each component of ∂M is an embedding. Suppose that* $\operatorname{int} C \neq \emptyset$, *where $C := \operatorname{conv} f(\partial M)$. Let $f(p) \in \partial C$, and U_p be a neighborhood of p in $\widetilde{M}$ such that $\widetilde{f}(U_p) \subset \partial K_p$, for some convex body $K_p \subset \mathbf{R}^{n+1}$ lying on the positive side of $f(U_p)$. Then $K_p \cap \operatorname{int} C \neq \emptyset$.*

Proof. First note that, for $p \in f^{-1}(\partial C)$, the condition $K_p \cap \operatorname{int} C = \emptyset$ is meaningful in the sense that it holds for every choice of U_p and K_p if it holds for any choice. This is because $f(p) \in K_p \cap C$, $K_p \cap C$ is convex, and the side of f on which K_p lies is determined.

Let X' be a component of the set

$$\{\, p \in f^{-1}(\partial C) \mid K_p \cap \operatorname{int} C = \emptyset \,\}.$$

Since the condition $K_p \cap \operatorname{int} C = \emptyset$ is both open and closed on the points $p \in f^{-1}(\partial C)$, it follows that X' is also a component of $f^{-1}(\partial C)$. Since $\partial M \subset f^{-1}(\partial C)$, then X' contains every component of ∂M that it intersects.

We must show $X' = \emptyset$. For any $p \in X'$, we denote by H_p a hyperplane that separates K_p and C; that is, K_p and C lie in the opposite closed half- spaces determined by H_p, say $C \subset H_p^+$ and $K_p \subset H_p^-$. Since C and K_p are both convex, the existence of a separating hyperplane follows from a well-known theorem in convexity.

Suppose there exists $p_0 \in X'$, and set $H = H_{p_0}$ for some choice of H_{p_0}. Let X be the component of $f^{-1}(H)$ containing p_0. Then (as in the proof of Lemma 3.7) the positive side of f is locally supported on a fixed side of H at every point of X. Since C lies on the other side of H, it follows that $(X \cap \partial M) \subset X'$.

We denote by ∂M_α, any component of ∂M that intersects X. Then X' intersects ∂M_α, so $X' \supset \partial M_\alpha$. Now observe that the condition

$$\mathrm{int}_{\partial C}(K_p \cap \partial C) \neq \emptyset$$

is open and closed on the points p of X'. Moreover, since ∂M_α is compact, there must exist a point $q \in \partial M_\alpha$ such that $f(q)$ is contained in the interior of no line segment in $f(\partial M_\alpha)$; for instance, we may let q be a point which is a maximizer for the distance of $f(\partial M_\alpha)$ from any fixed point in $\mathbf{R}^{n+1}$. It follows that $\mathrm{int}_{\partial C}(K_q \cap \partial C) \neq \emptyset$; for, otherwise, $f(\partial M_\alpha \cap U_q)$ would lie in an affine subspace of H_q and $f(q)$ would lie in the interior of a line segment in $f(\partial M_\alpha)$. Therefore $\mathrm{int}_{\partial C}(K_p \cap \partial C) \neq \emptyset$ for all $p \in \partial M_\alpha$. Consequently, the choice of H_p is uniquely determined for all $p \in \partial M_\alpha$. But then H_p is locally constant and hence constant on ∂M_α, so $H_p = H$. Therefore $f(\partial M_\alpha) \subset H$, and $\partial M_\alpha \subset X$. We conclude that X contains every component ∂M_α of ∂M that intersects X. Therefore by Lemma 3.7, the restriction of f to X is an embedding onto a compact convex subset K_H of H with finitely many open subsets D_α of $\mathrm{int}\, K_H$ removed, where D_α has boundary $f(\partial M_\alpha)$.

Let H^- denote the halfspace determined by H that does not contain C. Then X has a neighborhood V in M such that $f(V - X) \subset \mathrm{int}\, H^-$. We denote by H_t, a hyperplane parallel to H and lying in H^- at distance t from H, and by H_t^-, the side of H_t that lies in H. For t sufficiently small, we may choose V to be a component of $f^{-1}(H_t^+)$. But by Lemma 4.3, $f^{-1}(H_t^+)$ is connected. It follows that $\partial M \subset V$, and hence $\partial M \subset X$. In particular, $f(X)$ is a convex body in H.

Let Y_t be the closure of the component of $f^{-1}(\mathrm{int}\, H_t^-)$ that intersects V. By Lemma 4.2, f embeds the Y_t onto nested convex caps as $t \to 0$. Therefore f embeds the union of the Y_t onto a convex cap with boundary $f(\partial X)$. Thus $(\bigcup Y_t) \cup X$ is a compact manifold with boundary, and $M = (\bigcup Y_t) \cup X$. But then $f(\partial M) \subset H$, in contradiction to the assumption $\mathrm{int}\, C \neq \emptyset$. □

Proof(Proposition 6.4). Choose $R \in (0, R(f)/4)$. Set $C := \mathrm{conv}\, f(\partial M)$, and $M_C := f^{-1}(C)$. By Lemma 6.5, for the immersion f, the condition $K'_{p,R} \cap \mathrm{int}\, C \neq \emptyset$ holds for every $p \in M_C$. Further, by Proposition 6.2, the convex bodies $K'_{p,2R}$ vary lower semicontinuously in p. Together, these facts imply that for every p in some neighborhood W in $\widetilde{M}$ of M_C, there is a Euclidean ball of radius $\delta_0 > 0$ in $K'_{p,2R} \cap \mathrm{int}\, C$, where δ_0 is independent of p.

Now define

$$M^*_\epsilon := \{\, p \in M \mid \mathrm{dist}_f(p, M_C) \geqslant \epsilon \,\},$$

where dist_f denotes the intrinsic distance induced on M by f. Let

$$\lambda(\epsilon) := \inf_{p\in M_\epsilon^*} \mathrm{dist}(f(p), C),$$

where dist denotes the standard distance in $\mathbf{R}^{n+1}$. Then $\lambda(\epsilon) > 0$ because M_ϵ^* is compact and $f(M_\epsilon^*) \cap C = \emptyset$.

Now choose $\epsilon_0 > 0$ so that $M - M_{\epsilon_0}^* \subset W$, and set $\lambda_0 = \lambda(\epsilon_0)$. Let P be any convex polyhedron such that $C \subset \mathrm{int}\, P$, and $\mathrm{dist}_{\mathrm{H}}(P, C) \leqslant \lambda_0$. By Proposition 6.3, $R(f_P) > 2R$.

Set

$$M^+ := f_P^{-1}(\mathrm{int}\, P).$$

Since $M^+ \subset W$, then for the immersion f there is a Euclidean ball of radius $\delta_0 > 0$ in $K'_{p,2R} \cap \mathrm{int}\, C$, and hence in $K_{p,2R} \cap \mathrm{int}\, C$, where δ_0 is independent of $p \in M^+$ But since $C \subset \mathrm{int}\, P$, it follows from the definition of the clipping f_P that $K_{p,2R} \cap \mathrm{int}\, C$ is the same for f_P as it is for f. Therefore $r_{p,2R}(f_P) \geqslant \delta_0$ for all $p \in M^+$.

Let $U_R(p)$ be the open distance ball about $p \in M$ with radius R in $\widetilde{M}$, with respect to the metric induced by f_P. As in the proof of Proposition 6.3, we consider two cases: either (i) $U_R(p) \cap M^+ = \emptyset$, or (ii) $U_R(p) \cap M^+ \neq \emptyset$.

(Case i) Suppose $U_R(p) \cap M^+ = \emptyset$. Then $f_P(U_R(p)) \subset P$, and consequently $r_{p,R}(f_P) = \mathrm{inradius}(B_R(f_P(p)) \cap P)$. Choose $0 < \lambda_1 < R$, and set

$$\delta_1 := \inf\{\mathrm{inradius}(B_R(x) \cap C) \mid \mathrm{dist}(x, C) \leqslant \lambda_1\}.$$

Note that $\delta_1 > 0$, because $B_R(x) \cap C$ depends continuously on x. Now if $\mathrm{dist}_{\mathrm{H}}(P, C) \leq \lambda_1$, we have

$$\begin{aligned} r_{p,R}(f_P) &= \mathrm{inradius}(B_R(f_P(p)) \cap P) \\ &\geqslant \mathrm{inradius}(B_R(f_P(p)) \cap C) \ \geqslant\ \delta_1, \end{aligned}$$

where δ_1 is by definition independent of P.

(Case ii) Suppose $U_R(p) \cap M^+ \neq \emptyset$. Let $q \in U_R(p) \cap M^+$. Then $r_{q,2R}(f_P) \geqslant \delta_0$, where recall that δ_0 depends only on some fixed neighborhood W of M_C in $\widetilde{M}$. In particular, δ_0 is independent of P. By Lemma 6.1.3, $r_{p,R}(f_P) \geqslant \delta_0/3$.

Thus we have proved the proposition for $0 < R < R(f)/4$, $\lambda := \min\{\lambda_0, \lambda_1\}$, and $\delta = \min\{\delta_0/3, \delta_1\}$, which are all positive and independent of P. □

7. Convergence and Finiteness Theorem

Here we prove a noncollapsing convergence theorem for a sequence of locally convex immersed hypersurfaces, $f_k \colon M_k \to \mathbf{R}^{n+1}$. This result occupies a place between the Blaschke selection theorem and the Cheeger-Gromov compactness theorems for abstract Riemannian manifolds. Namely, it is an extrinsic analogue of the "Convergence Theorem of Riemannian Geometry"([27, Pg. 300]). Condition 1 below plays the role of the bounds on injectivity radius and sectional curvature in that theorem.

A corollary of Theorem 7.1 is a bound on the number of homeomorphism classes of locally convex, compact immersed hypersurfaces having fixed boundary and uniform bounds on radius of convexity, inradius of convexity and volume. By contrast, under the special boundary conditions of Theorem 1.2, we have at most two such classes, independently of such bounds. Note that finiteness theorems for Lipschitz homeomorphism classes can be derived in a more abstract Gromov-Hausdorff setting; see Petersen [27, p. 299], with reference to papers of Siebenmann and Sullivan in [7]. However, our setting is suited to a simpler approach.

Our framework for Theorem 7.1 allows a highly intuitive (if unavoidably technical) proof. The ingredients include: convergence of local convex pieces, by Blaschke selection; locally defined vector fields radiating from the centers of inradius balls in the limit convex pieces; and patching together these local vector fields to obtain a Lipschitz vector field v_k that is strongly transverse to f_k. Thus we use the Lipschitz structure of locally convex immersions, and our uniform radius of convexity bounds, to construct uniform, locally embedded collars for the f_k. For k sufficiently large, this collar structure yields explicit bi-Lipschitz homeomorphisms and convergence of immersions.

For each f_k, we fix a locally convex extension $\widetilde{f}_k\colon \widetilde{M}_k \to \mathbf{R}^{n+1}$. Set $\widetilde{d}_k := \operatorname{dist}_{\widetilde{f}_k}$, and $d_k := \widetilde{d}_k|M_k \times M_k$. Recall that a λ- *net* in a metric space M is a collection of points $p_i \in M$ such that every point of M is at distance less than λ from some p_i.

Theorem 7.1. *Let M_k be a sequence of compact n-manifolds whose boundaries ∂M_k, which we assume nonempty, are homeomorphically identified. Let $f_k\colon M_k \to \mathbf{R}^{n+1}$ be a sequence of one-sidedly locally convex immersions that are fixed on the boundary. Assume:*

(1) *there is a uniform lower bound $R > 0$ for the radii of convexity $R(f_k)$, such that the corresponding inradii of convexity $r_R(f_k)$ also have a uniform lower bound $r > 0$,*
(2) *for any $\lambda > 0$, the number of elements in some λ-net for the metric d_k on M_k has a uniform upper bound.*

Then there exists a one-sidedly locally convex immersion $f\colon M \to \mathbf{R}^{n+1}$ with locally convex extension $\widetilde{f}\colon \widetilde{M} \to \mathbf{R}^{n+1}$, and neighborhoods N_k of M_k in $\widetilde{M}_k$, such that after passing to a subsequence, there are homeomorphisms $f_k\colon M \to M_k$ with bi-Lipschitz extensions $h_k\colon \widetilde{M} \to N_k$ (Lipschitz constants depending only on R and r), such that $f_k \circ h_k \to f$.

We begin with an elementary lemma:

Lemma 7.2. *Suppose K is a convex body in $\mathbf{R}^{n+1}$, and $B_r(y) \subset K \subset B_R(x)$. Then:*

(1) *for some $\theta_0 = \theta_0(R, r) > 0$, and for any $z_1 \in \partial K$, K contains the truncated spherical cone*
$$T_{z_1} := \{\, z_2 \in \mathbf{R}^{n+1} | \parallel z_2 - z_1 \parallel \leq r,\ \angle z_2 z_1 y \leq \theta_0 \,\},$$
(2) *projection from y determines a distance-nonincreasing, bi-Lipschitz homeomorphism $\pi_y : \partial K \to \partial B_r(y)$, where the Lipschitz constant is determined by R and r.*

Proof. Since $K \subset B_{2R}(y)$, without loss of generality we take $x = y$.

1. Set $\theta_0 := \sin^{-1}(r/R) =$ half the angle subtended by $B_r(y)$ from a point on $\partial B_R(y)$. Clearly this is the required angle bound.

2. Given a segment γ in $B_R(y)$, we denote its length by ℓ and its angle subtended at y by α. Since $d_{\partial K}$ is induced from Euclidean distance, it suffices to show that there is an upper bound in terms of R and r for the ratio ℓ/α taken over all sufficiently short segments γ whose chords do not enter $B_r(y)$. For a given ℓ, the smallest subtended angle, say $\alpha(\ell)$, occurs when γ lies on the boundary of the cone over $B_r(y)$ from a point on $\partial B_R(y)$ and γ has an endpoint at the cone point. As in part 1, the cone angle is $2\theta = 2\sin^{-1}(r/R)$. Since $\tan\alpha(\ell) = \ell\sin\theta/(R-\ell\cos\theta)$, it follows that

$$\lim_{\ell\to 0}\ell/\alpha(\ell) = \lim_{\ell\to 0}\ell/\tan\alpha(\ell) = R^2/r.$$

□

Proof (Theorem 7.1). Let $\{p_i^k\}$ be a finite λ-net for d_k on M_k $(k = 0, 1, \ldots)$, where λ will be specified below. By condition 2, we may assume the number of points in $\{p_i^k\}$, say $m(\lambda)$, is independent of k. By condition 1, for each $\widetilde{f}_k$ there is a convex piece of Euclidean radius R centered at every point of M_k. In particular, there is such a convex piece K_i^k for $\widetilde{f}_k$ centered at p_i^k. By definition, this means that K_i^k is a convex body in $B_R(f_k(p_i^k))$, and $\widetilde{f}_k$ embeds an open neighborhood $U_i^k \subset \widetilde{M}_k$ of p_i^k onto $\partial K_i^k \cap \operatorname{int} B_R(f_k(p_i^k))$. In this proof, we shall refer to the neighborhood U_i^k as a *convex piece of* $\widetilde{M}_k$.

By condition 1, each K_i^k contains some ball $B_r(y_i^k)$. Since intrinsic distance is not less than extrinsic distance, U_i^k contains the open $\widetilde{d}_k$-ball in $\widetilde{M}_k$ of radius R centered at p_i^k. Note that two different points of $\widetilde{M}_k$ having the same $\widetilde{f}_k$-image lie at $\widetilde{d}_k$-distance at least R from each other.

By passing to iterated subsequences, we may assume the following *net convergence properties*. For each fixed i, the sequence $f_k(p_i^k)$ converges to a point $x_i \in \mathbf{R}^{n+1}$, the sequence y_i^k converges to a point $y_i \in \mathbf{R}^{n+1}$, and the convex bodies K_i^k converge to a convex set K_i in $B_R(x_i)$. Moreover, for each i and j, we may assume that the distances $\widetilde{d}_k(p_i^k, p_j^k)$ converge. These follow from condition 2 and the assumption of fixed boundary, which together confine the images of the f_k to a bounded region. Since K_i contains $B_r(y_i)$, K_i is a convex body. Further we may assume, after passing to a subsequence, that K_i^k contains $B_{r/2}(y_i)$, for all i and k.

Let $S_i = \partial K_i \cap \operatorname{int} B_R(x_i)$, so S_i is an embedded convex hypersurface in the interior of the ball $B_R(x_i)$. Let $d_{U_i^k}$ (resp. d_{S_i}) denote the intrinsic distance on U_i^k (resp. S_i). Let $B_{R/2}^k(p_i^k)$ (resp. $B_{R/2}^{S_i}(x_i)$) denote the corresponding closed distance balls. If $q_k, r_k \in B_{R/2}^k(p_i^k)$, then q_k and r_k are joined by a $\widetilde{d}_k$-minimizer whose $\widetilde{f}_k$-image cannot touch the boundary of $B_R(x_i)$. Therefore $\widetilde{d}_k(q_k, r_k) = d_{U_i^k}(q_k, r_k) = d_{\partial K_i^k}(f_k(q_k), f_k(r_k))$. Suppose $\widetilde{f}_k(q_k) \to z_1$, $\widetilde{f}_k(r_k) \to z_2$, where $q_k, r_k \in B_{R/2}^k(p_i^k)$ and $z_1, z_2 \in S_i$. Then $d_{S_i}(z_1, z_2) = d_{\partial K_i}(z_1, z_2)$. Since $d_{\partial K_i^k}(f_k(q_k), f_k(r_k))$ converges to $d_{\partial K_i}(z_1, z_2)$ (see [5, p. 81] or [3]), we have the following distance convergence formula:

$$\widetilde{d}_k(q_k, r_k) \to d_{\partial K_i}(z_1, z_2).$$

It is convenient to define a Lipschitz atlas, on an open subset of $\widetilde{M}_k$ containing M_k, by using the central projections $\pi_i : \mathbf{R}^{n+1} - B_{r/2}(y_i) \to \partial B_{r/2}(y_i)$ from the points y_i; or what is the same, the nearest-point projections to $B_{r/2}(y_i)$. Specifically, by Lemma 7.2.2, the map $\pi_i^k = \pi_i \circ \widetilde{f}_k : B^k_{R/2}(p_i^k) \to \partial B_{r/2}(y_i)$ is a distance-nonincreasing, bi-Lipschitz homeomorphism, with Lipschitz bound depending only on R and r, from $B^k_{R/2}(p_i^k)$, under the metric $\widetilde{d}_k$, onto its image in $\partial B_{r/2}(y_i)$.

Now we specify a net size λ, depending only on r and R, as follows. For all k and i, the image $\pi_i(B^{S_i}_{R/2}(x_i))$ contains a ball of some radius $5\lambda > 0$ about $\pi_i(x_i)$ in $\partial B_{r/2}(y_i)$. Thus $\lambda < R/10$. Let V_i^k (resp. $V_i'^k$) be the corresponding preimage in $B^k_{R/2}(p_i^k)$ under π_i^k of the open ball of radius 3λ (resp. 2λ) about $\pi_i(x_i)$ in $\partial B_{r/2}(y_i)$. We may assume $B^k_\lambda(p_i^k) \subset V_i'^k$ and $B^k_{2\lambda}(p_i^k) \subset V_i^k \subset B^k_{R/2}(p_i^k)$. Regarding the V_i^k as chart neighborhoods, with bi-Lipschitz chart maps onto balls in $\partial B_{r/2}(y_i)$, we obtain a Lipschitz atlas on the union of the V_i^k carrying the metric $\widetilde{d}_k$. The transition functions are bi-Lipschitz with constants depending only on R and r. Moreover, on chart neighborhoods, $\widetilde{d}_k$ is bi-Lipschitz equivalent to the chart distance, by Lemma 7.2.2. Therefore, since λ is a Lebesgue number for the atlas $\{V_i^k\}$, then $\widetilde{d}_k(p,q) < \lambda$ implies

$$\widetilde{d}_k(p,q) < c_0|\widetilde{f}_k(p) - \widetilde{f}_k(q)|, \tag{7.1}$$

where c_0 depends only on R and r.

Since π_i^k is distance-nonincreasing, we may pull back an appropriate function on the 3λ-ball about $\pi_i(x_i)$ in $\partial B_{r/2}(y_i)$ by $\widetilde{f}_k$, to obtain a Lipschitz function $\phi_i^k : V_i^k \to [0,1]$, taking value 1 on $V_i'^k$ and 0 on the complement of a neighborhood of $V_i'^k$ whose closure lies in V_i^k, where the Lipschitz constant depends only on R and r. Then we may define a vector field v_k as follows. Let $v_i^k : V_i^k \to \mathbf{R}^{n+1}$ be the radial vector field $v_i^k(p) := |\widetilde{f}_k(p) - y_i|^{-1}(\widetilde{f}_k(p) - y_i)$. Lemma 7.2.2 states that v_i^k is Lipschitz. Therefore $v_k = \Sigma_{0 \leq i \leq m(\lambda)} \phi_i^k v_i^k$ is a Lipschitz vector field defined on the union of the V_i^k. Since the Lipschitz constants of the ϕ_i^k and v_i^k depend only on R and r, as does the number of summands $m(\lambda)$ by condition 2, then so does the Lipschitz constant of v_k. Thus by (7.1), if $\widetilde{d}_k(p,q) < \lambda$ then

$$|v_k(p) - v_k(q)| \leq c_1 \widetilde{d}_k(p,q) \leq c_o c_1 |\widetilde{f}_k(p) - \widetilde{f}_k(q)|. \tag{7.2}$$

By the assumption of one-sided local convexity, at any point p in the union of the V_i^k, the convex hull of the rays determined by the unit vectors $\{v_i^k(p) : p \in V_i^k\}$ is an outward-pointing convex cone, which is supported by any support hyperplane for $\widetilde{f}_k$ at p. By Lemma 7.2.1, there is a positive lower bound, θ_0, for the angle between this cone and any support hyperplane at p. Therefore there is a positive lower bound, depending only on R and r, for the length of $v_k(p)$ for any p in the union of the $V_i'^k$.

Let N_k denote the union of the $V_i'^k$, and define $G_k \colon N_k \times [-\epsilon, \epsilon] \to \mathbf{R}^{n+1}$ by

$$G_k(p,t) := \widetilde{f}_k(p) + t v_k(p).$$

Now we prove:

Claim (†) *there exist $\delta \in (0, R)$ and $\epsilon > 0$ such that for every k and every $p \in N_k$, the restriction of G_k to $B^k_\delta(p) \times [-\epsilon, \epsilon]$ is an embedding.*

By (7.1), we may choose δ_0 satisfying $0 < \delta_0 < \min(\lambda, r)$, independently of k and $p \in N_k$, so that $B^k_\lambda(p)$ contains a convex piece U^k_p of $\widetilde{M}_k$ of Euclidean radius δ_0. That is, there is a convex body $K^k_p \subset B_{\delta_0}(\widetilde{f}_k(p))$ such that $\widetilde{f}_k$ embeds U^k_p onto $\partial K^k_p \cap \operatorname{int} B_{\delta_0}(\widetilde{f}_k(p))$. Since $\delta_0 < \lambda$, we have $K^k_p \supset (K^k_i \cap B_{\delta_0}(\widetilde{f}_k(p))$ whenever $p \in V^k_i$, and hence whenever $\phi^k_i(p) \neq 0$. Therefore by Lemma 7.2.1, for any i such that $\phi^k_i(p) \neq 0$, the convex body K^k_p contains the truncated spherical cone with vertex $\widetilde{f}_k(p)$, radius δ_0, angle θ_0 and axis vector $-v^k_i(p)$. Therefore K^k_p contains the convex hull of these truncated cones. It follows that there is a uniform δ, $0 < \delta < \delta_0$, such that K^k_p contains a truncated cone T^k_p with vertex $\widetilde{f}_k(p)$, radius δ, angle θ_0 and axis vector $-v_k(p)$. In particular, if $\widetilde{d}_k(p, q) < \delta$, then the angle between the line $L_k(p) := \{\widetilde{f}_k(p) + tv_k(p) | t \in \mathbf{R}\}$ and the vector $\widetilde{f}_k(q) - \widetilde{f}_k(p)$ is at least θ_0.

Now, writing $z = \widetilde{f}_k(p)$, Claim (†) may be rephrased as follows. For vectors $z_1, z_2, v, w \in \mathbf{R}^{n+1}$, suppose $|v - w| \leq c_0 c_1 |z_2 - z_1|$, and the angle θ between v and $z_2 - z_1$ satisfies $0 < \theta_0 \leq \theta \leq \pi - \theta_0$. Then there exists $\epsilon > 0$ such that $|s_2| > \epsilon$ whenever $z_2 + s_2 v_2$ lies on the line $\{z_1 + s_1 v_1\}$. To verify this, we may work without loss of generality in $\mathbf{R}^2$, with $z_1 = (0, 0)$, $v = (a, 0)$, $z_2 = t(\cos\theta, \sin\theta)$ where $t > 0$, and $z_2 + s_2 w = (b, 0)$. Then the second component of $v - w$ is $-t \sin\theta / s_2$, and so

$$t \sin\theta_0 / |s_2| \leq t \sin\theta / |s_2| \leq c_0 c_1 t.$$

Thus we may take $\epsilon = \sin\theta_0 / c_0 c_1$, and Claim (†) is verified.

Now we specify a new uniform net size $\lambda_0 = \delta/10$, for δ as in (†). Recall $\delta < \lambda < R/10$. We pass to a subsequence, also denoted by $\widetilde{f}_k$, with λ_0-nets $\{p^k_i\}$ satisfying the net convergence properties.

We may assume the Euclidean Hausdorff distance between the convex bodies K_i and K^k_i is as small as we please, uniformly in k and i, and similarly for the hypersurfaces $\widetilde{f}_k(B^k_{R/2}(p^k_i))$ and $B^{S_i}_{R/2}(x_i)$. We have seen that for $p \in N_k$, the line $L_k(p) := \{\widetilde{f}_k(p) + tv_k(p) | t \in \mathbf{R}\}$ is the axis of a truncated spherical cone T^k_p of uniform size, where T^k_p has vertex $\widetilde{f}_k(p)$ and lies in K^k_p. Moreover, by construction, $v_k(q)$ is arbitrarily close to $v_0(p)$ if $\widetilde{f}_k(q)$ is sufficiently close to $\widetilde{f}_0(p)$. It follows that if $p \in B^0_\delta(p^0_i)$, then we may assume the segment $\{\widetilde{f}_0(p) + tv_0(p) | -\epsilon \leq t \leq \epsilon\}$ strikes $\widetilde{f}_k(B^k_{R/2}(p^k_i))$ once only, at an angle bounded away from zero uniformly in k and i. Therefore each p^k_i has a neighborhood W^k_i, where $B^k_{\lambda_0}(p^k_i) \subset W^k_i \subset B^k_{R/2}(p^k_i)$, for which $\widetilde{f}_k(W^k_i)$ is the G_0-image of a section of $B^0_{2\lambda_0}(p^0_i) \times [-\epsilon, \epsilon]$. Moreover, this section has Lipschitz height function over the zero section, with Lipschitz constant independent of i and k.

Thus there is a bi-Lipschitz homeomorphism $h^k_i \colon W^0_i \to W^k_i$, where $W^0_i = B^0_{2\lambda_0}(p^0_i)$. Redefine the neighborhood N_k of M_k in $\widetilde{M}_k$ by $N_k := \cup_i W^k_i$. We now define a map $\widetilde{h}_k$ of N_0 onto N_k by setting $\widetilde{h}_k(p) := h^k_i(p)$ if $p \in W^0_i$. To see that $\widetilde{h}_k$ is well-defined (and hence continuous), observe that if $p \in W_i^0 \cap W_j^0$, then $\widetilde{d}_0(p_i^0, p_j^0) < 4\lambda_0$. Therefore we may assume $\widetilde{d}_k(p_i^k, p_j^k) < 5\lambda_0$ and $\widetilde{d}_k(h_i^k(p), h_j^k(p)) < 10\lambda_0 < R$. Therefore $h_i^k(p)$ and $h_j^k(p)$ lie on a convex piece of $\widetilde{M}_k$ with $\widetilde{f}_k(h_i^k(p)) = \widetilde{f}_k(h_j^k(p))$, hence $h_i^k(p) = h_j^k(p)$. The reverse argument shows that if $\widetilde{h}_k(p) = \widetilde{h}_k(q)$ then $\widetilde{d}_0(p, q) < 10\lambda_0 = \delta$, and so $p = q$. Therefore $\widetilde{h}_k$ is one-to-one. Thus $\widetilde{h}_k$ is a bi-Lipschitz homeomorphism of N_0 onto N_k, with uniform Lipschitz constants. Moreover, we have constructed a map $F_k \colon N_k \to N_0 \times [-\epsilon, \epsilon]$ for which $F_k \circ \widetilde{h}_k$ is a section of the bundle $N_0 \times [-\epsilon, \epsilon]$. Now the image of the limit section (which exists by construction) is our manifold $\widetilde{M}$, and the restriction of G_0 to $\widetilde{M}$ is our immersion $\widetilde{f}$. The composition of the projection from $\widetilde{M}$ to N_0 with $\widetilde{h}_k$ give the desired bi-Lipschitz equivalences of $\widetilde{M}$ with N_k; by abuse of notation, we denote these maps by $\widetilde{h}_k$ as well. □

Note 7.3. In Theorem 7.1, we may substitute for condition 2, the condition that the M_k have uniformly bounded volume. Indeed, if condition 2 fails, then for some $\lambda > 0$, the number of disjoint balls of radius $\lambda/2$ in M_k has no uniform bound. It easily follows from condition 1 and Lemma 7.2.1 that there is no uniform bound on volume.

8. Global Convexity; Gluings

The following proposition is key to proving the CHP, since a limit of clippings will satisfy its hypotheses:

Proposition 8.1. *Let $f \colon M \to \mathbf{R}^{n+1}$ be a locally convex immersion of a compact manifold M, and suppose that:*

(1) *f is one-to-one on each component ∂M_α of ∂M,*
(2) *$f(\partial M)$ lies on the boundary of a convex body C,*
(3) *∂M has a neighborhood V in M such that $f(V) \subset C$,*
(4) *f is one-sided on V.*

Then f is a convex embedding.

Proof. Let $\Gamma_\alpha := f(\partial M_\alpha)$. ∂C is homeomorphic to $\mathbf{S}^{n-1}$. So, by the generalized Jordan-Brouwer separation theorem [2, p. 353–355], $\partial C - \Gamma_\alpha$ has exactly two components, say D_α^- and D_α^+, and Γ_α is the boundary of each:

$$\partial C - \Gamma_\alpha = D_\alpha^+ \cup D_\alpha^-.$$

(Part 1) First, we show how D_α^- and D_α^+ may be distinguished from each other. Let $\widetilde{f}$ be a locally convex extension of f to a manifold $\widetilde{M}$ without boundary. Cover ∂M_α by finitely many open sets U_i in $\widetilde{M}$ such that: $\widetilde{f}|U_i$ is an embedding on the boundary of a convex body K_i in $\mathbf{R}^{n+1}$, where K_i lies on the positive side of $\widetilde{f}(U_i)$; and $U_i \cap \operatorname{int} M \subset V$ is connected.

(Case (i)) Suppose $K_i \cap \operatorname{int} C \neq \emptyset$ for some U_i, and hence for all U_i. (Here we are using the fact that the condition $K_p \cap \operatorname{int} C = \emptyset$ is both open and closed on the points $p \in \partial M$.) By convexity, for any $x \in \operatorname{int} K_i \cap \operatorname{int} C$, the rays through x determine a homeomorphism π_{ix} of ∂K_i onto ∂C, where the restriction of π_{ix} to

$\widetilde{f}(U_i \cap \partial M)$ is the identity. Define D_α^- to be the component of $\partial C - \Gamma_\alpha$ that does not contain the connected set $\pi_{ix}(f(U_i \cap \operatorname{int} M))$. Since π_{ix} varies continuously with $x \in \operatorname{int} K_i \cap \operatorname{int} C$, this choice of D_α^- is independent of $x \in \operatorname{int} K_i \cap \operatorname{int} C$. Moreover, the choice of D_α^- is independent of i. To see this, suppose $p \in \partial M \cap U_i \cap U_j$. Since $\widetilde{f}(U_i \cap U_j)$ lies on $\partial(K_i \cap K_j)$, then some neighborhood in K_i of $\widetilde{f}(p)$ lies in $K_i \cap K_j$. Since $\widetilde{f}(p)$ lies in C and $K_i \cap C$ is a nonempty convex body, then every neighborhood in K_i of $\widetilde{f}(p)$ intersects $\operatorname{int} C$. Therefore $\operatorname{int} K_i \cap \operatorname{int} K_j \cap \operatorname{int} C \neq \emptyset$. For $x \in \operatorname{int} K_i \cap \operatorname{int} K_j \cap \operatorname{int} C$, the choices of D_α^- respectively determined by π_{ix} and π_{jx} agree. Since ∂M_α is connected, it follows that the choices of D_α^- determined by any choices of i and j agree.

(Case (ii)) Suppose $K_i \cap \operatorname{int} C = \emptyset$ for all U_i. Since $f(U_i \cap M) \subset K_i \cap C$, then $f(U_i \cap M)$ lies in a hyperplane H_i separating K_i and C. Furthermore, H_i is independent of i, say $H_i = H$. Thus ∂M_α has a connected neighborhood U' in M which is embedded by f into $\partial C \cap H$. Let D_α^- be the component of $\partial C - \Gamma_\alpha$ that does not contain $f(U' \cup \operatorname{int} M)$.

Thus we have defined D_α^- for every α.

(Part 2) Now let $\overline{M}$ be the closed manifold obtained by gluing the closure of D_α^-, for each α, to M along ∂M_α. Further, let $\overline{f}$ be the natural extension of f to $\overline{M}$. It suffices to show that $\overline{f}$ is locally convex at the points of ∂M_α. Then $\overline{f}$ is locally convex everywhere, and by the theorem of van Heijenoort (Lemma 4.1), $\overline{f}$, and consequently f, is a convex embedding.

Let $p \in \partial M_\alpha \cap U_i$. In case (ii), $\overline{f}$ is locally convex at p because a neighborhood of p in $\overline{M}$ is embedded on ∂C. In case (i), let $\overline{U}_i$ be a neighborhood of p in $\overline{M}$, where $\overline{U}_i \cap M \subset U_i \cap M$. Then $\overline{f}(\partial M_\alpha \cap \overline{U}_i)$ lies on the boundary of the nonempty convex body $\operatorname{int} K_i \cap \operatorname{int} C$, and $\overline{f}(\overline{U}_i)$ lies in C.

Our choice of D_α^- implies that $\overline{f}(\overline{U}_i \cap \operatorname{int} M)$ and $\overline{f}(\overline{U}_i \cap D_\alpha^-)$ do not intersect, since $\pi_{ix}(f(U_i \cap \operatorname{int} M)) \cap D_\alpha^-) = \emptyset$. Therefore $\overline{f}$ is one-one on $\overline{U}_i$. Moreover, $\overline{f}(\overline{U}_i)$ is the graph of a continuous radial distance function in polar coordinates about $x \in \operatorname{int} K_i \cap \operatorname{int} C$. Thus for a sufficiently small Euclidean ball B about $\overline{f}(p)$, $\overline{f}(\overline{U}_i)$ separates B into two components. Let B^+ be the closure of the component that contains all segments joining $x \in \operatorname{int} K_i \cap \operatorname{int} C \cap \operatorname{int} B$ to the points of $\overline{f}(\overline{U}_i) \cap B$. For B sufficiently small, $B^+ \subset C$. Suppose a line segment γ with endpoints in B^+ leaves B^+. Since γ does not leave C or B, then γ must first leave B^+ and last reenter B^+ by leaving and entering K_i at points of $f(U_i \cap M)$. This is impossible since K_i is convex. Therefore $\overline{f}$ is locally convex at p, as required. □

9. Proof of the Main Theorems

We begin by showing that the strong CHP follows from the CHP:

Proposition 9.1. *Suppose all locally convex immersions $f\colon M \to \mathbf{R}^{n+1}$ (M compact) that fulfill conditions 1-3 of Theorem 1.2 possess the CHP. Then those that are locally strictly convex on a neighborhood of the boundary possess the strong CHP.*

Proof. We assume $\operatorname{int} C \neq \emptyset$, since otherwise the strong CHP holds by Theorem 5.1. Suppose $f\colon M \to \mathbf{R}^{n+1}$ satisfies conditions 1-3, and f is locally strictly convex on a neighborhood in M of ∂M. By assumption, $f(M) \cap \operatorname{int} C = \emptyset$. Assume, toward a contradiction, that there exists a point $p \in \operatorname{int} M$ such that $f(p) \in \partial C$. Then f locally supports C at p, either on the positive side of f (case (i)), or on its negative side (case (ii)). In case (i) there is a hyperplane H that supports C and locally supports the positive side of f at p, both in the same side, say H^+. In case (ii), there is a hyperplane H that supports C in one side, say H^-, and locally supports the positive side of f at p in H^+.

In either case, let $X \subset M$ be the connected component of $f^{-1}(H)$ containing p. By the strict local convexity assumption, $X \subset \operatorname{int} M$. By Lemma 3.4, f embeds X onto a closed convex subset of H. Then X has a neighborhood U in $\operatorname{int} M$, where U is a manifold with boundary satisfying $f(U) \subset H^+$ and $f(\partial U) \subset \operatorname{int} H^+$. Let d be the distance of $f(\partial U)$ to H, and let H_* be a hyperplane in $\operatorname{int} H^+$ that is parallel to H at distance less than d.

(Case (i)) Denote by H_*^-, the side of H_* containing H. Now we define a locally convex immersion f_* of M that agrees with f on the complement of $U \cap f^{-1}(H_*^-)$. Namely, we clip the restriction of f to U by H_*^+, as in Proposition 4.4. if H_* is sufficiently close to H then $f_*(M)$ intersects $\operatorname{int} C$. But this is a contradiction, because f_* and M satisfy conditions 1-3 of Theorem 1.1, and therefore possess the CHP.

(Case (ii)) In this case, we denote by H_*^+, the side of H_* containing H. By construction, the component of $f^{-1}(\operatorname{int} H_*^+)$ containing p lies in U, and hence in $\operatorname{int} M$. Moreover, $f(\partial M) \subset \operatorname{int} H_*^+$. Therefore $f^{-1}(H_*^+)$ is not connected, in contradiction to Lemma 4.3. □

We need the following elementary lemma; for a proof, see [30, p. 55].

Lemma 9.2. *Let $K \subset \mathbf{R}^{n+1}$ be a bounded convex set. Then for every $\epsilon > 0$ there exists a convex polyhedron $P \subset \mathbf{R}^{n+1}$ such that $K \subset \operatorname{int} P$ and $\operatorname{dist}_{\mathrm{H}}(K, P) \leqslant \epsilon$.*

Proof(Theorems 1.1 and 1.2). If f satisfies the hypothesis of Theorem 1.1, then, by Proposition 3.3, f is one-sidedly locally convex. Furthermore, since by assumption f has positive curvature on ∂M, f is locally strictly convex on a neighborhood in M of ∂M. Thus f satisfies all the hypothesis of Theorem 1.2. So it remains to prove Theorem 1.2.

Suppose that f satisfies the hypothesis of Theorem 1.2. Then, by Condition 2, either f is one-sided or is locally strictly convex on a neighborhood in M of ∂M. But recall that, by Proposition 3.5, strict convexity on the boundary implies one-sidedness. So we may assume that f is one-sided.

First assume $\operatorname{int} C \neq \emptyset$. By Lemma 9.2, for every $k \in \mathbb{N}$, there exists a convex polyhedron P_k such that $C \subset \operatorname{int} P_k$, and $\operatorname{dist}_{\mathrm{H}}(P_k, C) \leqslant \frac{1}{k}$. Let $f_k := f_{P_k}$ be the clipping of f by P_k, as defined in Proposition 4.4. By Propositions 6.3 and 6.4, the radii of convexity of f_k have uniform bounds. Further, recall that, by Proposition 4.4, f_k is distance-nonincreasing. Hence, by Theorem 7.1, we obtain a limit immersion of M, say f'. Since f' satisfies the hypotheses of Proposition 8.1, f' is a convex embedding. That is, $f'(M)$ lies on the boundary of its convex hull: $f'(M) \subset \partial \operatorname{conv} f'(M)$. So $f'(M) \cap \operatorname{int} \operatorname{conv} f'(M) = \emptyset$, and it follows that

$f'(M) \cap \operatorname{int}\operatorname{conv} f'(\partial M) = \emptyset$ as well. But note that $f'(\partial M) = f(\partial M)$. Thus we have $f'(M) \cap \operatorname{int} C = \emptyset$. Therefore, since by construction $f'(M) \cap \operatorname{int} C = f(M) \cap \operatorname{int} C$, it follows that $f(M) \cap \operatorname{int} C = \emptyset$. Further, if f is locally strictly convex on a neighborhood in M of ∂M, then, by Proposition 9.1, $f(\operatorname{int} M) \cap C = \emptyset$, which completes the proof of the convex hull property.

The auxiliary result with regard to the topology of M now easily follows. To see this, it is enough to note that $f'(M) \subset \partial C$, because by construction $f'(M) \subset C$, and, as we showed above, $f'(M) \cap \operatorname{int} C = \emptyset$. Thus, since f' is an embedding and $f'(\partial M) = f(\partial M)$, M is homeomorphic to the closure of a component of $\partial C - f(\partial M)$.

Now consider the case $\operatorname{int} C = \emptyset$. Then ∂M lies in a hyperplane H, and C is a convex body in H. By Theorem 5.1, if f is locally strictly convex on a neighborhood in M of ∂M, then f satisfies the conclusions of Theorem 1.2. It only remains to show that if f is merely one-sidedly locally convex, then f is an embedding on ∂M. Moreover, M may still be regarded as being homeomorphic to the closure of a component of $\partial C - f(\partial M)$ with boundary $f(\partial M)$, provided that we interpret ∂C to be the homeomorph of a doubly covered n-disk.

Take a component ∂M_1 of ∂M. Then (i) there exists a neighborhood V of ∂M_1 in M such that $f(V - \partial M_1)$ lies in an open halfspace $\operatorname{int} H^-$, or (ii) there exists a point $p \in \partial M_1$ and a neighborhood U of p in $\widetilde{M}$ such that H is a supporting hyperplane for $\widetilde{f}(U)$. In case (ii), let X be the component of $f^{-1}(H)$ which contains ∂M_1. Then X contains every component ∂M_α of ∂M that intersects X. By Lemma 3.7, the restriction of f to X is an embedding onto a compact convex body K_H of H with open subsets D_α of $\operatorname{int} K_H$ removed. If there are no such D_α, then $\partial M_1 = \partial M_\alpha$ and f embeds M into C, so we are done. Otherwise, X has a neighborhood V in M such that $f(V - X)$ lies in an open halfspace $\operatorname{int} H^-$.

In either case, let H_t, $0 < t < \epsilon$, be the hyperplane parallel to H in H^- at distance t from H. For t sufficiently small, let H_t^- be the halfspace determined by H_t that lies in H^-, and Y_t be the closure of the component of $f^{-1}(\operatorname{int} H_t^-)$ that intersects V. By Lemma 4.2, f embeds the Y_t onto nested convex caps as $t \to 0$. Therefore f embeds the union of the Y_t onto a convex cap, with boundary $f(\partial M_1)$ in case (i), and with boundary $f(\partial X)$ in case (ii). In case (i), $\bigcup Y_t$ is a compact manifold with boundary ∂M_1, and $M = \bigcup Y_t$. In case (ii), $(\bigcup Y_t) \cup X$ is a compact manifold with boundary $\bigcup_{\partial M_\alpha \subset X} \partial M_\alpha$, and $M = (\bigcup Y_t) \cup X$. In either case, the restriction of f to ∂M is an embedding. □

Note 9.3. When ∂M is connected, the image of ∂M limits the topology of M to at most two choices by the generalized Jordan-Brouwer separation theorem [2, p. 353–355]. If ∂M has more than one component, then the topology of M is uniquely determined, because then $\partial C - f(\partial M)$ has at most one connected component that is bounded by $f(\partial M)$. If ∂M has only one component but that component is homeomorphic to $\mathbf{S}^{n-1}$, then the topology of M is again uniquely determined. Indeed, if the closure of each component of $\partial C - f(\partial M)$ is a manifold with boundary, then each is homeomorphic to a ball by the generalized Schoënflies theorem.

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