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SHADOWS AND CONVEXITY OF SURFACES

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Abstract. We study the geometry and topology of immersed surfaces in Euclidean 3-space whose Gauss map satisfies a certain two-piece-property, and solve the "shadow problem" formulated by H. Wente.

1. Introduction

Let M be a closed oriented 2-dimensional manifold, $f\colon M \to \mathbf{R}^3$ be a smooth immersion into Euclidean 3-space, and $n\colon M \to \mathbf{S}^2$ be a unit normal vectorfield, or the Gauss map, induced by f. Then for every unit vector $u \in \mathbf{S}^2$ (corresponding to the direction of light) the *shadow*, S_u, is defined by

$$S_u := \{\, p \in M : \langle n(p), u \rangle > 0 \,\},$$

where $\langle \cdot, \cdot \rangle$ is the standard innerproduct. If f is a *convex embedding*, i.e., f maps M homeomorphically to the boundary of a convex body, then it is intuitively clear that S_u is a connected subset of M for each u. In 1978, motivated by problems concerning the stability of constant mean curvature surfaces [17], H. Wente appears to have been the first person to study the converse of this phenomenon, which has since become known as the "shadow problem" [13]: *Does connectedness of the shadows S_u imply that f is a convex embedding*? In this paper we prove:

Theorem 1.1. *f is a convex embedding if and only if, for every $u \in \mathbf{S}^2$, S_u is* simply *connected.*

Furthermore we show that the additional condition implied by the word "simply" in the above theorem is necessary:

Theorem 1.2. *There exists a smooth embedding of the torus, $f\colon \mathbf{S}^1 \times \mathbf{S}^1 \to \mathbf{R}^3$, such that for all $u \in \mathbf{S}^2$, S_u is connected.*

Thus, connectedness of the shadows in general is not strong enough to ensure convexity or even determine the topology; however, we can show:

Theorem 1.3. *If M is topologically a sphere, and, for every $u \in \mathbf{S}^2$, S_u is connected, then f must be a convex embedding.*

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In short, the answer to the above question is yes, provided that either the shadows are *simply* connected, or M is a sphere; otherwise, the answer is no. This settles Wente's shadow problem in 3-space. See [7] and [5] for motivations behind this problem and relations to constant mean curvature surfaces.

Note 1.4. The immersion $f\colon M \to \mathbf{S}^2$ has connected shadows if and only if for every great circle $C \subset \mathbf{S}^2$, $n^{-1}(\mathbf{S}^2 - C)$ has exactly two components. That is, the Gauss map satisfies a *two-piece-property* [3] similar to that formulated by T. Banchoff [2], and further developed by N. Kuiper [12].

Note 1.5. For a great circle $C \subset \mathbf{S}^2$, the number of components of $n^{-1}(\mathbf{S}^2 - C)$ has been called the *vision number* with respect to a direction perpendicular to C. This terminology is due to J. Choe, who conjectured [5, p. 210] that there always exists a direction with respect to which the vision number of $f\colon M \to \mathbf{R}^3$ is greater than or equal to $4 - \chi(M)$ where χ is the Euler characteristic. Theorem 1.2 gives a counterexample to this conjecture.

2. Regularity of Horizons and Shadow Boundaries

First we need to establish some basic regularity results regarding the generic behavior of shadows. For each $u \in \mathbf{S}^2$, define the *shadow function* $\sigma_u\colon M \to \mathbf{R}$ by

$$\sigma_u(p) := \langle n(p), u\rangle.$$

$H_u := \sigma_u^{-1}(0)$ is called the *horizon* [5] in the direction u. It is easy to see that in general $\partial S_u \neq H_u \neq \partial S_{-u}$, where ∂ denotes the boundary; however, using Sard's theorem, we can show

Proposition 2.1. *For almost all $u \in \mathbf{S}^2$ (in the sense of Lebesgue measure) H_u is a regular curve. Thus for these u, both ∂S_u and ∂S_{-u} are regular curves as well. Further, if H_u is connected, then $\partial S_u = H_u = \partial S_{-u}$.*

We say that $\Gamma \subset M$ is a *regular curve* if for each $p \in \Gamma$ there is an open neighborhood U of p in M and a homeomorphism $\varphi\colon U \to \mathbf{R}^2$ such that $\varphi(U \cap \Gamma) = \mathbf{R}$. In particular, unless stated otherwise, a regular curve needs not be differentiable.

Proof. Let T_pM be the tangent plane of M at p which we identify with a subspace of $\mathbf{R}^3$ (by identifying T_pM with $f_*(T_pM)$, and parallel translating the elements of $f_*(T_pM)$ to the origin in $\mathbf{R}^3$; f_* denotes the *differential* of f). Let $UTM := \{(p,u) : p \in M, u \in T_pM, \|u\| = 1\}$ denote the unit tangent bundle of M, and τ be the mapping given by

$$UTM \ni (p,u) \overset{\tau}{\longmapsto} u \in \mathbf{S}^2.$$

By Sard's Theorem almost every $u \in \mathbf{S}^2$ is a regular value of τ; consequently, for such u, $\tau^{-1}(u)$ is a regular curve in UTM.

Now let π be the mapping defined by

$$UTM \ni (p,u) \overset{\pi}{\longmapsto} p \in M,$$

and let u be a regular value of τ. Note that π is injective on $\tau^{-1}(u)$. As $\tau^{-1}(u)$ is compact, this implies that $\pi\colon \tau^{-1}(u) \to M$ is an embedding. Further note that

$$\pi(\tau^{-1}(u)) = \{p \in M : u \in T_pM\} = \{p \in M : \langle n(p), u\rangle = 0\} = H_u.$$

Thus H_u is a regular curve. But then, it follows that ∂S_u and ∂S_{-u} are each open in H_u, which yields that ∂S_u and ∂S_{-u} are both regular curves as well. Finally, since these shadow boundaries are also closed in H_u, it follows that whenever H_u is connected we have $\partial S_u = H_u = \partial S_{-u}$. □

Note 2.2. Suppose that there is an open set $U \subset \mathbf{S}^2$, such that, for all $u \in U$, both S_u and S_{-u} are simply connected. Then M is homeomorphic to $\mathbf{S}^2$; because, by the above proposition, there exists a $u_0 \in U$ such that H_{u_0} is a regular curve. Consequently the closures $\overline{S}_{u_0}$ and $\overline{S}_{-u_0}$ are homeomorphic to disks. Further, since by assumption $M - H_{u_0}$ is made up of a pair of simply connected components, H_{u_0} is connected. Thus by the above proposition $\partial S_{-u_0} = \partial S_{u_0}$. So M is homeomorphic to a pair of disks glued together along their boundaries.

By *smooth* we mean differentiable of class C^∞, and for convenience we always assume that the immersion $f\colon M \to \mathbf{R}^3$ is smooth, though in this paper it is enough that f be C^3.

Note 2.3. The embedding $\pi\colon \tau^{-1}(u) \to M$ in the above proposition is smooth, when u is a regular value of τ. In particular, H_u is smooth for almost all $u \in \mathbf{S}^2$. To see this let $(p,u) \in \tau^{-1}(u)$. Then $u \in T_pM$. Let $v \in T_pM$ with $\langle u, v\rangle = 0$. Then $c(t) := (p, \cos(t)u + \sin(t)v)$ parameterizes the fiber UT_pM of the unit tangent bundle. Note that

$$\tau_{*_{(p,u)}}(c'(0)) = \frac{d}{dt}\tau\big(\, p,\, \cos(t)u + \sin(t)v\,\big)\Big|_{t=0} = v \neq 0.$$

On the other hand,

$$T_{(p,u)}(\tau^{-1}(u)) = \{\, X \in T_{(p,u)}(UTM) : \tau_{*_{(p,u)}}(X) = 0\,\}.$$

Thus $c'(0) \notin T_{(p,u)}(\tau^{-1}(u))$, which implies that $\tau^{-1}(u)$ is never tangent to any of the fibers UT_pM of the unit tangent bundle. So $\pi|_{\tau^{-1}(u)}$ is a smooth immersion.

Next we need a local regularity result for the horizons and shadow boundaries. The *Gaussian curvature* $K\colon M \to \mathbf{R}$ is defined by $K(p) := \det(n_*(p))$.

Proposition 2.4. *If $K(p) \neq 0$ for some $p \in M$, then there exists a neighborhood U of p such that for all $u \in T_pM$, $H_u \cap U$ is a smooth regular curve and $\partial S_u \cap U = H_u \cap U = \partial S_{-u} \cap U$.*

Proof. Since $\det(n_{*_p}) = K(p) \neq 0$, then, by the inverse function theorem, n is a diffeomorphism between small neighborhoods U of p in M and V of $n(p)$ in $\mathbf{S}^2$. Let $\mathbf{S}^2_u := \{x \in \mathbf{S}^2 : \langle x, u\rangle > 0\}$. Then $\partial\mathbf{S}^2_u = \partial\mathbf{S}^2_{-u}$ is a regular curve. Thus, since $S_u = n^{-1}(\mathbf{S}^2_u)$ and $S_{-u} = n^{-1}(\mathbf{S}^2_{-u})$, the proof follows. □

Note 2.5. If $K(p) = 0$, then H_u may not be regular for *all* $u \in T_pM$; however, typically H_u will be regular for most $u \in T_pM$; because, for $u \in T_pM$, the differential of σ_u at p is given by

$$(d\sigma_u)_p(\cdot) = \langle\, \cdot\,, n_{*_p}(u)\rangle.$$

So if $n_{*_p}(u) \neq 0$, e.g., u is not an *asymptotic direction*, then $d\sigma_u$ is nonzero at p. Consequently, by the implicit function theorem, $\sigma_u^{-1}(\sigma_u(p)) = \sigma_u^{-1}(0) = H_u$ is a smooth regular curve near p.

3. Critical Points of Height functions

The next set of preliminary results we need involves some basic applications of Morse theory [14]. For every $u \in \mathbf{S}^2$, let the *height function* $h_u\colon M \to \mathbf{R}$, associated to the immersion $f\colon M \to \mathbf{R}^3$, be defined by

$$h_u(p) := \langle f(p), u\rangle.$$

Recall that p is a *critical point* of h_u if the differential map $(dh_u)_p\colon T_pM \to \mathbf{R}$ is zero. Since $(dh_u)_p(\cdot) = \langle \cdot, u\rangle$, it follows that p is a critical point of h_u if and only if $u = \pm n(p)$. If all of its critical points are nondegenerate, h_u is a *Morse function.*

Lemma 3.1. *(i) h_u is a Morse function if and only if $K \neq 0$ at all critical points of h_u. (ii) h_u is a Morse function for almost all $u \in \mathbf{S}^2$. (iii) The set $U \subset \mathbf{S}^2$ such that for all $u \in U$ h_u is a Morse function is open.*

Though the above is fairly well-known (e.g. see [3, pp. 11–12]), we include a brief proof for completeness.

Proof. If p is a critical point of h_u, then, as a standard computation shows, the Hessian of h_u is given by

$$\operatorname{Hess} h_u(\cdot,\cdot) = \pm\langle\, \cdot\,, n_{*_p}(\cdot)\rangle.$$

Thus h_u is a Morse function if and only if at each critical point p, $K(p) = \det(n_{*_p}) \neq 0$. This is equivalent to requiring that both u and $-u$ be regular values of n, because p is a critical point of h_u if and only if $u = \pm n(p)$. Let $U \subset \mathbf{S}^2$ be the set of all such values. Then, by Sard's theorem, $\mathbf{S}^2 - U$ has measure zero. Further, since M is compact, and the set of critical points of n is closed, it follows that the set of critical values of n is closed as well, so U is open. □

The following is implicit in a paper of Chern and Lashof [4].

Lemma 3.2. *If f is not a convex embedding, then there exists a Morse height function h_u with at least 3 critical points.* □

Proof. Let $\#C(h_u)$ denote the number of critical points of h_u. Since p is a critical point of h_u if and only if $n(p) = \pm u$, we have:

$$\int_{\mathbf{S}^2} \#C(h_u)\, du = \int_{\mathbf{S}^2} \#n^{-1}(\pm u)\, du = 2\int_M |\det(n_*)|\, dV = 2\int_M |K|\, dV.$$

The second equality above is just an application of the area formula [6, Thm 3.2.3], where dV denotes the volume element on M. Suppose that f is not a convex embedding. Then, by a well-known theorem of Chern and Lashof [4],

$$\int_M |K|\, dV > 4\pi.$$

Combining the above expressions yields a lower bound for the average number of critical points:

$$\frac{1}{4\pi}\int_{\mathbf{S}^2} \#C(h_u)\, du > 2.$$

So since, by Lemma 3.1, h_u is a Morse function for almost every $u \in \mathbf{S}^2$, it follows that there exists a Morse function such that $\#C(h_u) > 2$. □

4. Triplets on the boundaries of simply connected domains

Here we develop some elementary topological methods whose motivation will become more clear in the next section.

Definition 4.1. By a *domain* we mean a connected open subset $\Omega \subset M$. We say Ω is *adjacent* to a triplet of points $\{p_1, p_2, p_3\} \subset M$ if $p_i \in \partial\Omega$. Ω is *regular* near p_i if there are open neighborhoods U_i of p_i and homeomorphisms $\varphi_i \colon U_i \to \mathbf{R}^2$ which map $U_i \cap \Omega$ into the upper half-plane. A simple closed curve $T \subset \overline{\Omega}$ is a *triangle* of Ω (with vertices at $\{p_1, p_2, p_3\}$) if $p_i \in T$, and $T - \{p_1, p_2, p_3\} \subset \Omega$.

The following lemma, though quite elementary, is more subtle than it might at first appear (see Note 4.3).

Lemma 4.2. *Every domain Ω adjacent to $\{p_1, p_2, p_3\}$ admits a triangle. Further if Ω is* simply *connected and regular near p_i, then any pair of such triangles may be homotoped to each other through a family of triangles of Ω.*

Proof. Since Ω is open and connected, there exists a regular arc $A_{12} \subset \Omega$ whose end points are p_1 and p_2. Since A_{12} is regular, there exists a component $(\Omega - A_{12})^+$ of $\Omega - A_{12}$ which contains p_3 in its closure. Let $A_{23} \subset (\Omega - A_{12})^+$ be a regular arc with end points on p_2 and p_3. Then, similarly, there exists a component $((\Omega - A_{12})^+ - A_{23})^+$ of $(\Omega - A_{12})^+ - A_{23}$ which contains p_1 in its closure. Finally, let $A_{31} \subset ((\Omega - A_{12})^+ - A_{23})^+$ be a regular arc with end points at p_3 and p_1. The union of these three arcs, and their endpoints, gives the desired triangle.

Now suppose that Ω is simply connected and regular near p_i. Let T and T' be a pair of triangles of Ω, and let A_{12} and A'_{12} be arcs of T and T' respectively which connect p_1 and p_2. Since Ω is regular near p_i, we may homotope A_{12} (while keeping its end points fixed) by a small perturbation near p_1 so that A_{12} and A'_{12} coincide along a segment near p_1. Similarly, we may assume that they coincide near p_2 as well. Then it remains to homotope proper subarcs of A_{12} and A'_{12} which coincide at a pair of end points in Ω. Since Ω is simply connected, these subarcs may be homotoped to each other while keeping the end points fixed. Thus A_{12} and A'_{12} are homotopic through a family of arcs of Ω with end points at p_1 and p_2. Other arcs of T may be similarly homotoped to their counterparts in T', which completes the proof. $\square$

Note 4.3. Without the regularity assumption near p_i, the second claim in the above lemma is not true in general: Suppose for instance that $\Omega \subset \mathbf{R}^2$ is an open disk of radius 1 centered at the origin, and with segment $[0, 1)$ removed. Set $p_1 = (0, 0)$, $p_2 = (1/2, 0)$, and $p_3 = (1, 0)$. Then a triangle of Ω which lies above the x-axis may not be homotoped to one lying below the x-axis.

Proposition 4.4. *For a fixed orientation of M, every simply connected domain Ω which is adjacent to and regular near a triple of (distinct) points $\{p_1, p_2, p_3\} \subset M$ uniquely determines a permutation α_Ω of $\{p_1, p_2, p_3\}$ such that (i) if Ω and Ω' have a triangle in common, then $\alpha_\Omega = \alpha_{\Omega'}$; and (ii) if $\partial\Omega = \partial\Omega'$ is a regular curve, and Ω and Ω' are distinct, then $\alpha_\Omega \neq \alpha_{\Omega'}$.*

Proof. By Lemma 4.2 there exists a triangle T of Ω. T bounds a simply connected subdomain U of Ω. Since M is oriented, U inherits a preferred sense of orientation, which in turn induces an orientation, or a sense of direction, on T. This direction induces a permutation of $\{p_1, p_2, p_3\}$ in the obvious way: If as we move along T and pass p_1 we reach p_2 before reaching p_3, then we set the induced permutation α_Ω to be the cycle (p_1, p_2, p_3); otherwise, the induced permutation is the cycle (p_1, p_3, p_2). It is clear that these permutations depend continuously on T. Thus, since by Lemma 4.2, all triangles of Ω are homotopic, it follows that α_Ω does not depend on the choice of T and is therefore well defined; and furthermore, if Ω and Ω' have a triangle in common then $\alpha_\Omega = \alpha_{\Omega'}$.

Now suppose that $\partial\Omega = \partial\Omega'$ is a regular curve, and Ω and Ω' are distinct. Then Ω and Ω' induce opposite orientations on $\partial\Omega$ which in turn gives rise to distinct permutations of $\{p_1, p_2, p_3\}$ (since Ω is simply connected, $\partial\Omega$ is connected). But by small perturbations, $\partial\Omega$ may be homotoped to a triangle of Ω, just as well as it may be homotoped to a triangle of Ω'. Thus the orientations which Ω and Ω' induce on $\partial\Omega$ are consistent with the orientations which Ω and Ω' induce on their own triangles respectively. So $\alpha_\Omega \neq \alpha_{\Omega'}$. $\square$

5. Proof of Theorem 1.1

First we show that if f is a convex embedding, then S_u is simply connected for all $u \in \mathbf{S}^2$. To see this let Π be a plane perpendicular to u and let $\pi \colon \mathbf{R}^3 \to \Pi$ be the orthogonal projection. Then $D := \pi(f(M))$ is a convex subset of Π with interior points. In particular, $\mathrm{int}(D)$ is homeomorphic to an open disk. Since $f(M)$ is convex and by definition $\langle n(p), u\rangle > 0$ for all $p \in S_u$, it is not hard to verify that $f(S_u)$ is a graph over $\mathrm{int}(D)$. Thus $\pi \circ f : S_u \to \mathrm{int}(D)$ is a homeomorphism.

Now we prove the other direction: Assume that for every $u \in \mathbf{S}^2$, S_u is simply connected; we have to show that f is a convex embedding. The proof is by contradiction:

Lemma 5.1. *If f is not a convex embedding, then there exists a pair of orthogonal vectors u_0, $v_0 \in \mathbf{S}^2$ such that (i) h_{u_0} is a Morse function with at least 3 critical points, and (ii) $\partial S_{v_0} = H_{v_0} = \partial S_{-v_0}$ is a regular curve.*

Proof. By Lemma 3.2, there exists a unit vector $u \in \mathbf{S}^2$ such that the corresponding height function h_u is a Morse function and has at least three critical points. Further, it follows from Lemma 3.1, that this u may be chosen from an open set $U \subset \mathbf{S}^2$.

Let $u^\perp := \{v \in \mathbf{S}^2 : \langle u, v\rangle = 0\}$. Then $U^\perp := \cup_{u\in U} u^\perp$ is open. Consequently, by Proposition 2.1, there exits a $v_0 \in u_0^\perp \subset U^\perp$ such that H_{v_0} is a regular curve. Further, since the complement of H_{v_0} consists of a pair of simply connected domains, H_{v_0} is connected. Thus, again by Proposition 2.1, $\partial S_{v_0} = H_{v_0} = \partial S_{-v_0}$ is a regular curve. □

Let $\widehat{v}_0 \in \mathbf{S}^2$ be a vector orthogonal to both u_0 and v_0, and set

$$v(\theta) := \cos(\theta)\, v_0 + \sin(\theta)\, \widehat{v}_0. \tag{1}$$

Let p_i, $i = 1, 2, 3$, be a fixed triple of (distinct) critical points of h_{u_0}.

Lemma 5.2. *For all $\theta \in \mathbf{R}$, $S_{v(\theta)}$ is a domain adjacent to and regular near p_i.*

Proof. If p_i is a critical point of h_{u_0}, then $n(p_i) = \pm u_0$. So $\sigma_{v(\theta)}(p_i) = \langle v(\theta), \pm u_0\rangle = 0$, which yields that $p_i \in H_{v(\theta)}$. Since h_{u_0} is a Morse function, then, by Lemma 3.1, $K(p_i) \neq 0$. So by Proposition 2.4, there exists a neighborhood U_i of p_i such that $\partial S_{v(\theta)} \cap U_i = H_{v(\theta)} \cap U_i = \partial S_{-v(\theta)} \cap U_i$, which completes the proof. □

It now follows from Proposition 4.4 that each $S_{v(\theta)}$ induces a permutation of $\{p_1, p_2, p_3\}$ which we denote by $\alpha_\theta := \alpha_{(S_{v(\theta)})}$. Further, by the same proposition and since $\partial S_{v_0} = \partial S_{-v_0}$ is a regular curve, it follows that $\alpha_0 \neq \alpha_\pi$. On the other hand, letting Sym denote the symmetric group, we claim that the mapping

$$\mathbf{R} \ni \theta \longmapsto \alpha_\theta \in \mathrm{Sym}\big(\{p_1, p_2, p_3\}\big)$$

is locally constant, which, since $[0, \pi]$ is connected, would imply that $\alpha_0 = \alpha_\pi$. This contradiction, which would complete the proof, follows from Proposition 4.4 and the following:

Lemma 5.3. *For each $\theta_0 \in \mathbf{R}$ there exists an $\epsilon > 0$ such that if $|\theta - \theta_0| < \epsilon$ then $S_{v(\theta)}$ and $S_{v(\theta_0)}$ have a common triangle (with vertices at $\{p_1, p_2, p_3\}$).*

Proof. Recall that, since h_{u_0} is a Morse function, then, by Lemma 3.1, $K(p_i) \neq 0$ which yields that n is a local diffeomorphism at p_i. Therefore, by Proposition 2.4, in a neighborhood W of $\{p_1, p_2, p_3\}$, $\partial S_{v(\theta)} = H_{v(\theta)} = n^{-1}(v^\perp(\theta))$ where $v^\perp(\theta)$ denotes the great circle in $\mathbf{S}^2$ orthogonal to $v(\theta)$. So, since $v^\perp(\theta)$ depends continuously on θ, it follows that, in W, $\partial S_{v(\theta)}$ depends continuously on θ as well.

Let T be a triangle of $S_{v(\theta_0)}$. Since $S_{v(\theta_0)}$ is open, after a perturbation of T we may assume that the arcs of T are smooth and meet $\partial S_{v(\theta_0)}$ transversely (recall that, by Proposition 2.4, $\partial S_{v(\theta_0)}$ is smooth near p_i). Thus, by the above paragraph, it follows that if $|\theta - \theta_0| < \epsilon_1$, for some sufficiently small $\epsilon_1 > 0$, then T meets $\partial S_{v(\theta)}$ transversely as well. Then it follows that for some neighborhood W of $\{p_1, p_2, p_3\}$, $(T - \{p_1, p_2, p_3\}) \cap W \subset S_{v(\theta)}$ for all θ such that $|\theta - \theta_0| < \epsilon_1$.

Next note that $T - W$ is compact, and the mapping $\theta \mapsto \sigma_{v(\theta)}$ is continuous; therefore, since by assumption $\sigma_{v(\theta_0)} > 0$ on $T - W$, it follows that there exists an

$\epsilon_2 > 0$ such that $\sigma_{v(\theta)} > 0$ on $T - W$ for all θ such that $|\theta - \theta_0| < \epsilon_2$. This yields that $T - W \subset S_{v(\theta)}$ for all θ such that $|\theta - \theta_0| < \epsilon_2$.

From the previous two paragraphs it follows that, setting $\epsilon := \min\{\epsilon_1, \epsilon_2\}$, we have $(T - \{p_1, p_2, p_3\}) \subset S_{v(\theta)}$ for all θ such that $|\theta - \theta_0| < \epsilon$, which completes the proof. □

Note 5.4. Theorem 1.1 does not remain valid if the shadows are defined as the sets where $\langle n(p), u\rangle \geq 0$. For instance, the standard torus of revolution would be a counterexample.

Note 5.5. Theorem 1.1 does not remain valid without the compactness assumption; the hyperbolic paraboloid given by the graph of $z = xy$ would be a counterexample. This follows because here the unit normal vectorfield n is a homeomorphism into a hemisphere. Thus the preimage of any open hemisphere under n is simply connected.

6. Proof of Theorem 1.2

Definition 6.1. We say an immersion $\gamma\colon \mathbf{S}^1 \simeq \mathbf{R}/2\pi \to \mathbf{R}^3$ is a *skew loop* if it has no pair of distinct parallel tangent lines, i.e,

$$\gamma'(t) \times \gamma'(s) \neq 0$$

for all t, $s \in [0, 2\pi)$, $t \neq s$.

A specific example of a skew loop, formulated by Ralph Howard, is as follows:

Example 6.2. Let $\gamma(t) := (x(t), y(t), z(t))$, where

$$\begin{aligned} x(t) &:= -\cos(t) - \frac{1}{20}\cos(4t) + \frac{1}{10}\cos(2t), \\ y(t) &:= +\sin(t) + \frac{1}{10}\sin(2t) + \frac{1}{20}\sin(4t), \\ z(t) &:= -\frac{46}{75}\sin(3t) - \frac{2}{15}\cos(3t)\sin(3t), \end{aligned}$$

and $t \in [0, 2\pi]$. A computation of the tangential indicatrix $T(t) := \gamma'(t)/\|\gamma'(t)\|$ shows that $T(t) \neq \pm T(s)$ for all t, $s \in [0, 2\pi)$, $t \neq s$. Thus γ is a skew loop. Figure 1 shows the pictures of a tube built around $\gamma(\mathbf{S}^1)$.

If $\gamma\colon \mathbf{S}^1 \to \mathbf{R}^3$ is an immersion, then the unit normal bundle of γ consists of all pairs $(p, \nu) \in \mathbf{S}^1 \times \mathbf{S}^2$ such that $\langle \gamma'(p), \nu\rangle = 0$. Since this bundle is homeomorphic to a torus, the following proposition yields Theorem 1.2.

Proposition 6.3. *Let $\gamma\colon \mathbf{S}^1 \to \mathbf{R}^3$ be a skew loop and M be the unit normal bundle of γ. For $\epsilon > 0$, define $f_\epsilon\colon M \to \mathbf{R}^3$ by*

$$f_\epsilon(p, \nu) := \gamma(p) + \epsilon\,\nu.$$

Then, for ϵ sufficiently small , f_ϵ is a smooth immersion, and for all $u \in \mathbf{S}^2$, S_u is connected. If γ is an embedding, then f_ϵ is an embedding as well.

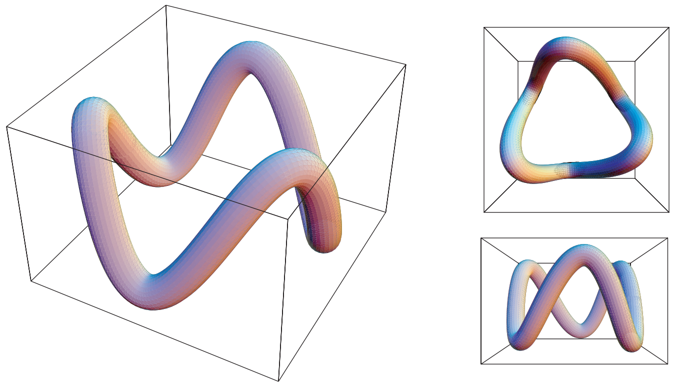

FIGURE 1

Proof. That f_ϵ is a smooth immersion and is an embedding when γ is embedded follows from the tubular neighborhood theorem. Let $n\colon M \to \mathbf{S}^2$ be the unit normal vector field given by $n(p,\nu) = \nu$, and $\pi\colon M \to \mathbf{S}^1$ be given by $\pi(p,\nu) = p$. For every $p \in \mathbf{S}^1$, let $F_p := \pi^{-1}(p)$ be the corresponding fiber. Note that n embeds F_p into the great circle in $\mathbf{S}^2$ which lies in the plane perpendicular to $T(p)$. Further recall that $S_u = n^{-1}(\mathbf{S}^2_u)$ where $\mathbf{S}^2_u$ is the open hemisphere determined by u. Thus there are only two possibilities for each $p \in \mathbf{S}^1$: either F_p intersects S_u in an open half-circle, or F_p is disjoint from S_u. The latter occurs if and only if $T(p)$ is parallel to u, which, since γ is skew, can occur at most once. Hence, it follows that S_u is either homeomorphic to a disk or an annulus. In particular, S_u is connected for every $u \in \mathbf{S}^2$. □

Question 6.4. Let M be a closed oriented 2-dimensional manifold with topological genus $g(M) \geq 2$. Does there exist an embedding, or an immersion, $f\colon M \to \mathbf{R}^3$ such that S_u is connected for all $u \in \mathbf{S}^2$?

Note 6.5. Skew loops were first discovered by B. Segre [16] to disprove a conjecture of H. Steinhaus (see also [15]). More recently, it has been shown that there exists a skew loop in each knot class [18], and every pair of knots may be realized with the same tangential indicatrix [1].

Note 6.6. A general procedure for constructing skew loops is as follows. Let $T \subset \mathbf{S}^2$ be a smooth simple closed curve such that (i) the origin is contained in the interior of the convex hull of T, $(0,0,0) \in \operatorname{int}\operatorname{conv} T$, and (ii) T does not contain any pair of antipodal points, $T \cap -T = \emptyset$. Figure 2 shows an example. Let $T(s)$, $s \in \mathbf{R}$, denote a periodic parameterization of T by arclength. So, assuming T has total length L, we have $T(s+L) = T(s)$. Since $(0,0,0) \in \operatorname{int}\operatorname{conv} T$, there exists a function $\rho(s)$ with period L such that $\int_0^L \rho(s)T(s)\,ds = 0$ [10, p. 168]. Set

$$\gamma(t) := \int_0^t \rho(s)T(s)\,ds.$$

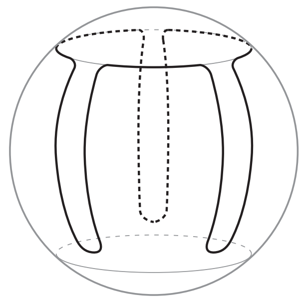

FIGURE 2

Then $\gamma(t+L) = \gamma(t)$. Further, $\gamma'(t)/\|\gamma'(t)\| = T(t)$. Thus γ is a closed curve whose tangential spherical image coincides with T. Hence γ is a skew loop.

Note 6.7. With the sole exception of ellipsoids, every closed surface immersed in $\mathbf{R}^3$ admits a skew loop [8].

7. Proof of Theorem 1.3

We follow a modified outline of the proof of Theorem 1.1, which again proceeds by contradiction. Suppose that M is homeomorphic to $\mathbf{S}^2$ and S_u is connected for all $u \in \mathbf{S}^2$. If f is not a convex embedding, let u_0 and v_0 be as in Lemma 5.1, and $v(\theta)$ be as defined by (1).

Definition 7.1. The *augmented shadow* $\widetilde{S}_{v(\theta)}$ is the union of $S_{v(\theta)}$ with all components X of $H_{v(\theta)}$ such that $U - X \subset S_{v(\theta)}$ for an open neighborhood U of X.

Then $\widetilde{S}_{v(\theta)}$ satisfies the conditions of the following lemma:

Lemma 7.2. *If $U \subset \mathbf{S}^2$ is a connected open set, and $\mathbf{S}^2 - U$ is also connected and has an interior point, then U is simply connected.*

Proof. Let p be an interior point of $\mathbf{S}^2 - U$. Then the stereographic projection maps U into a connected open set with connected complement. Thus, by [9, Thm. 11.4.1], U is simply connected. □

So $\widetilde{S}_{v(\theta)}$ is simply connected. Further:

Lemma 7.3. *For all $\theta \in \mathbf{R}$, $\widetilde{S}_{v(\theta)}$ is a domain adjacent to and regular near p_i.*

Proof. This follows just as in the proof of Lemma 5.2, once we observe that whenever $\partial S_{v(\theta)} = H_{v(\theta)} = \partial S_{-v(\theta)}$ is regular in some open neighborhood, then $\partial \widetilde{S}_{v(\theta)}$, and $\partial S_{v(\theta)}$ coincide within that neighborhood. □

Thus each θ induces a permutation $\widetilde{\alpha}_\theta := \alpha_{(\widetilde{S}_{v(\theta)})}$ of $\{p_1, p_2, p_3\}$ which satisfies the enumerated properties in Proposition 4.4. In particular $\widetilde{\alpha}_0 \neq \widetilde{\alpha}_\pi$, because since $\partial S_{v(0)} = \partial S_{-v(0)}$ is by Lemma 5.1 a regular curve, it follows that $\partial \widetilde{S}_{v(0)} = \partial \widetilde{S}_{-v(0)}$ is a regular curve as well. So it remains to verify the following lemma which shows that $\theta \mapsto \widetilde{\alpha}_\theta$ is locally constant. This would yield that $\widetilde{\alpha}_0 = \widetilde{\alpha}_\pi$ which is the desired contradiction.

Lemma 7.4. *For each $\theta_0 \in \mathbf{R}$ there exists an $\epsilon > 0$ such that if $|\theta - \theta_0| < \epsilon$ then $\widetilde{S}_{v(\theta)}$ and $\widetilde{S}_{v(\theta_0)}$ have a common triangle (with vertices at $\{p_1, p_2, p_3\}$).*

Proof. This is an immediate consequence of Lemma 5.3 where it was proved that $S_{v(\theta)}$ and $S_{v(\theta_0)}$ have a triangle in common (the proof of Lemma 5.3 makes no use of the simply connectedness assumption on $S_{v(\theta)}$). □

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