# OPTIMAL SMOOTHING FOR CONVEX POLYTOPES

MOHAMMAD GHOMI

Abstract. It is proved that given a convex polytope $P$ in $\mathbf{R}^n$, together with a collection of compact convex subsets in the interior of each facet of $P$, there exists a smooth convex body arbitrarily close to $P$ which coincides with each facet precisely along the prescribed sets, and has positive curvature elsewhere.

## 1. Introduction

It has been known since the foundational work of H. Minkowski [9], see [1, p. 39], that the boundary of every convex polytope $P$ in Euclidean space $\mathbf{R}^n$ may be approximated, in the sense of Hausdorff distance, by an analytic convex hypersurface. There have been also some refinements of this theorem due to P. Hammer [7] and W. Firey [3] who extended it to algebraic hypersurfaces. Though these approximations are as smooth as one could wish, for certain purposes they may have a drawback: they do not coincide with $P$ along any open subset. Thus in this paper we are led to develop a smoothing procedure which preserves $P$ along prescribed regions:

**Theorem 1.1.** *Let $P \subset \mathbf{R}^n$ be a convex polytope, with interior points, and facets $F_i$, $i = 1, \dots, k$. Let $X_i$ be a compact convex subset in the interior of $F_i$. Then for every $\epsilon > 0$ there exists a convex body $K \subset P$ with smooth ($C^\infty$) boundary $\partial K$ such that*

1. $\partial K \cap F_i = X_i$,
2. $\partial K - \cup_i X_i$ *has positive curvature,*
3. $\mathrm{dist}(K, P) \leq \epsilon$.

*where* dist *denotes Hausdorff distance. Furthermore, if $\cup_i X_i$ is symmetric with respect to some rigid motion in $\mathbf{R}^n$, then there exists a convex body $K$, satisfying the above properties, which has the same symmetry.*

The above smoothing may be considered "optimal" in the sense that it preserves the boundary of $P$ precisely as much or as little as desired. In the case where each $X_i$ is a point, the above has been proved by W. Weil [12], using a certain convolution first devised by C. Berg, and further studied by R. Schneider [11, 10]. Our proof also employs this convolution together with some recent

1991 *Mathematics Subject Classification.* 53A07, 52B11, 53C45.

*Key words and phrases.* Smooth approximation, convex polytopes, support function, convolution, Gaussian curvature.

The author was partially supported by the NSF grant DMS-0204190, and CAREER award DMS-0332333.

results on strictly convex submanifolds [4]. The above may be of interest in studying Brownian motion in convex polygons [8], constructing "subsolutions" for Monge-Ampére equations [4], smoothing of convex functions [5], and approximating general convex bodies [6]. The above theorem improves [4, Thm 1.2.4], where a similar smoothing had been constructed under the additional requirement that $X_i$ is smooth and has positively curved boundary.

By a *convex body* $K \subset \mathbf{R}^n$ we mean a compact convex set with interior points. A *polytope* $P \subset \mathbf{R}^n$ is a convex body which is the intersection of finitely many closed half-spaces. A *facet* $F_i$ of $P$ is the intersection of $P$ with a support hyperplane $H_i$ provided that $F_i$ has interior points in $H_i$. By *smooth* we always mean differentiable of class $C^\infty$. A point $p$ in the boundary $\partial K$ is a *smooth point* if an open neighborhood of $p$ in $\partial K$ admits a $C^\infty$ parametrization, e.g., it is the graph of a $C^\infty$ (convex) function over a support hyperplane of $K$ at $p$. If this function has positive definite hessian, then we say that $K$ has *positive curvature* at $p$.

**Note 1.2.** It is easy to satisfy property 1 of Theorem 1.1, if we require that $\partial K$ be only differentiable of class $C^{1,1}$. To see this let $\nu_i$ be the outward unit normal to the facet $F_i$, $\delta > 0$, and $X_i^\delta := X_i - \delta\nu_i$ be the translation of $X_i$ into $P$. Let $\overline{P} := \mathrm{conv}(\cup_i X_i^\delta)$ be the convex hull of these translations. An elementary computation shows that if

$$\delta < \inf\left\{ \frac{\langle x_j - x_i, \nu_j\rangle}{1 - \langle \nu_i, \nu_j\rangle} : x_i \in X_i,\, x_j \in X_j,\, i \neq j \right\},$$

then $\cup_i X_i^\delta \subset \partial\overline{P}$, where $\langle\cdot,\cdot\rangle$ denotes the standard inner product in $\mathbf{R}^n$. Consequently $K := \overline{P} + \delta B^n$, the outer parallel body of $\overline{P}$ at the distance $\delta$, is the desired object ($B^n$ denotes the unit ball in $\mathbf{R}^n$).

**Note 1.3.** Proving Theorem 1.1 is not difficult if we weaken condition 1 to $X_i \subset K \cap F_i$, and disregard condition 2. To see this suppose that $P$ contains the origin of $\mathbf{R}^n$ in its interior, and let $\rho\colon \mathbf{R}^n \to \mathbf{R}$, given by

$$\rho(x) := \inf\{\, \lambda > 0 : x \in \lambda P \,\}, \tag{1}$$

be the *distance function* of $P$. Then $\rho$ is a convex piecewise linear function with $\rho^{-1}([0,1]) = P$. Let $\widetilde{\rho}$ be the convolution of $\rho$ with a positive and centrally symmetric approximate identity function $\theta_\epsilon\colon \mathbf{R}^n \to \mathbf{R}$ with support inside a ball of radius $\epsilon$. Choose $\epsilon$ sufficiently small so that an $\epsilon$-neighborhood of $X_i$, in the affine hull of $F_i$, lies in $F_i$. Then $K := \widetilde{\rho}^{-1}([0,1])$ is the desired body; because, the convolution preserves convexity and fixes $\rho$ over any compact subset of an open region where $\rho$ is linear.

To prove Theorem 1.1 we require a pair of propositions which are proved in the next two sections.

## 2. Smooth Convex Functions with Prescribed Minima

We say a $C^2$ convex function $f\colon \mathbf{R}^n \to \mathbf{R}$ is *strictly convex* on a subset $U \subset \mathbf{R}^n$ if the Hessian of $f$ is positive definite on $U$. Recall that, for every $p \in U$, $\operatorname{Hess} f_p$

is the bilinear form on $\mathbf{R}^n \times \mathbf{R}^n$ given by

$$\operatorname{Hess} f_p(v, w) := \sum_{i,j=1}^{n} D_{ij} f(p) v_i w_j.$$

Note that if $f$ has positive definite hessian, then the graph of $f$ contains no line segments. Thus our definition of strict convexity is stronger than the one which is commonly used in convexity texts.

**Proposition 2.1.** *For every compact convex subset $X \subset \mathbf{R}^n$, there exists a smooth nonnegative convex function $f\colon \mathbf{R}^n \to \mathbf{R}$ such that $f^{-1}(\{0\}) = X$, and $f$ is strictly convex on $\mathbf{R}^n - X$.*

*Proof.* After a translation, we may assume that the origin $o$ of $\mathbf{R}^n$ is contained in $X$. Let $h\colon \mathbf{R}^n \to \mathbf{R}$ be the *support function* of $X$, that is

$$h(\cdot) := \sup_{x \in X} \langle x, \cdot \rangle. \tag{2}$$

Note that, for every $u$ in the sphere $\mathbf{S}^{n-1}$, $h(u)$ is the distance between $o$ and the support hyperplane

$$H_u := \{\, p \in \mathbf{R}^n : \langle p, u \rangle = h(u) \,\}.$$

Let $g\colon \mathbf{R} \to \mathbf{R}$ be any smooth function which is strictly convex on $(0, \infty)$, but vanishes on $(-\infty, 0]$. For instance, we may set:

$$g(x) := \begin{cases} x^2 \exp\left(\frac{-1}{x^2}\right), & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Define $\phi\colon \mathbf{S}^{n-1} \times \mathbf{R}^n \to \mathbf{R}$ by

$$\phi(u, p) := \begin{cases} g\Big(\langle p, u \rangle - h(u)\Big), & \text{if } \langle p, u \rangle > h(u); \\ 0, & \text{otherwise.} \end{cases}$$

Thus, for every $u \in \mathbf{S}^{n-1}$, $\phi(u, \cdot\,)$ is a smooth convex function which vanishes on $X$, but is positive in the half space $\langle p, u \rangle > h(u)$. Set

$$f(p) := \int_{\mathbf{S}^{n-1}} \phi(u, p)\, du. \tag{3}$$

Since $\phi$ is smooth, $f$ is smooth, and one easily verifies that it is convex as well, using the linearity of integrals. Further, it is clear that $f$ vanishes on $X$. On the other hand, if $p \notin X$, then there exists a support hyperplane $H_{u_0}$ which separates $p$ and $X$, because $X$ is convex. Thus, $\phi(u, p) > 0$ for all $u$ in a neighborhood of $u_0$. Since $\phi \geq 0$ everywhere, this yields that $f(p) > 0$. So $f$ vanishes precisely on $X$.

It remains to check that the Hessian of $f$ is positive definite on $\mathbf{R}^n - X$. To this end recall that

$$\operatorname{Hess} f_p(v, v) = \frac{d^2}{dt^2} f(p + tv)\Big|_{t=0}. \tag{4}$$

Next note that $t \mapsto \phi(u, p+tv)$ is convex. Thus, $d^2\phi(u, p+tv)/dt^2 \geq 0$, which yields that, for every $p$, $v \in \mathbf{R}^n$ and $U \subset \mathbf{S}^{n-1}$,

$$\frac{d^2}{dt^2} f(p+tv) = \int_{\mathbf{S}^{n-1}} \frac{d^2}{dt^2}\phi(u, p+tv)\, du \geq \int_U \frac{d^2}{dt^2}\phi(u, p+tv)\, du. \tag{5}$$

For each $p \in \mathbf{R}^n - X$ there exists a $u_p \in \mathbf{S}^{n-1}$ such that $H_{u_p}$ separates $p$ and $X$. Then $\langle p, u_p\rangle > h(u_p)$. So there exists an open neighborhood $U_p \subset \mathbf{S}^{n-1}$ and an $\epsilon_p > 0$ such that for all $(u, t) \in U_p \times (-\epsilon_p, \epsilon_p)$, and $v \in \mathbf{S}^{n-1}$, $\langle p + tv, u\rangle > h(u)$. Consequently, for these values, the definition of $\phi$ yields that

$$\phi(u, p+tv) = g\Big(\langle p+tv, u\rangle - h(u)\Big).$$

When $\langle p+tv, u\rangle - h(u) > 0$, the above is strictly convex in $t$, in which case

$$\frac{d^2}{dt^2}\phi(u, p+tv)\Big|_{t=0} > 0.$$

Thus in (5) if we set $U := U_p$, then $d^2 f(p+tv)/dt^2|_{t=0} > 0$, for all $p \in \mathbf{R}^n - X$ and $v \in \mathbf{S}^{n-1}$. So by (4) Hess $f_p$ is positive definite on $\mathbf{R}^n - X$. □

**Note 2.2.** For $\epsilon > 0$, let $X_\epsilon := f^{-1}([0, \epsilon])$, where $f$ is as in (3). This yields a family of convex bodies with smooth boundary which, as $\epsilon \to 0$, converges to $X$ in the sense of Hausdorff distance.

## 3. Completion of Strictly Convex Patches

Recall that the support function of a convex body, as defined by (2), is a convex and positively homogeneous function $h\colon \mathbf{R}^n \to \mathbf{R}$. Conversely, every such function uniquely determines a convex body

$$K = \{\, x \in \mathbf{R}^n : \langle x, p\rangle \leq h(p), \text{ for all } p \in \mathbf{R}^n \,\},$$

[11, Thm. 1.7.1]. We say $v \in \mathbf{S}^{n-1}$ is a *support vector* for $p \in \partial K$, if $K$ lies on one side of the support hyperplane $H$ which is orthogonal to $v$ and passes through $p$. Further, if $p + v$ lies in the halfspace of $H$ not containing $K$, then we say that $v$ is an *outward* support vector. When $p$ is a smooth point of $\partial K$, the (unique) support hyperplane of $K$ at $p$ is denoted by $T_p\partial K$, and is called the *tangent hyperplane* of $K$ at $p$.

**Lemma 3.1.** *Let $K \subset \mathbf{R}^n$ be a convex body with support function $h$, and $v_0 \in \mathbf{S}^{n-1}$ be an outward support vector for $p_0 \in \partial K$. Then the following are equivalent:*

1. *$p_0$ is a smooth point of $\partial K$, and $\partial K$ has positive curvature at $p_0$.*
2. *$v_0$ is a smooth point of $h$, and $h$ is strictly convex on $T_{v_0}\mathbf{S}^{n-1}$.*

Though the above is essentially known, e.g. see [11, p. 103–109], we include a concise proof for lack of an explicit reference.

*Proof.* (**1** $\Rightarrow$ **2**). Let $U \subset \partial K$ be an open neighborhood of $p_0$ which is smooth and positively curved. Then the inverse function theorem implies that the outward unit normal, or the *Gauss map*, $\nu\colon U \to \nu(U) \subset \mathbf{S}^{n-1}$, is a diffeomorphism. Consequently, setting $V := \nu(U)$, we obtain a one-to-one correspondence

$$\partial K \supset U \ni p \longleftrightarrow v \in V \subset \mathbf{S}^2.$$

In particular, using the above convention, we may write

$$h(v) = \langle p, v\rangle.$$

Thus $h|_V$ is smooth, which, since $h$ is homogeneous, yields that $h$ is smooth on (an open neighborhood of) $V$. Further, the above equation yields that the gradient of $h$ on $V$ is given by

$$\operatorname{grad} h(v) := \big(D_1 h(v), \dots, D_n h(v)\big) = p.$$

It is a basic fact in differential geometry that, since $\partial K$ has positive curvature on $U$, for every $p \in U$ there exists a basis $e_i = e_i(p)$, $1 \leq i \leq n-1$, for the tangent hyperplane $T_p\partial K$ such that

$$d\nu_p(e_i) = k_i e_i,$$

where $d$ is the *differential map*, and $k_i = k_i(p) > 0$ ($e_i$ are the "principle directions" and $k_i$ are the corresponding "principal curvatures").

Note that $T_p\partial K$ is parallel to $T_v\mathbf{S}^{n-1}$. Thus $\{e_i\}$ also forms a basis for $T_v\mathbf{S}^{n-1}$, and using the last two equations above, we have

$$\operatorname{Hess} h_v(e_i, e_j) = \big\langle D_{e_i} \operatorname{grad} h(v), e_j\big\rangle = \big\langle d\nu_v^{-1}(e_i), e_j\big\rangle = \begin{cases} \frac{1}{k_i}, & \text{if } i = j; \\ 0, & \text{otherwise.}\end{cases}$$

So we conclude that $h$ is strictly convex on $T_v\mathbf{S}^{n-1}$.

($\mathbf{2} \Rightarrow \mathbf{1}$) Let $V \subset \mathbf{S}^{n-1}$ be an open neighborhood of $v_0$ where $h$ is smooth and strictly convex on $T_v\mathbf{S}^{n-1}$ for all $v \in V$. Define $f\colon V \to \mathbf{R}^n$ by

$$f(v) := \operatorname{grad} h_v.$$

Since the restriction of $\operatorname{Hess} h_v$ to $T_v\mathbf{S}^{n-1}$ is positive definite, for every nonzero vector $x \in T_v\mathbf{S}^{n-1}$ we have

$$\big\langle df_v(x), x\big\rangle = \big\langle D_x \operatorname{grad} h(v), x\big\rangle = \operatorname{Hess} h_v(x, x) > 0. \tag{6}$$

So $df_v$ is nondegenerate which yields that $f\colon V \to f(V) \subset \partial K$ is a diffeomorphism, assuming $V$ is sufficiently small. In particular, $U := f(V)$ is a smooth open subset of $\partial K$. Now define $\nu\colon U \to \mathbf{S}^{n-1}$ by $\nu(f(v)) = v$. For all $v \in V$, and $x \in T_v\mathbf{S}^{n-1}$,

$$\langle df_v(x), v\rangle = \langle x, D_v \operatorname{grad} h(v)\rangle = 0$$

because, since $h$ is homogenous, $D_v \operatorname{grad} h(v) = 0$. So $v$ is orthogonal to $T_{f(v)}\partial K$, which yields that $\nu$ is the Gauss map of $U$. Since $\nu \circ f$ is the identity, and $df_{v_0}$ is nondegenerate, it follows that $d\nu_{p_0} = (df_{v_0})^{-1}$. So the eigenvalues of $d\nu_{p_0}$ are reciprocal of those of $df_{v_0}$, which are positive by (6). So $\partial K$ has positive curvature at $p_0$. □

Let $K \subset \mathbf{R}^n$ be a convex body with support function $h$. For $\epsilon > 0$, let $\theta_\epsilon \colon [0,\infty) \to [0,\infty)$ be a smooth function with support $\mathrm{supp}(\theta_\epsilon) \subset [\epsilon/2, \epsilon]$, $\int_{\mathbf{R}^n} \theta_\epsilon(\|x\|)dx = 1$, and set

$$\widetilde{h^\epsilon}(p) := \int_{\mathbf{R}^n} h(p + \|p\|x)\, \theta_\epsilon(\|x\|)dx, \tag{7}$$

where $\|\cdot\| := \langle \cdot, \cdot \rangle^{\frac{1}{2}}$ denotes the standard norm in $\mathbf{R}^n$. It is not difficult to show that $\widetilde{h^\epsilon}$ is convex and positively homogeneous; thus it determines a convex body $\widetilde{K^\epsilon}$ which we call the *Schneider transform* of $K$ [11, p. 158]. We say that the radii of curvature of $K$ are *bounded below* if there exists an $r > 0$ such that through every point $p \in \partial K$ there passes a ball $B$ of radius $r$ contained inside $K$ (one may also say that $B$ "rolls freely" inside $K$).

The following lemma is also known, but again a proof is included because the author is not aware of an explicit reference.

**Lemma 3.2.** *Let $K \subset \mathbf{R}^n$ be a convex body whose radii of curvature are bounded below. Then the Schneider transform of $K$ is smooth, and has positive curvature.*

*Proof.* Suppose that the radii of curvature of $K$ are bounded below by $r$. Set

$$L := \{\, p \in K : B^n(p, r) \subset K \,\},$$

where $B^n(p, r)$ denotes the ball of radius $r$ centered at $p$. Then $L$ is a convex body, and $K = L + B^n(o, r)$, where $+$ denotes Minkowski addition. So, $h_K = h_L + h_{B^n(o,r)}$, which in turn yields

$$\widetilde{h^\epsilon}_K(u) = \widetilde{h^\epsilon}_L(u) + \widetilde{h^\epsilon}_{B^n(o,r)}(u) = \widetilde{h^\epsilon}_L(u) + r\|u\|.$$

Note that the restriction of $\|\cdot\|$ to $T_p\mathbf{S}^{n-1}$ is strictly convex, for all $p \in \mathbf{S}^{n-1}$. Thus $\widetilde{h^\epsilon}_K$ is strictly convex on the tangent hyperplanes of the sphere, which, by Lemma 3.1, yields that $\widetilde{K}$ is smooth and has positive curvature. □

We say a smooth hypersurface $M \subset \mathbf{R}^n$ is *strictly convex* if, for all $p \in M$, (i) $M$ lies on one side the tangent hyperplane $T_pM$, (ii) $M \cap T_pM = \{p\}$, and (iii) $M$ has positive curvature at $p$. Unless stated otherwise, our hypersurfaces may be disconnected and may have boundary.

**Proposition 3.3.** *Let $\widetilde{M} \subset \mathbf{R}^n$ be a smooth strictly convex hypersurface without boundary, and $M \subset \widetilde{M}$ be compact. Then $M$ lies on the boundary of a smooth convex body with positive curvature.*

The above is a special case of the main result of [4]. Since the special case may be treated much more concisely, however, we include a proof:

*Proof.* Let $U \subset \widetilde{M}$ be an open subset with compact closure $\overline{U}$, and $U \supset M$. Let $\nu \colon \widetilde{M} \to \mathbf{S}^{n-1}$ be the Gauss map, and, for small $r > 0$, define the inner parallel hypersurface of $\overline{U}$ by

$$\overline{U}_r := \{\, p_r := p - r\nu(p) \;:\; p \in \overline{U} \,\}.$$

Since the curvature of $\overline{U}_r$ depends continuously on $r$, and $\overline{U}$ is compact, $\overline{U}_r$ has positive curvature (for $r$ sufficiently small). Thus $\overline{U}_r$ lies locally on one side of each of its tangent hyperplanes. Equivalently, if we define $f_r\colon \overline{U}\times\overline{U}\to\mathbf{R}$ as

$$f_r(p,q) := \big\langle p_r - q_r, \nu(q_r)\big\rangle,$$

the signed distance between $p_r$ and $T_{q_r}\overline{U}$, then $f_r \le 0$ on an open neighborhood $A$ of the diagonal of $\overline{U}\times\overline{U}$. Since by assumption $\overline{U}$ is strictly convex, $f_0 < 0$ on $B := \overline{U}\times\overline{U} - A$. So, since $B$ is compact, it follows that $f_r < 0$ on $B$ as well. Consequently $\overline{U}_r$ lies globally on each side of its tangent hyperplanes, or, equivalently, $\overline{U}_r \subset \partial\,\mathrm{conv}(\overline{U}_r)$. Thus setting

$$K := \mathrm{conv}(\overline{U}_r) + B^n(o,r),$$

we obtain a convex body with $\overline{U}\subset\partial K$.

Let $V\subset U$ be an open set with $M\subset V$ and $\overline{V}\subset U$. Set $U' := \nu(U)$, and $V' := \nu(V)$. Then $U'$ and $V'$ are open in $\mathbf{S}^{n-1}$, because, since the curvature of $U$ is nonzero, $\nu$ is a local diffeomorphism. Let $\overline{\phi}\colon \mathbf{S}^{n-1}\to\mathbf{R}$ be a smooth function with support $\mathrm{supp}(\overline{\phi})\subset U'$, and $\overline{\phi}|_{\overline{V'}}\equiv 1$. Let $\phi$ be the extension of $\overline{\phi}$ to $\mathbf{R}^n$ given by $\phi(o) := 0$, and $\phi(p) := \overline{\phi}(p/\|p\|)$, when $p\neq o$. Define $\overline{h}\colon \mathbf{R}^n\to\mathbf{R}$ by

$$\overline{h}^{\epsilon}(p) := \widetilde{h}^{\epsilon}(p) + \phi(p)\big(h(p) - \widetilde{h}^{\epsilon}(p)\big),$$

where $h$ is the support function of $K$ and $\widetilde{h}^{\epsilon}$ is as in (7). We claim that there exists an $\epsilon > 0$, giving an $\overline{h}^{\epsilon}$ such that

$$\overline{K}^{\epsilon} := \{\, x\in\mathbf{R}^n : \langle x,p\rangle \le \overline{h}^{\epsilon}(p),\ \text{for all}\ p\in\mathbf{R}^n \,\}$$

is the desired body.

To establish the above claim, with an eye towards applying Lemmas 3.1 and 3.2, we first show that $\overline{K}^{\epsilon}$ is a convex body with support function $\overline{h}^{\epsilon}$. To this end, it suffices to check that $\overline{h}^{\epsilon}$ is positively homogeneous and convex. Homogeneity of $\overline{h}^{\epsilon}$ is immediate from the definition. Thus to see convexity, it suffices to show that $\mathrm{Hess}\,\overline{h}^{\epsilon}_p$ is nonnegative semidefinite for all $p\in\mathbf{S}^{n-1}$. Since $\overline{h}^{\epsilon}|_{\mathbf{S}^{n}-U'} = \widetilde{h}^{\epsilon}$, and $\widetilde{h}^{\epsilon}$ is convex, we need to check this only for $p\in U'$. To this end, note that, for each $p\in\overline{U}'$, $h|_{T_p\mathbf{S}^{n-1}}$ is strictly convex. Further, by construction,

$$\|h - \overline{h}^{\epsilon}\|_{C^2(\overline{U}')} \to 0,$$

as $\epsilon\to 0$. So, for every $p\in\overline{U}'$, there exists an $\epsilon(p) > 0$ such that $\overline{h}^{\epsilon}|_{T_p\mathbf{S}^{n-1}}$ is strictly convex. Since $\overline{U}'$ is compact and $\epsilon(p)$ depends on the size of the eigenvalues of the Hessian matrix of $\overline{h}^{\epsilon}|_{T_p\mathbf{S}^{n-1}}$, which in turn depend continuously on $p$, it follows that there is an $\epsilon > 0$ such that $\overline{h}^{\epsilon}|_{T_p\mathbf{S}^{n-1}}$ is strictly convex for all $p\in\overline{U}'$. Next we show that $\partial K$ is smooth and positively curved. To this end, by Lemma 3.1, we need to check that $\overline{h}^{\epsilon}|_{T_p\mathbf{S}^{n-1}}$ is strictly convex for all $p\in\mathbf{S}^{n-1}$. For $p\in U'$, this was verified above. For $p\in\mathbf{S}^{n-1}-U'$, note that $\overline{h}^{\epsilon} = \widetilde{h}^{\epsilon}$ on the cone spanned by $\mathbf{S}^{n-1}-U'$. So it is enough to check that $\widetilde{h}^{\epsilon}|_{T_p\mathbf{S}^{n-1}}$ is strictly convex. By Lemmas 3.2 and 3.1, this follows from the boundedness of the radii of curvature from below.

Finally, it remains to show that $M \subset \partial \overline{K}^{\epsilon}$. Since $M \subset U$, which is smooth in $\partial K$, we have $h(p) = \langle \nu^{-1}(p), p \rangle$, for all $p \in U'$. Consequently $\operatorname{grad} h(p) = \nu^{-1}(p)$. Thus

$$\nu^{-1}(p) = \operatorname{grad} h(p) = \operatorname{grad} \overline{h}^{\epsilon}(p) = \overline{\nu}^{-1}(p),$$

where $\overline{\nu}$ is the Gauss map of $\overline{K}^{\epsilon}$. So $M \subset \overline{\nu}^{-1}(U') \subset \partial \overline{K}^{\epsilon}$. □

## 4. Proof of Theorem 1.1

By Proposition 2.1, for every facet $F_i$ of $P$ there exists a smooth convex function $f_i \colon F_i \to \mathbf{R}$ with $f_i^{-1}(\{0\}) = X_i$. Let $\nu_i$ be the outward unit normal of $P$ at $F_i$ and set

$$\mathrm{Plate}_i := \big\{\, p - f_i(p)\, \nu_i \;\; : p \in U_\delta(X_i) \,\big\},$$

where $U_\delta(X_i)$ is a $\delta$-neighborhood of $X_i$ in the affine hull $\operatorname{aff}(F_i)$, i.e., the hyperplane in $\mathbf{R}^n$ which contains $F_i$.. Set

$$\mathrm{Plates} := \cup_i \, \mathrm{Plate}_i \,.$$

Since by assumption $X_j$ lies in the relative interior of $F_j$, we may choose $\delta > 0$ small enough so that

$$\operatorname{aff}(F_i) \cap \mathrm{Plate}_j = \emptyset, \tag{8}$$

for all $i \neq j$. Now define $d_i \colon \mathrm{Plate}_i \to \mathbf{R}$ by

$$d_i(p) := \inf \big\{\, |\langle x - p, \nu(p) \rangle| : x \in (\mathrm{Plates} - \mathrm{Plate}_i) \,\big\},$$

where $\nu \colon \mathrm{Plates} \to \mathbf{S}^{n-1}$ is the outward unit normal. Note that $d_i(p)$ is the distance between $T_p \,\mathrm{Plate}_i$ and $\mathrm{Plates} - \mathrm{Plate}_i$. Further, if $p \in X_i$, then $T_p \,\mathrm{Plate}_i = \operatorname{aff}(F_i)$. Thus (8) implies $d_i > 0$ on $X_i$. So, since $d_i$ is continuous and $X_i$ is compact, there exists $\delta_i > 0$ such that $d_i > 0$ on $U_{\delta_i}(X_i)$. Set $\delta := \min_i \delta_i$. Then Plates lies on one side of each of its tangent hyperplanes; or, equivalently, it lies on the boundary of its own convex hull:

$$\mathrm{Plates} \subset \partial(\operatorname{conv} \mathrm{Plates}), \tag{9}$$

where we also use the fact that each $\mathrm{Plate}_i$ is a convex hypersurface. Next define

$$\mathrm{Rim}_i := \big\{\, p - f_i(p)\, \nu_i \;\; : p \in U_\delta(X_i) - U_{\delta/2}(X_i) \,\big\},$$

and set

$$\mathrm{Rims} := \cup_i \, \mathrm{Rim}_i \,.$$

Since $f_i$ has positive definite Hessian on $F_i - X_i$, it follows from (9) that Rims is a strictly convex hypersurface. Thus, by Proposition 3.3, Rims lies on the boundary of a smooth convex body $L \subset \mathbf{R}^n$ with positive curvature.

Let $\Gamma_i^1$ and $\Gamma_i^2$ be the boundary components of $\mathrm{Rim}_i$, i.e., the graphs over $\partial(U_\delta(X_i))$ and $\partial(U_{\delta/2}(X_i))$ respectively. Note that since $U_\delta(X_i)$ is a convex body in $F_i$, $\Gamma_i^1$ is homeomorphic to $\mathbf{S}^{n-2}$. Thus, since $\partial L$ is homeomorphic to $\mathbf{S}^{n-1}$, it follows from the Jordan-Brouwer separation theorem that $\partial L - \Gamma_i^1$ has precisely two (connected) components. Let $C_i$ be the component of $\partial L - \Gamma_i^1$ which contains $\Gamma_i^2$. Set

$$C := \partial L - \cup_i C_i.$$

Since each $C_i$ is topologically a disk, and $C_i \cap C_j = \emptyset$, whenever $i \neq j$, it follows that $C$ is connected. Further note that by construction $\partial X = \partial\,\text{Plates}$, and the interior of $X$ is disjoint from Rims. Thus $\text{Plates} \cup C$ is a smooth closed hypersurface with nonnegative curvature. It follows then from a theorem of Chern and Lashof [2, Thm. 4] that $\text{Plates} \cup C$ bounds a convex body $K$. Further, by construction, $K \cap F_i = X_i$, and $\partial K - \cup_i X_i$ has positive curvature.

To push $K$ within an $\epsilon$ distance of $P$, choose in the interior of each $F_i$ a compact convex subset $Y_i$ such that $X_i \subset Y_i$. By the above construction, there exists then a smooth convex body $\overline{K}$ with $Y_i \subset \partial\overline{K}$. Choosing $Y_i$ sufficiently large, we may assume that $\text{dist}(\overline{K}, P) \leq \epsilon/2$. Suppose that $o \in \text{int}\,\overline{K}$ and let $\overline{\rho}$, $\rho$ be the distance functions of $\overline{K}$ and $K$ respectively, as defined by (1). For $\lambda \in [0,1)$, set

$$\rho_\lambda := \lambda\,\overline{\rho} + (1-\lambda)\,\rho.$$

Then $K_\lambda := \rho_\lambda^{-1}([0,1])$ is a smooth convex body, because $\overline{\rho}$ and $\rho$ are both smooth convex functions. Further note that since $\rho, \overline{\rho} \geq 1$ on $F_i$, it follows that $\rho_\lambda(x) = 1$ at $x \in F_i$, if and only if $\rho(x) = 1 = \overline{\rho}(x)$. Consequently

$$\partial K_\lambda \cap F_i = \big(\partial\overline{K} \cap F_i\big) \cap \big(\partial K \cap F_i\big) = Y_i \cap X_i = X_i.$$

Next we check that $\partial K_\lambda$ has positive curvature in the complement of $X := \cup_i X_i$. Let $\nu$ be the Gauss map of $K_\lambda$. Since $\partial K_\lambda$ is a level set of $\rho_\lambda$, for every $e_i$, $e_j \in T_p\partial K_\lambda$ we have

$$\langle d\nu_p(e_i), e_j\rangle = \left\langle D_{e_i}\frac{\text{grad}(\rho_\lambda)_p}{\|\,\text{grad}(\rho_\lambda)_p\|}, e_j\right\rangle = \frac{1}{\|\,\text{grad}(\rho_\lambda)_p\|}\,\text{Hess}(\rho_\lambda)_p(e_i, e_j).$$

Thus $\partial K_\lambda$ is positively curved at $p$, if and only if $\rho_\lambda$ is strictly convex on $T_p\partial K_\lambda$. Since $\rho_\lambda$ is homogeneous, this is equivalent to $\rho_\lambda$ being strictly convex on $T_{\nu(p)}\mathbf{S}^{n-1}$. If $p \notin X$, then the point on $K$ with outward normal $\nu(p)$ is also disjoint from $X$, and thus has positive curvature by construction. Consequently, $\rho$ is strictly convex on $T_{\nu(p)}\mathbf{S}^{n-1}$, which yields that $\rho_\lambda$ is also strictly convex. So $\partial K_\lambda$ has positive curvature on the complement of $X$. Now note that $\rho_\lambda \to \overline{\rho}$ as $\lambda \to 1$. Thus there exists a $\lambda_0 < 1$ such that $\text{dist}(K_{\lambda_0}, \overline{K}) \leq \epsilon/2$. The triangle inequality yields

$$\text{dist}(K_{\lambda_0}, P) \leq \text{dist}(K_{\lambda_0}, \overline{K}) + \text{dist}(\overline{K}, P) \leq \epsilon.$$

Finally, suppose that $X$ is symmetric with respect to some rigid motion $m \in O(n)$, i.e., $m(X) = X$. To make sure that $K_{\lambda_0}$ inherits the same symmetry, we may repeat the above procedure after replacing $\rho$ and $\overline{\rho}$ by

$$\frac{1}{2}(\rho + \rho \circ m), \qquad \text{and} \qquad \frac{1}{2}(\overline{\rho} + \overline{\rho} \circ m),$$

respectively.

## Acknowledgments

The author thanks Ralph Howard for helpful comments, specially with regard to the proof of Proposition 2.1. Further, he is grateful to the editors and the referee for a detailed reading of this work, and suggestions for an improved exposition.

## References

[1] T. Bonnesen, and W. Fenchel, *Theory of convex bodies*, BCS Associates, Moscow, Idaho, 1987.
[2] S. S. Chern, and R. K. Lashof, *On the total curvature of immersed manifolds*, Amer. J. Math. **79** (1957), 306–318.
[3] W. Firey, Approximating convex bodies by algebraic ones. Arch. Math. (Basel) **25** (1974), 424–425.
[4] M. Ghomi, *Strictly convex submanifolds and hypersurfaces of positive curvature*, J. Differential Geom, **57** (2001) 239–271.
[5] ——, *The problem of optimal smoothing for convex functions*, Proc. Amer. Math. Soc., **130** (2002) 2255–2259.
[6] P. Gruber, *Aspects of approximation of convex bodies*, Handbook of convex geometry, Vol. A, 319–345, North-Holland, Amsterdam, 1993.
[7] P. Hammer, *Approximation of convex surfaces by algebraic surfaces*, Mathematika **10** (1963) 64–71.
[8] L. Helms, *Brownian motion in a closed convex polygon with normal reflection*, Ann. Acad. Sci. Fenn. Ser. A I Math. 17 (1992), no. 2, 199–209.
[9] H. Minkowski, *Volumen und Oberfläche.* Math. Ann., **57**(1903), 447–495.
[10] R. Schneider, *Smooth approximation of convex bodies*, Rend. Circ. Mat. Palermo (2) **33** (1984), no. 3, 436–440.
[11] ——, *Convex Bodies: The Brunn-Minkowski Theory*, Encyclopedia of mathematics and its applications, v. 44, Cambridge University Press, Cambridge, UK, 1993.
[12] W. Weil, *Einschachtelung konvexer Körper*, Arch. Math., **26**(1975), 666–9.

School of Mathematics, Georgia Institute of Technologies, Atlanta, GA 30332

*E-mail address*: ghomi@math.gatech.edu

*URL*: www.math.gatech.edu/∼ghomi