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STRICTLY CONVEX SUBMANIFOLDS AND HYPERSURFACES OF POSITIVE CURVATURE

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Abstract. We construct smooth closed hypersurfaces of positive curvature with prescribed submanifolds and tangent planes. Further, we develop some applications to boundary value problems via Monge-Ampére equations, smoothing of convex polytopes, and an extension of Hadamard's ovaloid theorem to hypersurfaces with boundary.

1. Introduction

1.1. **The main theorem.** We say a C^2 submanifold $M \subset \mathbf{R}^m$ is *strictly convex* if through every point of M there passes a *nonsingular support hyperplane*, i.e., a hyperplane with contact of order one with respect to which M lies strictly on one side. For instance, if M lies on a convex surface with positive curvature, which we call an *ovaloid*, then M is strictly convex. In this paper we prove the converse, which yields the following characterization:

Theorem 1.1.1. *Let $M \subset \mathbf{R}^m$ be a smooth (C^∞) compact embedded submanifold, possibly with boundary; then, M lies in a smooth ovaloid if, and only if, M is strictly convex. Furthermore, if M is strictly convex, then*

1. *Any finite number of nonsingular support hyperplanes at distinct points of M may be extended to a smooth distribution of nonsingular support hyperplanes along M.*
2. *For every smooth distribution of nonsingular support hyperplanes along M there exists a smooth ovaloid which contains M and is tangent to the given distribution.*
3. *This ovaloid may be constructed within an arbitrary small distance of the convex hull of M.*
4. *If M is symmetric with respect to some rotation or reflection in $\mathbf{R}^m$, then there exists a smooth ovaloid, containing M, which has the same symmetry.*

Finally, if M is strictly convex, but is only of class C^k, for some $k \geqslant 2$, then there exists an ovaloid, containing M, which is also C^k.

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1.2. **Applications.** In his list of open problems, S.-T. Yau asks for conditions for a Jordan curve $\Gamma \subset \mathbf{R}^3$ to bound a disk with a given metric of positive curvature [32, Prob. 26]. H. Rosenberg has shown that—in addition to the elementary requirement that Γ have no inflection points—a necessary condition is that the self linking number of Γ be zero [25]; however, these conditions are not sufficient, as was shown by H. Gluck and L. Pan [8]. Our main theorem gives a sufficient condition:

Theorem 1.2.1. *Let $\Gamma \subset \mathbf{R}^3$ be a smooth simple closed curve without inflection points. Suppose that through every point p of Γ there passes a plane H such that $H \cap \Gamma = \{p\}$. Then Γ bounds a smooth convex embedded surface of positive curvature.*

Proof. After a small rotation of H about the tangent line of Γ at p, we may assume that the principal normal of Γ at p does not lie in H. Then the order of contact between H and Γ is one. Thus Γ is strictly convex. So, by the main theorem, Γ lies on a smooth ovaloid $O \subset \mathbf{R}^3$. Consequently, by Jordan's curve theorem, Γ bounds a surface of positive curvature. □

The conditions of the above theorem are quite delicate, as there exists a simple closed curve without inflection points which lies on the boundary of a convex body, but bounds no surfaces of positive curvature, see Figure 1 and Appendix A.

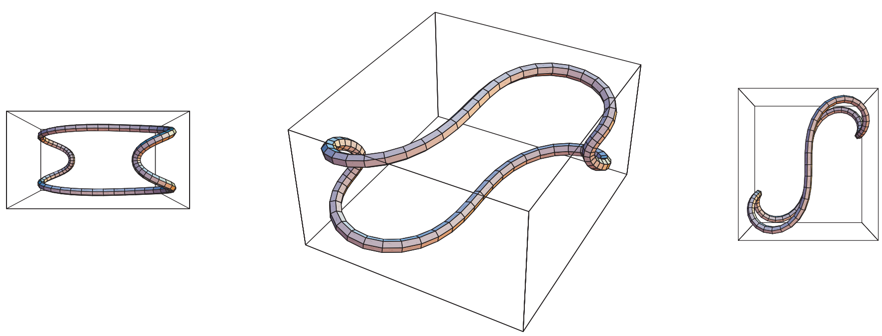

FIGURE 1

From the point of view of PDE's, Theorem 1.1.1 gives a "subsolution" for a Dirichlet problem involving Monge-Ampére equations on a spherical domain. These equations have been studied by B. Guan and J. Spruck [11], and, based on their work, we can show:

Theorem 1.2.2. *Let $\Gamma \subset \mathbf{R}^m$ be a smooth strictly convex closed submanifold of codimension 2. Then there exists an $\epsilon > 0$ such that for every $0 < K \leqslant \epsilon$, Γ bounds a smooth convex hypersurface of constant curvature K.*

Proof. By the main theorem, Γ lies on an ovaloid O. Thus, by Jordan-Brouwer's separation theorem, there exists a connected region $M \subset O$ such that $\Gamma = \partial M$. So Γ bounds a strictly convex hypersurface. By [11, Cor. 0.1], every strictly convex hypersurface may be deformed to one with constant positive curvature without perturbing the boundary. □

Note that a collection of closed curves without inflection points in $\mathbf{R}^3$, each of which is embedded in the interior of a different face of a convex polytope, gives an example of a strictly convex submanifold of codimension 2. Thus every such configuration bounds a surface of constant positive curvature by the above theorem. This generalizes [18, Thm. 1.1] and [11, Cor. 0.3].

The next application is concerned with an optimal regularity result which we prove using the theory of *degenerate* Monge-Ampére equations, as developed in [5] and [13].

Theorem 1.2.3. *Let $\Gamma \subset \mathbf{R}^m$ be a $C^{3,1}$ strictly convex closed submanifold of codimension 2, and C be the boundary of the convex hull of Γ. Then the closure of each component of $C - \Gamma$ is $C^{1,1}$.*

Proof. By the main theorem, there exists a C^2 ovaloid O containing Γ. Let M be the closure of a component of $C - \Gamma$, and choose the origin o of $\mathbf{R}^m$ so that

$$o \in \operatorname{conv}(O) - \operatorname{conv}(M),$$

where "conv" denotes the convex hull. Then M projects radially and injectively into a domain $\overline{U}$ contained in an open hemisphere of $\mathbf{S}^{m-1}$. Choose the direction of the m^{th} axis in $\mathbf{R}^m$ so that $\overline{U}$ is contained in the upper hemisphere $(\mathbf{S}^{m-1})^+$. Note that ∂U is a $C^{3,1}$ embedded submanifold of $(\mathbf{S}^{m-1})^+$, and there exists a function $\phi \in C^{3,1}(\partial U)$ such that $\partial M = \{\, \phi(x)x : x \in \partial U \,\}$. For every $\rho \in C^2(\overline{U})$, let $S_\rho := \{\, \rho(x)x : x \in \overline{U} \,\}$. We say ρ is strictly convex, if S_ρ is strictly convex. Set

$$A := \{\, \rho \in C^2(\overline{U}) : \rho \text{ is strictly convex, and } \rho\big|_{\partial U} = \phi \,\},$$

and let $\underline{\rho} = \inf A$. Then, by the definition of convex hull, it follows that $S_{\underline{\rho}} = M$. Thus we need to show that $\underline{\rho} \in C^{1,1}(\overline{U})$. Let $\pi\colon (\mathbf{S}^{m-1})^+ \to T_{n.p.}\mathbf{S}^{m-1}$ be the stereographic projection, where $n.p. := (0, 0, \dots, 1)$. Let $\overline{\Omega} := \pi(\overline{U})$ and set

$$\tilde{u}_\rho(x) := \frac{\sqrt{1 + \|x\|^2}}{\rho(\pi^{-1}(x))}.$$

We claim that $\tilde{u}_{\underline{\rho}} \in C^{1,1}(\overline{\Omega})$, which would complete the proof.

To establish the claim, note that the mapping $\rho \mapsto \tilde{u}_\rho$ is monotone, i.e., $\rho_1 \leqslant \rho_2$ if and only if $\tilde{u}_{\rho_1} \geqslant \tilde{u}_{\rho_2}$. Further, as is well known (e.g., see [24, p. 827] or [11, p. 617]), this mapping preserves the positiveness of curvature of the corresponding graphs. It follows that $\tilde{u}_{\underline{\rho}}$ is the supremum of all strictly convex functions on Ω which satisfy the same boundary values. So $\tilde{u}_{\underline{\rho}}$ will be locally convex, but not strictly locally convex; in particular, $\tilde{u}_{\underline{\rho}}$ is a (weak) solution to the degenerate Monge-Ampére equation

$$\det(\operatorname{Hess} \tilde{u}_{\underline{\rho}}) = 0$$

on Ω [5, p. 20]. By our main theorem, the above equation has a C^2 "subsolution", e.g., $\det(\operatorname{Hess} \tilde{u}_{\rho_O}) > 0$, where ρ_O is the radial function corresponding to O. Hence, by [13, Thm. 1.2], $\tilde{u}_{\underline{\rho}} \in C^{1,1}(\overline{\Omega})$, . $\square$

The above generalizes an earlier result of L. Caffarelli, L. Nirenberg, and J. Spruck [5], and also of N. V. Krylov [20], who had studied degenerate Monge-Ampére equations over convex planar domains. The optimality of the above theorem follows from a pair of examples in [5], one of which is due to J. Urbas.

The next application is concerned with smoothing convex polytopes. It has been known since H. Minkowski [22], see [2, p. 39], that the boundary of every convex polytope may be approximated by a smooth convex hypersurface. We show that this smoothing may be achieved in an optimal way (for a somewhat more general result see [10]):

Theorem 1.2.4. *Let $P \subset \mathbf{R}^m$ be a convex polytope with facets F_i, $i = 1, \dots, k$. Let $X_i \subset \operatorname{int} F_i$ be a connected convex subset with smooth and positively curved boundary. Then, there exists a convex body $K \subset P$ with smooth boundary such that $K \cap F_i = X_i$.*

Proof. Suppose $F_i \subset \mathbf{R}^{m-1} \times \{0\}$ and the positive direction of the m^{th} axis points into the interior of P. Define $f_i \colon F_i \to \mathbf{R}$ by

$$f_i(x) := \begin{cases} 0, & \text{if } x \in X_i; \\ \exp\left(\frac{-1}{\operatorname{dist}^2(x, X_i)}\right), & \text{otherwise.} \end{cases}$$

Let $Plate_i$ be the graph of f_i over a δ-neighborhood of X_i, $U_\delta(X_i)$. Then $Plate_i$ is a smooth convex surface with $Plate_i \cap F_i = X_i$. Let C_i be the graph of f_i over $U_\delta(X_i) - U_{\delta/2}(X_i)$, and set $C := \cup_{i=1}^k C_i$. Let H_i be the hyperplane spanned by F_i. Clearly, if δ is small, $H_i \cap C_j = \emptyset$, for all $i \neq j$. So, by continuity of tangent planes, there exists a $\delta > 0$ such that $\cup_{i=1}^k Plate_i$ is supported at each point by a tangent plane. Since by construction C has positive curvature, it follows that, for sufficiently small δ, C is strictly convex. So, by the main theorem, C may be extended to a smooth ovaloid O. C_i divides O into a pair of regions, say O_i^+ and O_i^-. Let O_i^+ be the region neighboring the boundary of $Plate_i$. Set

$$O' := \left(\bigcup_{i=1}^{k} Plate_i\right) \cup \left(\bigcap_{i=1}^{k} O_i^+\right),$$

then O' is a smooth closed hypersurface with $O' \cap F_i = X_i$. Further, since O' has everywhere nonnegative curvature, O' is convex [4]. $\square$

Finally, we mention a result which may be regarded as an extension of Hadamard's theorem [14, 17], on convexity of closed surfaces of positive curvature, to hypersurfaces with boundary. This has been generalized in various directions by several authors [29, 15, 4, 26, 19, 31, 6, 7]; however, in these generalizations, it is necessary to assume that the given submanifold is complete and without boundary.

Theorem 1.2.5. *Let $M \subset \mathbf{R}^m$, $m \geqslant 3$, be a compact connected immersed C^k hypersurface with positive curvature; then, M may be extended to a C^k ovaloid if, and only if, each boundary component of M lies strictly on one side of the tangent hyperplanes of M at that component.*

Proof. Let Γ_i be a component of ∂M. By assumption, for every $p \in \Gamma_i$, $T_pM \cap \Gamma_i = \{p\}$. Thus, since M has positive curvature, and Γ_i is compact, there exists $\delta > 0$ such that $C_i := \{x \in M : \operatorname{dist}(x, \Gamma_i) \leqslant \delta\}$ is strictly convex (see Lemma 3.1.6). Then, by the main theorem, C_i lies on an ovaloid O_i. Γ_i separates O_i into a pair of regions, say O_i^+ and O_i^-. Let O_i^+ be the region which contains C_i, and set

$$\overline{M} := M \cup \left(\bigcup_{i=1}^{k} O_i^- \right).$$

Then $\overline{M}$ is a closed hypersurface with positive curvature; therefore, by Hadamard's theorem, $\overline{M}$ is strictly convex. $\square$

In closing this subsection, we should mention a paper of W. Weil [30] who showed that given a convex polytope P, it is possible to inscribe a smooth ovaloid inside P which touches the interior of each facet at prescribed points. This is an immediate implication of our main theorem, when M is discrete.

1.3. **Outline of the proof.** Given a strictly convex compact $C^{k\geqslant 2}$ submanifold $M \subset \mathbf{R}^m$, we give a constructive proof of the existence of a C^k ovaloid O containing M in four steps.

Step 1. By extending the outward unit normal of a nonsingular support hyperplane to a small neighborhood of the point of contact, it is possible to slide each nonsingular support hyperplane locally. Using a partition of unity, we then construct a C^{k-1} *nonsingular support*, i.e., a unit normal vector field given by a C^{k-1} mapping $\sigma\colon M \to \mathbf{S}^{m-1}$ which generates nonsingular support hyperplanes along M. Further, it is possible to construct σ so that small perturbations of M along σ are C^k. When σ has this additional property, we say that σ is *proper*.

Step 2. By perturbing M inward a distance of ϵ along σ, and then building a tubular hypersurface of radius ϵ around the perturbed submanifold, we will show that there exists a C^k *strictly convex patch* P, i.e., a compact embedded hypersurface with boundary, which contains M in its interior and is tangent to every hyperplane generated by σ. We do this by using a variation of the endpoint map, based on σ, to embed a portion of the unit normal bundle of M.

Step 3. We will show that every strictly convex patch can be extended to a C^1 *ovaloid* O, i.e., a closed hypersurface whose radii of curvature are well bounded (see Section 2.2). We construct O by (i) forming the inner parallel hypersurface of P at a small distance ϵ, (ii) taking the intersection of all balls of a sufficiently large radius containing the perturbed hypersurface, and (iii) forming the outer parallel body of the intersection at the distance ϵ.

Step 4. By applying a certain convolution, due to R. Schneider, to the support function of O, and then a gluing with the aid of a fixed bump function on the sphere, we construct a sequence O_i of C^k closed hypersurfaces which contain M and converge to O. We will show that, for every i, O_i has uniformly bounded positive curvature except in a small neighborhood of M with fixed radius; however, it turns out that these small neighborhoods converge to P up to the second order; therefore, this sequence will eventually have positive curvature near M as well; thus, producing the desired ovaloid.

2. Definitions and Background

2.1. **Nonsingular support and height functions.** We say a C^2 submanifold $M \subset \mathbf{R}^m$ has *contact of order one* with a hyperplane H at $p \in M$ if for every C^2 curve $\gamma\colon (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$, and $\gamma'(0) \neq 0$ we have $\langle \gamma'(0), \sigma \rangle = 0$, but $\langle \gamma''(0), \sigma \rangle \neq 0$, where σ is a unit normal vector to H. H is a *nonsingular support* hyperplane of M if M lies on one side of H, $H \cap M = \{p\}$, and H has contact of order one with M. If through every $p \in M$ passes a nonsingular support hyperplane, we say M is *strictly convex.*

For every $p \in M$, let $\sigma_p \in \mathbf{S}^{m-1}$ be a unit vector associated with p; then, the *height function* $l_p\colon M \to \mathbf{R}$ is given by

$$l_p(\,\cdot\,) := \langle\, \cdot, \sigma_p \rangle.$$

We say that σ_p is a *nonsingular support vector* of M if p is the unique absolute maximum and a nondegenerate critical point of l_p, i.e.,

$$l_p(x) < l_p(p), \qquad (\operatorname{grad} l_p)_p = 0, \quad \text{and} \quad (\operatorname{Hess} l_p)_p(X_p, X_p) \neq 0, \tag{1}$$

for all $x \in M - \{p\}$, $p \in M$, and $X_p \in T_pM - \{0\}$.

The hyperplane through p with unit normal σ_p is given by $H_p := \{\, x \in \mathbf{R}^m : l_p(x) = l_p(p) \,\}$, and $H_p^- := \{\, x \in \mathbf{R}^m : l_p(x) \leqslant l_p(p) \,\}$ is the halfspace which does not contain $p + \sigma_p$. Thus by (1) $M \subset H_p^-$, and $M \cap H_p = \{p\}$. Further,

$$0 = \langle (\operatorname{grad} l_p)_p, X_p \rangle = X_p(l_p) = (l_p \circ \gamma)'(0) = \langle \gamma'(0), \sigma_p \rangle = \langle X_p, \sigma_p \rangle,$$

where $\gamma\colon (-\epsilon, \epsilon) \to M$ is a curve with $\gamma(0) = p$, and $\gamma'(0) = X_p$; and

$$0 \neq (\operatorname{Hess} l_p)_p(X_p, X_p) = X_p(X l_p) = \langle \nabla_{X_p} X, \sigma_p \rangle = \langle \gamma''(0), \sigma_p \rangle,$$

where ∇ is the flat connection on $\mathbf{R}^m$. So M is strictly convex, if and only if for every $p \in M$, there exists a nonsingular support vector $\sigma_p \in \mathbf{S}^{m-1}$.

Next we comment on the curvature of M. Recall that

$$(\operatorname{Hess} l_p)_p(X_p, Y_p) = \langle \nabla_{X_p} Y, \sigma_p \rangle = \langle Y_p, -\nabla_{X_p} \sigma \rangle = \langle Y_p, A_{\sigma_p} X_p \rangle,$$

where $A_{\sigma_p}\colon T_pM \to T_pM$, $A_{\sigma_p}(X_p) := -(\nabla_{X_p}\sigma)^\top$, is the *shape operator*, and σ is a differentiable extension of σ_p. The above shows that, since $(\operatorname{Hess} l_p)_p$ is negative definite, the eigenvalues of A_{σ_p}, i.e., the *principal curvatures* of M at p in the direction σ_p, are negative:

$$k_i(p, \sigma_p) < 0.$$

2.2. **Ovaloids and their support functions.** By an *ovaloid* we mean a closed convex hypersurface with *bounded radii of curvature*, i.e., through every point of O there passes a ball containing O and a ball contained in O. In other words O *rolls freely* in a ball and a ball rolls freely in O. Since O is (at least) C^1, the outward unit normal $\sigma\colon O \to \mathbf{S}^{m-1}$, a.k.a. the *Gauss map*, is well defined. If O is C^k, then

σ is a C^{k-1} diffeomorphism. Let $\pi(p) := p/\|p\|$. The *support function* $h\colon \mathbf{R}^m \to \mathbf{R}$ of O is defined by, $h(0) := 0$, and

$$h(p) := \langle \sigma^{-1} \circ \pi(p),\, p \rangle,$$

when $p \neq 0$. If $p = (x_1 \ldots, x_m)$ and $\sigma^{-1} \circ \pi(p) = (y_1, \ldots, y_m)$, then $h(p) = \sum_{i=1}^m x_i y_i$, and consequently $\partial h/\partial x_i = y_i$. Thus, for $p \neq 0$,

$$(\operatorname{grad} h)_p = \sigma^{-1} \circ \pi(p)$$

which shows that $\operatorname{grad} h$ is C^{k-1}; therefore, h is C^k on $\mathbf{R}^m - \{0\}$. Let E_p^i, $1 \leqslant i \leqslant m-1$, be the *principal directions* of O at $\sigma^{-1} \circ \pi(p)$, i.e., the eigenvectors of the shape operator $A_{\sigma(p)}$. Then

$$\sigma_*(E_p^i) = \nabla_{E_p^i} \sigma = -A_{\sigma(p)}(E_p^i) = -k_i\, E_p^i.$$

Let $E_p^m := \pi(p)$. Then $\{E_p^1, \ldots, E_p^m\}$ is an orthonormal basis for $\mathbf{R}^m$. Note that $\pi_*(E_p^i) = E_p^i$, if $i \neq m$, and $\pi_*(E_p^m) = 0$. Thus, if O is C^2,

$$\begin{aligned} (\operatorname{Hess} h)_p(E_p^i, E_p^j) &= \langle \nabla_{E_p^i}(\sigma^{-1} \circ \pi), E_p^j \rangle = \langle (\sigma_*)^{-1}(\pi_*(E_p^i)), E_p^j \rangle \\ &= \begin{cases} r_i > 0, & \text{if } 1 \leqslant i = j \leqslant m-1, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where $r_i := -1/k_i$ are the *principal radii of curvature* of O at $\sigma^{-1} \circ \pi(p)$. So we conclude that to every $C^{k \geqslant 2}$ ovaloid $O \subset \mathbf{R}^m$ there is associated a function $h\colon \mathbf{R}^m \to \mathbf{R}$, which is (i) C^k on $\mathbf{R}^m - \{0\}$, (ii) positively homogeneous, and (iii) convex; moreover, (iv) the restriction of h to every hyperplane tangent to the sphere is strictly convex (because E_p^i, $1 \leqslant i \leqslant m-1$, is a basis for $T_p\mathbf{S}^{m-1}$).

Conversely, for every $C^{k \geqslant 2}$ function $h\colon \mathbf{R}^m \to \mathbf{R}$ which satisfies these four properties, there exists a unique C^k ovaloid with support function h. To see this define $f\colon \mathbf{S}^{m-1} \to \mathbf{R}^m$ by

$$f(p) := (\operatorname{grad} h)_p.$$

We claim that $O := f(\mathbf{S}^{m-1})$ is the desired ovaloid. By our assumptions, for every $p \in \mathbf{S}^{m-1}$ there exists an orthonormal basis E_p^i, with respect to which the Hessian matrix of h is diagonal with all the main entries, except the last one, positive. Thus, for $1 \leqslant i, j \leqslant m-1$,

$$\langle f_*(E_p^i), E_p^j \rangle = \langle \nabla_{E_p^i}(\operatorname{grad} h), E_p^j \rangle = (\operatorname{Hess} h)_p(E_p^i, E_p^j)$$

vanishes if and only if $i \neq j$, and is positive otherwise. This yields that f_* is nondegenerate. So O is a closed immersed hypersurface. Let $\sigma\colon O \to \mathbf{S}^{m-1}$ be defined by $\sigma(f(p)) := p$. Then $\langle \sigma(f(p)), f_*(E_p^i) \rangle = \langle E_p^m, f_*(E_p^i) \rangle = (\operatorname{Hess} h)_p(E_p^m, E_p^i) = 0$. Thus σ is the Gauss map. So since σ is C^{k-1}, O is C^k. Further, since $\sigma_* \circ f_* = id$ and f_* is an immersion, it follows that σ is also an immersion and therefore O has nonvanishing curvature, which, since O is closed, must be positive. Then, by Hadamard's theorem, O is convex. Finally since the curvature of O is positive, O is an ovaloid by Blaschke's rolling theorem [3].

2.3. **Schneider's transform.** A convex body is a compact convex subset with interior points. We denote the space of convex bodies in $\mathbf{R}^m$ by $\mathcal{K}^m$. This space is closed under *Minkowski sum* $K+K'$, and *scalar multiplication* λK where $\lambda>0$. Further, $(\mathcal{K}^m, \mathrm{dist}_{\mathrm{H}})$ is a locally compact metric space where $\mathrm{dist}_{\mathrm{H}}$ denotes the *Hausdorff distance*:

$$\mathrm{dist}_{\mathrm{H}}(K,L) := \inf\{\, r \geqslant 0 : K \subset L + rB^m \text{ and } \; L \subset K + rB^m \,\}.$$

B^m denotes the unit ball in $\mathbf{R}^m$.

To every $K\in\mathcal{K}^m$, there is associated a *support function* $h_K\colon \mathbf{R}^m\to\mathbf{R}$ defined by $h_K(p) := \sup\{\langle p, x\rangle : x\in K\}$. Let $\theta_\epsilon\colon [0,\infty)\to[0,\infty)$ be a smooth function with $\mathrm{supp}(\theta_\epsilon)\subset[\epsilon/2,\epsilon]$, and $\int_{\mathbf{R}^m}\theta_\epsilon(\|x\|)dx = 1$. Then the *Schneider's convolution* of h [27] is given by

$$\widetilde{h^{\epsilon}_K}(p) := \int_{\mathbf{R}^m} h_K(p+\|p\|x)\theta_\epsilon(\|x\|)dx.$$

Since every positively homogeneous convex function on $\mathbf{R}^m$ is the support function of a convex body, the above defines an endomorphism $T_\epsilon : \mathcal{K}^m\to\mathcal{K}^m$ by

$$h_{T_\epsilon(K)} := \widetilde{h^{\epsilon}_K}.$$

Then $\mathrm{dist}_{\mathrm{H}}(K, T_\epsilon(K))\leqslant\epsilon$. Further $T(K)$ has smooth (C^∞) support function. This implies that the boundary of $T(K)$ is smooth provided that the radii of curvature of $T(K)$ are bounded below (see the last paragraph in Section 2.2). If ∂K is an ovaloid, then a ball rolls freely in K, and since T preserves balls and inclusion, it follows that a ball rolls freely in $T(K)$ as well. Thus we conclude that the Schneider transform of an ovaloid is a smooth ovaloid.

2.4. **Regularity of the distance function.** Let $M^n\subset\mathbf{R}^m$ be a $C^{k\geqslant 2}$ compact embedded submanifold. The *normal bundle* of M is given by

$$NM := \{\,(x,v) : x\in M, v\in T_xM^{\perp}\,\},$$

and has a canonical C^{k-1} structure. Let $\mathrm{end}\colon NM\to\mathbf{R}^m$ be the *end point map*,

$$\mathrm{end}(x,v) := x+v.$$

Set $N_\epsilon M := \{\,(x,v)\in NM : \|v\|<\epsilon\}$ and $Tube_\epsilon M := \mathrm{end}(N_\epsilon M)$, then, by the tubular neighborhood theorem, there exists an $\epsilon>0$ such that $\mathrm{end}\colon N_\epsilon M\to Tube_\epsilon M$ is a C^{k-1} diffeomorphism. For $0<r<\epsilon$ set

$$B_rM := \{\,(x,v)\in NM : \|v\| = r\,\}, \quad \text{and} \quad S_rM := \mathrm{end}(B_rM),$$

S_rM is called a *tubular hypersurface* of M. Define $d\colon\mathbf{R}^m\to\mathbf{R}$ by

$$d(p) := \mathrm{dist}_{\mathbf{R}^m}(p, M) = \inf\{\|x-p\| : x\in M\}.$$

We claim that $S_rM = d^{-1}(r)$, and d, restricted to $Tube_\epsilon M - M$ is a C^k submersion. This would show that S_rM is C^k. To prove this, define

$$x(p) := \pi_1(\mathrm{end}^{-1}(p)), \quad \text{and} \quad v(p) := \pi_2(\mathrm{end}^{-1}(p)),$$

where $NM\ni(x,v)\overset{\pi_1}{\longmapsto} x\in M$, and $NM\ni(x,v)\overset{\pi_2}{\longmapsto} v\in\mathbf{R}^m$. Clearly, x and v are C^{k-1}. Further, since $x(p)+v(p) = \mathrm{end}(x(p), v(p)) = p$, it follows that

$v(p) = p - x(p)$, i.e., $p - x(p)$ is perpendicular to M. This implies that $d(p) = \|p - x(p)\| = \|v(p)\|$. So $d|_{Tube_\epsilon M - M}$ is C^{k-1}; therefore, the gradient of d is well-defined for all $p \in Tube_\epsilon M - M$. We claim that

$$(\operatorname{grad} d)_p = \frac{v(p)}{\|v(p)\|},$$

which, since v is C^{k-1}, would yield that d is a C^k submersion.

By the *generalized Gauss' lemma*, for every $p \in S_r M$, $v(p)$ is perpendicular to $S_r M$; therefore, since $S_r M$ is a level hypersurface of d, $(\operatorname{grad} d)_p$ must be parallel to $v(p)$. Thus $(\operatorname{grad} d)_p = \langle (\operatorname{grad} d)_p, v(p)\rangle \frac{v(p)}{\|v(p)\|^2}$. Let $\gamma(t) := p + tv(p)$, then $\langle \operatorname{grad} d_p, v(p)\rangle = (d \circ \gamma)'(0)$. But $d \circ \gamma(t) = \|\gamma(t) - x(\gamma(t))\| = \|\gamma(t) - x(p)\| = \|p + t(p - x(p)) - x(p)\| = (1+t)\|p - x(p)\| = (1+t)\|v(p)\|$. Thus $(d \circ \gamma)'(0) = \|v(p)\|$ which yields the above formula.

3. Proof of the Main Theorem

The proof is divided into four propositions, corresponding to the steps outlined in Section 1.3, which are developed in the next four subsections.

Item 4 of the main theorem follows once we establish the existence of any ovaloid O containing M: Choose the origin of the coordinate system inside O. Suppose M is symmetric with respect to some orthogonal transformation $g \in \mathcal{O}(m)$, i.e., $g(M) = M$. We wish to show that there exists an ovaloid $\overline{O}$, containing M, such that $g(\overline{O}) = \overline{O}$. Let $\rho\colon \mathbf{S}^{m-1} \to \mathbf{R}$ be the function such that $O = \{\rho(x)x : x \in \mathbf{S}^{m-1}\}$. Then the radial graph of $\overline{\rho} := (\rho + \rho \circ g^{-1})/2$ is the desired ovaloid.

3.1. Construction of a section of the normal bundle.

Proposition 3.1.1. *Every compact $C^{k\geqslant 2}$ embedded strictly convex submanifold $M \subset \mathbf{R}^m$ admits a C^{k-1} proper nonsingular support.*

Recall that by nonsingular support we mean a mapping $\sigma\colon M \to \mathbf{S}^{m-1}$ such that, for all $p \in M$, $\sigma(p)$ is a nonsingular support vector (Section 2.1). Further, σ is proper if a small perturbation of M along σ yields a submanifold with the same degree of regularity as M. To prove the above we need the following lemmas:

Lemma 3.1.2. *If $\sigma_1 \dots \sigma_N$ are nonsingular support vectors of M at a fixed point, then any normalized convex combination of $\sigma_1 \dots \sigma_N$ is also a nonsingular support vector.*

Proof. Let $l_i(\cdot) := \langle \cdot, \sigma_i\rangle$, $1 \leqslant i \leqslant N$. By assumption, there exists a point $p \in M$ such that p is the unique maximum and a nondegenerate critical point of l_i. Let $\sigma := \sum_{i=1}^N c_i\sigma_i$, where $c_i \geqslant 0$ and $\sum_{i=1}^N c_i = 1$. We have to show that $\widehat{\sigma} := \sigma/\|\sigma\|$ is well-defined and p is the unique maximum and a nondegenerate critical point of $\widehat{l}(\cdot) := \langle \cdot, \widehat{\sigma}\rangle$. Let $l(\cdot) := \langle \cdot, \sigma\rangle$. Then

$$l(x) = \sum_{i=1}^N c_i l_i(x) < \sum_{i=1}^N c_i l_i(p) = l(p).$$

So we conclude that $l \not\equiv 0$, which yields that $\sigma \neq 0$. Consequently $\widehat{\sigma}$ is well-defined. Next we show that p is the strict absolute maximum of $\widehat{l}$. To see this, let $x \in M - \{p\}$, then

$$\widehat{l}(x) = \frac{1}{\|\sigma\|} l(x) < \frac{1}{\|\sigma\|} l(p) = \widehat{l}(p).$$

So it only remains to show that p is a nondegenerate critical point of $\widehat{l}$. This follows because $\widehat{l} = \frac{1}{\|\sigma\|} \sum_{i=1}^{N} c_i l_i$, $c_i > 0$, and the operators grad and Hess are linear. □

Let $BM := \{(p,v) : p \in M, v \in T_pM^{\perp}, \text{and } \|v\| = 1\}$ denote the unit normal bundle of M, and $\pi \colon BM \to M$ be given by $\pi(p,v) := p$. By a fiber of BM we mean $\pi^{-1}(p)$, $p \in M$.

Lemma 3.1.3. *Let $M \subset \mathbf{R}^m$ be a compact embedded $C^{k \geqslant 2}$ submanifold, and suppose $\sigma \colon M \to \mathbf{S}^{m-1}$ is a C^{ℓ} nonsingular support, $1 \leqslant \ell \leqslant k$, then σ is a C^{ℓ} embedding. In particular, $\overline{\sigma} \colon M \to BM$, given by $\overline{\sigma}(p) := (p, \sigma(p))$, is transverse to the fibers of BM.*

Proof. Since M is compact, it suffices to show that σ is a one-to-one immersion. By assumption, $l_p(x) < l_p(p)$ for all $p \in M$ and $x \in M - \{p\}$, where $l_p(\cdot) := \langle \cdot, \sigma(p) \rangle$. If $p \neq q$, then

$$\langle q - p, \sigma(p) - \sigma(q) \rangle = (l_p(q) - l_p(p)) + (l_q(p) - l_q(q)) < 0,$$

which implies $\sigma(p) \neq \sigma(q)$. So it remains to show that σ is an immersion. Let E_p^i be the principal directions of M at p with respect to $\sigma(p)$, and recall that

$$\langle \sigma_*(E_p^i), E_p^j \rangle = -\langle A_{\sigma(p)}(E_p^i), E_p^j \rangle = \begin{cases} -k_i(p, \sigma(p)) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

If $X_p \in T_pM$, then $X_p = \sum_{i=1}^{n} c_i E_p^i$, for some $c_i \in \mathbf{R}$. Consequently,

$$\langle \sigma_*(X_p), X_p \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \langle \sigma_*(E_p^i), E_p^j \rangle = -\sum_{i=1}^{n} c_i^2 k_i(p, \sigma(p)).$$

Thus σ_* is nondegenerate; because, $k_i < 0$ (Section 2.1). So $\overline{\sigma} \colon M \to \overline{\sigma}(M)$ is a local diffeomorphism, by the inverse function theorem. Since $\pi \circ \overline{\sigma}(p) = p$, it follows then that $\pi \colon \overline{\sigma}(M) \to M$ is a local diffeomorphism as well. In particular $\pi_{*_{(p,v)}}$ is nondegenerate for all $(p,v) \in \overline{\sigma}(M)$. So $\overline{\sigma}(M)$ may not be tangent to the fibers of BM. □

Lemma 3.1.4. *Let $M \subset \mathbf{R}^m$ be a $C^{k \geqslant 1}$ embedded submanifold, then any unit vector normal to M may be extended locally to a C^{k-1} unit normal vector field.*

Proof. Let $\xi_p \in T_pM^{\perp}$, and U be a small open neighborhood of p. Define $\sigma \colon U \to \mathbf{S}^{m-1}$ by $\sigma(q) := f(q)/\|f(q)\|$ where $f(q) := \mathrm{Proj}_{T_qM^{\perp}}(\xi_p)$, the projection of ξ_p into $T_qM^{\perp}$. If $\phi \colon U \to \mathbf{R}^n$ is a C^k local chart, then $X^i(q) := \partial(\phi^{-1})/\partial x_i|_{\phi(q)}$, $1 \leqslant i \leqslant n$, gives a C^{k-1} moving basis for T_qM, and

$$f(q) = \xi_p - \sum_{i=1}^{n} \Big(\sum_{j=1}^{n} g^{ij}(q) \langle \xi_p, X_q^j \rangle \Big) X_q^i,$$

where g^{ij} are the entries of the inverse of the matrix (g_{ij}), and $g_{ij}(p) := \langle X_p^i, X_p^j \rangle$ are the coefficients of the metric tensor. Thus f, and consequently σ, is C^{k-1}.

□

Lemma 3.1.5. *Let $M \subset \mathbf{R}^m$ be a compact C^2 embedded submanifold; then, any continuous distribution of locally supporting hyperplanes with contact of order one along M* uniformly *locally strictly support M.*

Proof. By assumption, there exists a continuous unit normal vector field $\sigma\colon M \to \mathbf{S}^{m-1}$ such that for every $p \in M$, there exists a $\delta_p > 0$ such that

$$l_p(q) < l_p(p), \quad \text{for all} \quad q \in U_{\delta_p}(p),$$

where $l_p(\cdot) := \langle \cdot, \sigma(p) \rangle$, and $U_{\delta_p}(p) \subset M$ is an open neighborhood of p with radius δ_p. We want to show that there exists a $\delta > 0$, independent of p, such that $l_p(q) < l_p(p)$ for all $q \in U_\delta(p)$. Identify a neighborhood of p in M with Euclidean n-space via normal coordinates; then, by Taylor's theorem,

$$l_p(q) - l_p(p) = \frac{1}{2}(\operatorname{Hess} l_p)_p(p-q, p-q) + o_p(\|p-q\|^2).$$

Since we are using normal coordinates, $\|p - q\| = \operatorname{dist}_M(p, q)$, so we may think of o_p as a function on M. Note that since $p \mapsto l_p$ is continuous, $p \mapsto o_p$ is continuous as well. Let $k := \sup_{p \in M}\{k_1, \dots, k_n\}$, where $k_i := k_i(p, \sigma(p))$ are the the principal curvatures, which are all negative because $(\operatorname{Hess} l_p)_p$ is negative definite (see Section 2.1). So

$$l_p(q) - l_p(p) \leqslant \frac{1}{2} k \|p-q\|^2 + o_p(\|p-q\|^2).$$

Since $p \mapsto o_p$ is continuous, and $k < 0$, there exists a $\delta > 0$, such that for all $p \in M$, $o_p(\|p-q\|^2)/\|p-q\|^2 < -k/2$, whenever $q \in U_\delta(p)$. So $l_p(q) - l_p(p) < 0$ for all $p \in M$, and $q \in U_\delta(p)$. □

Lemma 3.1.6. *Let σ_p be a nonsingular support vector of M at p, $V \subset M$ be a neighborhood of p, and $\sigma\colon V \to \mathbf{S}^{m-1}$ be a continuous extension of σ_p to a unit normal vector field. Then there exists a neighborhood U of p, such that $\sigma(q)$ is a nonsingular support vector for all $q \in U$.*

Proof. Let $l_q\colon V \to \mathbf{R}$ be given by $l_q(\,\cdot\,) := \langle\,\cdot\,, \sigma(q)\rangle$. We have to show that there exists a neighborhood U of p such that (i) $l_q(x) < l_q(q)$ for all $x \in M - \{q\}$, (ii) $(\operatorname{grad} l_q)_q = 0$, and (iii) $(\operatorname{Hess} l_q)_q \neq 0$. (ii) follows because $\sigma(q) \in T_qM^\perp$, and (iii) follows because $(\operatorname{Hess} l_p)_p$ is negative definite. To see (i), let $L\colon V \times M \to \mathbf{R}$ be defined by

$$L(q, x) := l_q(x) - l_q(q).$$

We have to show that $L|_A < 0$, where

$$A := \{\, (q,x) \in U \times M : q \neq x \,\}.$$

This is done by partitioning A into:

$$B_r := \{\, (q,x) \in A : \mathrm{dist}(q,x) < r \,\}, \quad \text{and} \quad C_r := A - B_r,$$

where 'dist' is the intrinsic distance in M. There exists an $r > 0$ such that $L|_{B_r} < 0$, by Lemma 3.1.5. So it remains to show that $L|_{C_r} < 0$. Set

$$D := \{\, (p,x) : x \in M, \text{ and } \ \mathrm{dist}(x,p) \geqslant \frac{r}{2} \,\}.$$

Since $\sigma(p)$ is a nonsingular support, $L|_D < 0$; therefore, by compactness of D, L must be negative over some open neighborhood $W_\delta(D)$ with radius $\delta < \frac{r}{2}$. Set

$$U := \{\, x \in M : \mathrm{dist}(x,p) < \delta \,\}.$$

Then we claim that $C_r \subset W_\delta(D)$, which is all we need. Let $(q_0, x_0) \in C_r$, then $\mathrm{dist}(q_0,x_0) > r$ and $\mathrm{dist}(q_0,p) < \delta$. Now $\mathrm{dist}(x_0,p) \geqslant \mathrm{dist}(q_0,x_0) - \mathrm{dist}(p,q_0) \geqslant r - \delta > \frac{r}{2}$. Thus, $(p,x_0) \in D$; and, therefore,

$$\overline{\mathrm{dist}}((q_0,x_0),D) \leqslant \overline{\mathrm{dist}}((q_0,x_0),(p,x_0)) = \mathrm{dist}(q_0,p) < \delta,$$

where $\overline{\mathrm{dist}}((p,q),(p',q')) := (\mathrm{dist}^2(p,p') + \mathrm{dist}^2(q,q'))^{\frac{1}{2}}$. So $(q_0,x_0) \in W_\delta(D)$. □

Lemma 3.1.7. *Let $\sigma\colon M \to \mathbf{S}^{m-1}$ be a continuous nonsingular support. Then there exists an $\epsilon > 0$ such that every unit normal vector field $\widetilde{\sigma}\colon M \to \mathbf{S}^{m-1}$ with $\sup_{p\in M} \|\sigma(p) - \widetilde{\sigma}(p)\| \leqslant \epsilon$ is also a nonsingular support.*

Proof. Let $l_p(\cdot) := \langle \cdot, \sigma(p)\rangle$, and $\widetilde{l}_p(\cdot) := \langle \cdot, \widetilde{\sigma}(p)\rangle$. Since $\mathrm{Hess}\, l_p$ is negative definite, there exists an $\epsilon_1(p) > 0$ such that, if $\|\sigma(p) - \widetilde{\sigma}(p)\| \leqslant \epsilon_1(p)$, then $(\mathrm{Hess}\,\widetilde{l}_p)_p$ is negative definite as well. So there exists a neighborhood U of p such that $\widetilde{l}_p(q) - \widetilde{l}_p(p) < 0$ for all $q \in U - \{p\}$. Further, since $M - U$ is compact, and $l_p(q) - l_p(p) < 0$ on $M - U$, it follows that there exists an $\epsilon_2(p) > 0$ such that, if $\|\sigma(p) - \widetilde{\sigma}(p)\| \leqslant \epsilon_2(p)$, then $\widetilde{l}_p(q) - \widetilde{l}_p(p) < 0$ on $M - U$. Set $\epsilon(p) := \min\{\epsilon_1(p), \epsilon_2(p)\}$. Then if $\|\sigma(p) - \widetilde{\sigma}(p)\| \leqslant \epsilon(p)$, it follows that $\widetilde{\sigma}(p)$ is a nonsingular support vector. Let $\overline{\epsilon}(p)$ be the supremum of all such $\epsilon(p)$. Then $\overline{\epsilon}(p) > 0$. Further, by Lemma 3.1.6, $p \mapsto \overline{\epsilon}(p)$ is lower semicontinuous. Thus $\epsilon := \inf_{p\in M} \overline{\epsilon}(p) > 0$, which is the desired estimate. □

Lemma 3.1.8. *Let $M \subset \mathbf{R}^m$ be a compact C^k embedded submanifold, then the tubular hypersurface of M at a small distance is also C^k.*

Proof. See Section 2.4. □

Proof of Proposition 3.1.1. By assumption, for every $p \in M$ there exists a nonsingular support vector $\xi_p \in \mathbf{S}^{m-1}$. Thus, by Lemmas 3.1.4 and 3.1.6, for every $p \in M$, there exists an open neighborhood U^p of p and a mapping $\sigma^p\colon U^p \to \mathbf{S}^{m-1}$ which is a C^{k-1} nonsingular support with $\sigma^p(p) = \xi_p$. Since M is compact, there exists a finite subcover $\mathcal{U} = \{U^{p_i}\}$, $1 \leqslant i \leqslant N$. Given any finite collection of points

$x_j \in M$, $1 \leqslant j \leqslant n$, we may assume that $U^{x_j} \in \mathcal{U}$, and U^{x_j} is the only element of $\mathcal{U}$ containing x_j. Let $\{\phi_i\}$ be a C^k partition of unity subordinate to $\mathcal{U}$, and set

$$\sigma(p) := \frac{\sum_{i=1}^{N} \phi_i(p)\ \sigma^{p_i}(p)}{\| \sum_{i=1}^{N} \phi_i(p)\ \sigma^{p_i}(p)\|}.$$

Then $\sigma\colon M \to \mathbf{S}^{m-1}$ is well-defined and is a C^{k-1} nonsingular support by Lemma 3.1.2. Further, by construction $\sigma(x_j) = \xi_{x_j}$.

We claim that after a perturbation, we may assume that σ is proper, which would complete the proof. To see this, let $f\colon M \to \mathbf{R}^m$ be given by $f(p) := p + \epsilon\,\sigma(p)$. By the tubular neighborhood theorem and Lemma 3.1.3, f is a C^{k-1} embedding, which yields that $M_\epsilon := f(M)$ is a C^{k-1} submanifold for small ϵ. Recall that the tubular hypersurface $S_\epsilon(M)$ is C^k (Section 2.4). So by a perturbation of M_ϵ in $S_\epsilon(M)$, keeping $f(p_i)$ fixed, we may construct a C^k submanifold $\widetilde{M}_\epsilon \subset S_\epsilon(M)$ which is C^1-close to M_ϵ [16]. Since, by Lemma 3.1.3 , M_ϵ is transversal to the fibers of $S_\epsilon(M)$, and M_ϵ and $\widetilde{M}_\epsilon$ are C^1-close, $\widetilde{M}_\epsilon$ meets the fibers of $S_\epsilon M$ transversely as well. Thus $\widetilde{M}_\epsilon$ determines a normal vector field $\widetilde{\sigma}\colon M \to \mathbf{S}^{n-1}$, which is a nonsingular support by Lemma 3.1.7, and is proper by construction. ☐

3.2. **Construction of a germ of the solution.**

Proposition 3.2.1. *Let $M \subset \mathbf{R}^m$ be a compact $C^{k\geqslant 2}$ embedded submanifold, then for every C^{k-1} nonsingular support $\sigma\colon M \to \mathbf{S}^{m-1}$ there exists a C^{k-1} strictly convex patch P containing M and tangent to all the hyperplanes generated by σ. Furthermore, if σ is proper, then we may construct P so that it is C^k.*

Recall that by a patch we mean a compact connected hypersurface with boundary which contains M in its interior.

Lemma 3.2.2. *Let $f\colon M \to N$ be an immersion, and $A \subset M$ be a compact subset. If f restricted to A is one-to-one, then f is one-to-one in an open neighborhood of A. In particular, there exists an open neighborhood U of A such that f restricted to U is an embedding.*

Proof. Let $U_i := \{p \in M : \mathrm{dist}(p, A) < \frac{1}{i}\}$, $i \in \mathbb{N}$. If $f|_{\overline{U_i}}$ is one-to-one for some i then we are done; otherwise, there exist u_i, $v_i \in U_i$ such that $u_i \neq v_i$, but $f(u_i) = f(v_i)$. Since M is compact, after a passing to a subsequence, we may assume that u_i and v_i converge respectively to a_u and a_v, which, since A is closed, belong to A. Since f is continuous $f(a_u) = f(a_v)$. So, since $f|_A$ is one-to-one, $a_u = a_v$. Thus f is not one-to-one in any neighborhood of a_u, which is a contradiction, because f is an immersion. ☐

Proof of Proposition 3.2.1. Let BM denote the unit normal bundle of M, $r > 0$, and define $f\colon BM \to \mathbf{R}^m$ by

$$f(p, v) := p + r(v - \sigma(p)),$$

where $\sigma\colon M \to \mathbf{S}^{m-1}$ is a given C^{k-1} nonsingular support. Let $\overline{\sigma}\colon M \to BM$ be given by $\overline{\sigma}(p) := (p, \sigma(p))$, and let $U \subset BM$ be an open neighborhood of $\overline{\sigma}(M)$. We claim that there exist U and r such that

$$P := f(\overline{U})$$

is the desired patch. It is clear that (i) $M \subset P$. So it remains to check: (ii) P is embedded, (iii) P is tangent to all the hyperplanes generated by σ, (iv) P has everywhere positive curvature, (v) P lies strictly on one side of all of its tangent planes, and (vi) P is at least C^{k-1} and, if σ is proper, then P is C^k.

(ii) It suffices to show that f is a one-to-one immersion in a neighborhood of $\overline{\sigma}(M)$. Since, by Lemma 3.1.3, σ is an embedding, and $f(p,\sigma(p)) = p$, it follows that $f|_{\overline{\sigma}(M)}$ is one-to-one; therefore, by Lemma 3.2.2, it suffices to show that f is an immersion on $\overline{\sigma}(M)$. Let $Z \in T_{\overline{\sigma}(p)}BM - \{0\}$, and $\gamma\colon (-\epsilon,\epsilon) \to BM$ be a curve with $\gamma(0) = \overline{\sigma}(p)$, and $\gamma'(0) = Z$. Then

$$\gamma(t) = \overline{\sigma}(q(t)) = (q(t), \sigma(q(t))) =: (q(t), v(t))$$

where $q(t)$ is a curve in M with $q(0) = p$, and $v(t) := \sigma(q(t))$ is a curve in $\mathbf{S}^{m-1}$ with $v(0) = \sigma(q(0)) = \sigma(p)$. Also note that $Z = (q'(0), v'(0)) =: (X, V)$, where $X \in T_pM$ and $V \in T_{\sigma(p)}\mathbf{S}^{m-1}$. We have

$$f_*(Z) = (f \circ \gamma)'(0) = q'(0) + r\,(v'(0) - (\sigma \circ q)'(0)) = X + r\,(V - \sigma_*(X)).$$

Since $Z \neq 0$, X and V cannot vanish simultaneously; therefore, if $X = 0$, then $f_*(Z) = rV \neq 0$. On the other hand, if $X \neq 0$ we have

$$\langle f_*(Z), X\rangle = \|X\|^2 + r(\langle V, X\rangle - \langle \sigma_*(X), X\rangle).$$

Since $v(t) = \sigma(q(t)) \in T_{q(t)}M^{\perp}$, $\langle v(t), q'(t)\rangle = 0$. So $\langle v'(t), q'(t)\rangle = -\langle v(t), q''(t)\rangle$, which yields

$$\langle V, X\rangle = -\langle v(0), q''(0)\rangle = -(\mathrm{Hess}\, l_p)_p(X,X) = -\langle A_{\sigma(p)}(X), X\rangle,$$

where $l_p(\cdot) := \langle \cdot, \sigma(p)\rangle$. Also, recall that

$$\langle \sigma_*(X), X\rangle = \langle (\nabla_X \sigma)^{\perp}, X\rangle = -\langle A_{\sigma(p)}(X), X\rangle.$$

Combining the three previous calculations, we get

$$\langle f_*(Z), X\rangle = \|X\|^2.$$

So $f_*(Z) \neq 0$, for all $Z \in T_{\overline{\sigma}(p)}BM - \{0\}$.

(iii) We show that $\sigma(p) \in (T_pP)^{\perp}$, for all $p \in M$. Let $\gamma\colon (-\epsilon,\epsilon) \to P$ be a curve with $\gamma(0) = p$ and $\gamma'(0) = W$. Since $\gamma(t) = f(q(t), v(t)) = q(t) + r(v(t) - \sigma(q(t)))$,

$$\begin{aligned} \langle \sigma(p), W\rangle &= \langle \sigma(p), q'(0)\rangle + r(\langle \sigma(p), v'(0)\rangle - \langle \sigma(p), \sigma_*(q'(0))\rangle) \\ &=: \langle \sigma(p), X\rangle + r(\langle \sigma(p), V\rangle - \langle \sigma(p), \sigma_*(X)\rangle), \end{aligned}$$

where $X \in T_pM$ and $V \in T_{\sigma(p)}\mathbf{S}^{m-1}$. Since $\sigma(p) \in T_{\sigma(p)}M^{\perp}$, $\langle \sigma(p), X\rangle = 0$. Also $\sigma(p) \in (T_{\sigma(p)}\mathbf{S}^{m-1})^{\perp}$, thus $\langle \sigma(p), V\rangle = 0$. Finally, $\sigma_*(X) \in T_{\sigma(p)}\mathbf{S}^{m-1}$, so $\langle \sigma(p), \sigma_*(X)\rangle = 0$. Hence $\langle \sigma(p), W\rangle = 0$.

(iv) Let $p \in M$ and $l_p\colon P \to \mathbf{R}$ be given by $l_p(\cdot) := \langle \cdot, \sigma(p)\rangle$. It is enough to show that $(\mathrm{Hess}\, l_p)_p$ is negative definite. For $W \in T_pP$,

$$(\mathrm{Hess}\, l_p)_p(W,W) = (l \circ \gamma)''(0),$$

where γ is a curve on P as in step (iii). Note that

$$(l \circ \gamma)''(0) = \langle \sigma(p), q''(0) \rangle + r\langle \sigma(p), v''(0) \rangle - r\langle \sigma(p), (\sigma \circ q)''(0) \rangle.$$

We need three calculations for each of the terms in the last sentence. First,

$$\langle \sigma(p), q''(0) \rangle = (\operatorname{Hess} l_p)_p(q'(0), q'(0)) = \langle A_{\sigma(p)}(X), X \rangle.$$

Secondly, since $\|v(t)\| = 1$, $\langle v(t), v'(t) \rangle = 0$, and thus

$$\langle \sigma(p), v''(0) \rangle = \langle v(0), v''(0) \rangle = -\langle v'(0), v'(0) \rangle = -\|V\|^2.$$

Thirdly, since $\|\sigma \circ q(t)\| = 1$, $\langle (\sigma \circ q)'(t), \sigma \circ q(t) \rangle = 0$, so

$$\langle \sigma(p), (\sigma \circ q)''(0) \rangle = -\langle (\sigma \circ q)'(0), (\sigma \circ q)'(0) \rangle = -\|\sigma_*(X)\|^2.$$

Combining the five preceding calculations we obtain

$$(\operatorname{Hess} l_p)_p(W, W) = \langle A_{\sigma(p)}(X), X \rangle - r\|V\|^2 + r\|\sigma_*(X)\|^2.$$

Let $k := \sup_{p \in M}\{k_1(p, \sigma(p)), \ldots, k_n(p, \sigma(p))\}$, then $\langle A_{\sigma(p)}(X), X \rangle \leqslant k\|X\|^2$. Further, let $\lambda(p)$ be the norm of the linear operator σ_{*_p}, and set $\lambda := \sup_{p \in M} \lambda(p)$, then $\|\sigma_*(X)\| \leqslant \lambda\|X\|$. So, assuming $W \neq 0$, we have

$$(\operatorname{Hess} l_p)_p(W, W) < (k + r\lambda^2)\|X\|^2.$$

Thus if we set $r < -k/\lambda^2$, then $(\operatorname{Hess} l_p)_p$ is negative definite.

(v) Now, since P has positive curvature and M is strictly convex, it follows that P lies on one side of all of its tangent planes at points $p \in M$, assuming U is small. Thus, since M is compact, Lemma 3.1.6 yields that P is strictly convex.

(vi) Since by assumption σ is proper, the perturbations $M_\epsilon := \{p - \epsilon\sigma(p) : p \in M\}$ are C^k for small ϵ; therefore, since P is a segment of the tube around the perturbed submanifold, i.e., $P \subset Tube_\epsilon M_\epsilon$, P will be C^k as well; because, the distance function of a C^k submanifold is C^k everywhere except at the focal points (Section 2.4). Since f is an embedding, P does not contain any of the focal points; therefore, the distance function of M_ϵ is a C^k submersion when restricted to a neighborhood of P. Hence, P is C^k. □

3.3. **Extension of the germ to a weak solution.**

Proposition 3.3.1. *Every C^k strictly convex patch P may be extended to a C^1 ovaloid O. Moreover, we can construct O so that it is C^k in an open neighborhood of P, and arbitrarily close to the convex hull of P.*

Recall that by an ovaloid we mean a closed hypersurface with bounded radii of curvature (Section 2.2). To prove the above, we need the following lemmas :

Lemma 3.3.2. *Let $P \subset \mathbf{R}^m$ be a C^k strictly convex patch, then there exists a $\delta > 0$ such that, for all $r < \delta$, the inner parallel hypersurface of P at the distance r is a C^k strictly convex patch.*

Proof. P, being a strictly convex hypersurface, has a unique C^{k-1} nonsingular strict support σ which is just its outward unit normal vector field. The inner parallel hypersurface of P at the distance r is given by

$$\overline{P} := \{\, x - r\sigma(x) : x \in P\,\}.$$

We have to show that there exists a $\delta > 0$ such that, for all $r < \delta$, (i) $\overline{P}$ is a C^k embedded hypersurface, (ii) $\overline{P}$ has everywhere positive curvature, and (iii) $\overline{P}$ lies strictly on one side of all of its tangent planes.

(i) Define $f\colon P \to \mathbf{R}^m$ by

$$f(p) := p - r\sigma(p),$$

we claim that there exists a $\delta_1 > 0$ such that, for every $r < \delta_1$, f is an embedding. This would show that P is a C^{k-1} embedded submanifold. Then we may use the distance function to show that P is C^k (Section 2.4).

Since P is compact, it is sufficient to show that f is a one-to-one immersion. Let E_p^i be the principal directions of P at p, and $\gamma\colon(-\epsilon,\epsilon) \to P$ be a curve with $\gamma(0) = p$ and $\gamma'(0) = E_p^i$, then

$$f_*(E_p^i) = \gamma'(0) - r(\sigma\circ\gamma)'(0) = E_p^i - r\sigma_*(E_p^i) = (1 + rk_i(p))E^i(p).$$

Thus $f_*(E_p^i) \neq 0$ if $r \neq -1/k_i(p)$. In particular, if $0 \leqslant r < \lambda$, where

$$\lambda := \inf_{p\in M}\{r_1(p), \dots, r_n(p)\},$$

and $r_i(p) := 1/|k_i(p)|$ are the principal curvatures, then it follows that f is an immersion. It remains to show that f is one-to-one. To see this let $F : P\times\mathbf{R} \to \mathbf{R}^m$ be defined by $F(p,r) := p - r\sigma(p)$. By the previous paragraph, $F|_{P\times(-\lambda,\lambda)}$ is an immersion. Thus, by Lemma 3.2.2, $F|_{P\times(-\epsilon,\epsilon)}$, must be an embedding for some small $\epsilon > 0$; because, $F|_{P\times\{0\}}$ is one-to-one. In particular, if $\delta_1 := \min\{\epsilon, \lambda\}$, then f is a C^{k-1} embedding for every $0 \leqslant r \leqslant \delta_1$.

(ii) Assuming that $0 < r < \delta_1$, we now show that all principal curvatures of $\overline{P}$ are nonzero and have the same sign. has positive curvature. Let $\overline{\sigma}\colon \overline{P} \to \mathbf{S}^{m-1}$ be defined by $\overline{\sigma}(f(p)) := \sigma(p)$. We claim that $\overline{\sigma}$ is the Gauss map of $\overline{P}$ i.e., $\overline{\sigma}(f(p)) \in T_{f(p)}\overline{P}^{\perp}$. To see this, let E_p^i be the principal directions of P at p, and note that $\{f_*(E_p^i)\}$ forms a basis for $T_{f(p)}\overline{P}$; because, f_* is an immersion. Thus all we need is to check that:

$$\langle\overline{\sigma}(f(p)), f_*(E_p^i)\rangle = \langle\sigma(p), (1 + rk_i(p))E_p^i\rangle = (1 + rk_i(p))\langle\sigma(p), E_p^i\rangle = 0.$$

Next note that $(1 + rk_i(p))\overline{\sigma}_*(E_p^i) = \overline{\sigma}_*(f_*(E_p^i)) = \sigma_*(E_p^i) = -k_i(p)E_p^i$. So

$$\overline{\sigma_*}(E_p^i) = \frac{-k_i(p)}{1 + rk_i(p)}E_p^i.$$

Since $\overline{\sigma}(f(p)) = \sigma(p)$, T_pP and $T_{f(p)}\overline{P}$ are parallel. Hence the principal directions of $\overline{P}$ at $f(p)$ are the E_p^i with corresponding principal curvatures $-k_i(p)/(1 + rk_i(p))$. These curvatures are well-defined, because $r < \delta_1$; and all have the same sign, because all k_i have the same sign.

(iii) We show that there exists a $0 < \delta \leqslant \delta_1$ such that, if $0 < r < \delta$ then $\overline{P}$ is strictly supported by its tangent hyperplanes. For $\overline{p} \in \overline{P}$ define $\bar{l}_{\overline{p}} \colon \overline{P} \to \mathbf{R}$ by

$$\bar{l}_{\overline{p}}(\,\cdot\,) = \langle\,\cdot, \overline{\sigma}(\overline{p})\rangle,$$

and set $\overline{L}(\overline{p},\overline{q}) := \bar{l}_{\overline{p}}(\overline{q}) - \bar{l}_{\overline{p}}(\overline{p})$ We have to show that $\overline{L}|_A < 0$, where $A := \overline{P} \times \overline{P} - \Delta(\overline{P}\times\overline{P})$. Partition A into $B_\alpha := \{(\overline{p},\overline{q}) \in A : \mathrm{dist}_{\overline{P}}(\overline{p},\overline{q}) < \alpha\}$, and its complement C_α. Since $\overline{P}$ is compact and has positive curvature, it follows, by Lemma 3.1.5, that there exists an $\alpha > 0$ such that $\overline{L}|_{B_\alpha} < 0$. So it remains to show that $\overline{L}|_{C_\alpha} < 0$. Note that $\overline{q} = f(q) = q - r\sigma(q)$ for some $q \in P$, and $\overline{\sigma}(\overline{p}) = \overline{\sigma}(f(p)) = \sigma(p)$, by definition of $\overline{\sigma}$. Thus

$$\bar{l}_{\overline{p}}(\overline{q}) - \bar{l}_{\overline{p}}(\overline{p}) = l_p(q) - l_p(p) + r(1 - \langle\sigma(q),\sigma(p)\rangle) \leqslant l_p(q) - l_p(p) + 2r,$$

where $l_p(\cdot) := \langle\cdot,\sigma(p)\rangle$. So recalling that $L(p,q) := l_p(q) - l_p(p)$, we have

$$\overline{L}(\overline{p},\overline{q}) \leqslant L(p,q) + 2r.$$

If $(\overline{p},\overline{q}) \in \overline{C}_\alpha$, then $(p,q) \in C_{\alpha'} := \{(p,q) \in P \times P : \mathrm{dist}_P(p,q) \geqslant \alpha'\}$. Let $\eta := \sup\{\,L(p,q) : (p,q) \in C_{\alpha'}\}$. Since P is strictly convex, $\eta < 0$. Further the above inequality shows that if $r < -\eta/2$, then $\overline{L}|_{\overline{C}_\alpha} < 0$. Hence $\delta := \min\{\delta_1, -\eta/2\}$ is the desired estimate. □

Lemma 3.3.3. *For every strictly convex patch P there exists a $\Delta > 0$ such that, if $R > \Delta$, through every point of P there passes a sphere of radius R containing P.*

Proof. Let $\sigma \colon P \to \mathbf{S}^{m-1}$ be the outward unit normal P and $B^{p,r}$ be a ball of radius r centered at $p - r\sigma(p)$. Then $p \in \partial B^{p,r}$. We show that there exists $0 < \Delta < \infty$ such that $P \subset B^{p,r}$ for all $p \in P$, if $r > \Delta$. Let $\lambda := \sup_{p\in P}\{\,r_1(p),\ldots,r_n(p)\,\}$, where $r_i(p) := 1/|k_i(p)|$ are the principal radii of curvature of P. If $r > \lambda$, then every $p \in P$ has an open neighborhood $U_{\delta_p}(p) \subset P$ such that $U_{\delta_p}(p) \subset B^{p,r}$. Since P is C^2 and compact δ may be chosen independently of p. Define $f \colon P \to \mathbf{R}$ by

$$f(p) := \sup\left\{\ \frac{\|q-p\|^2}{-2\langle q-p,\sigma(p)\rangle}\ \middle|\ q \in P - U_\delta(p)\ \right\}.$$

Note that $|\langle q-p,\sigma(p)\rangle| = \mathrm{dist}(T_pP, q) > 0$, for all $q \in P - \{p\}$; because, P is strictly convex. Hence f is well-defined, and continuous. Let $\eta := \sup f$. If $r > \eta$, then, for all $q \in U_\delta(p)$, $\|q - (p - r\sigma(p))\|^2 = \|q-p\|^2 + 2r\langle q-p,\sigma(p)\rangle + r^2 \leqslant r^2$, which implies $P - U_\delta(p) \subset B^{p,r}$. Thus $\Delta := \max\{\lambda,\eta\}$ is the desired estimate. □

Lemma 3.3.4. *Let K be an arbitrary intersection of balls of fixed radius R, then through every point in the boundary of K there passes a sphere of radius R containing K.*

Proof. Let $\{B_i\}_{i\in I}$ be an arbitrary collection of balls of fixed radius R, and set $K := \cap_{i\in I}B_i$. We have to show that for every $x \in \partial K$, there is a ball B of radius R such that $K \subset B$ and $x \in \partial B$. If $x \in \partial B_i$ for some $i \in I$, then we are done. So suppose that $x \in \mathrm{int}\,B_i$ for every $i \in I$. Then for every $n \in \mathbb{N}$, there is a ball, say B_n, such that $U_{1/n}(x) \cap B_n \neq \emptyset$, where $U_{1/n}(x)$ is a neighborhood of radius $1/n$ about x. If $K \neq \emptyset$ then the sequence $\{B_n\}_{n\in\mathbb{N}}$ is bounded. So by Blaschke's selection principal there is a subsequence $B_{n'}$ converging to some body B in the sense of Hausdorff distance. It is easy to verify that B is the desired ball. □

Lemma 3.3.5. *Let $A \subset \mathbf{R}^m$ be a compact subset, and K_R be the intersection of all balls of radius R containing A. Then for every $\epsilon > 0$, there exists an $R < \infty$ such that* $\mathrm{dist}_{\mathrm{H}}(K_R, \mathrm{conv}\, A) < \epsilon$.

Proof. Fix an $\epsilon > 0$, and suppose $\mathrm{dist}(p, \mathrm{conv}\, A) \geqslant \epsilon$ for some $p \in \mathbf{R}^m$. It is enough to show that for every such point p, there exists a ball B of radius R, depending only on ϵ, such that $\mathrm{conv}\, A \subset B$, but $p \notin B$; we derive the following estimate:

$$R \geqslant \frac{\epsilon}{4} + \frac{\delta^2}{\epsilon},$$

where $\delta := \mathrm{diam}(A)$, the largest distance between pairs of points in A. To see this let p' be the (unique) point of $\mathrm{conv}\, A$ which is closest to p, and let l be the line determined by p, and p'. Let H be the hyperplane which contains p' and is perpendicular to l, and H^+ be the half space which does not contain p. Then $\mathrm{conv}\, A \subset H^+$. Let $B^+ := B(p', \delta) \cap H^+$. Then $\mathrm{conv}\, A \subset B^+$, because $\mathrm{diam}(\mathrm{conv}\, A) = \mathrm{diam}(A) = \delta$. Choose a point o on l which lies in H^+, and suppose $\mathrm{dist}(o, p') + \epsilon/2 = R$. Then $p \notin B(o, R)$. Furthermore, for $\mathrm{conv}\, A$ to be contained in $B(o, R)$, we must have $\sup_{x \in \mathrm{conv}\, A} \mathrm{dist}(x, o) \leqslant R$. Since $\mathrm{conv}\, A \subset B^+$, we have

$$\sup_{x \in \mathrm{conv}\, A} \mathrm{dist}(x, o) \leqslant \sup_{x \in B^+} \mathrm{dist}(x, o) = ((R - \frac{\epsilon}{2})^2 + \delta^2)^{\frac{1}{2}}.$$

The last equality follows because the farthest point of B^+ with respect to p lies on the great circle which bounds the intersection of B^+ with H. Setting the right hand side of the above less than or equal to R, we obtain the desired estimate. □

Lemma 3.3.6. *Every C^k strictly convex patch P may be extended to a C^k strictly convex hypersurface without boundary. In particular, P is contained in the interior of a C^k strictly convex patch.*

Proof. Using the notion of the *double* of a manifold, see [23], it can be shown that every compact embedded hypersurface with boundary may be extended along its boundaries. Thus there exists an embedded hypersurface S without boundary such that $P \subset S$. Since P is compact and has positive curvature, there exists an open neighborhood $U \subset S$, $P \subset U$, such that U has positive curvature. We claim that if U is sufficiently small, then U is strictly convex. Since U has positive curvature, it suffices to show that U lies on one side of all its tangent hyperplanes. To see this, suppose U is small enough so that it has compact closure $\overline{U}$. Then U is uniformly locally strictly convex by Lemma 3.1.5, i.e., every $q \in U$ has a neighborhood of radius $\delta > 0$ which lies on one side of the tangent hyperplane at q. It can be shown that there exists an $0 < \epsilon < \delta$ such that if the radius of U with respect to P is less than ϵ, then U is strictly convex; the proof is similar to that of Lemma 3.3.4. □

Proof of Proposition 3.3.1. Let $\overline{P}$ be the inner parallel hypersurface of P at the distance r. By Lemma 3.3.2, $\overline{P}$ is strictly convex when r is sufficiently small;

therefore, by Lemma 3.3.3, through every point $\overline{p} \in \overline{P}$ there passes a ball $B_{\overline{p}}$, of some fixed radius R, such that $\overline{p} \in \partial B_{\overline{p}}$ and $\overline{P} \subset B_{\overline{p}}$. Let

$$\overline{K} := \bigcap_{\overline{p} \in \overline{P}} B_{\overline{p}}, \quad K := \overline{K} + rB^m,$$

where B^m denotes the unit ball, and set

$$O := \partial K.$$

Then O is a closed convex hypersurface. We claim that O is the desired ovaloid. It is clear that $P \subset O$. Further, by construction, the radii of curvature of O are bounded above and below by $R+r$ and r respectively. Since K is convex, $\mathrm{dist}_{\mathrm{H}}(O, \mathrm{conv}\, P) = \mathrm{dist}_{\mathrm{H}}(K, \mathrm{conv}\, P)$. Since $P \subset \mathrm{conv}\, P$, $\mathrm{dist}_{\mathrm{H}}(K, \mathrm{conv}\, P) \leqslant \mathrm{dist}_{\mathrm{H}}(K, P)$. Further,

$$\mathrm{dist}_{\mathrm{H}}(K, P) \leqslant \mathrm{dist}_{\mathrm{H}}(K, \overline{K}) + \mathrm{dist}_{\mathrm{H}}(\overline{K}, \overline{P}) + \mathrm{dist}_{\mathrm{H}}(\overline{P}, P) = \mathrm{dist}_{\mathrm{H}}(\overline{K}, \overline{P}) + 2r.$$

By Lemma 3.3.5, $\mathrm{dist}_{\mathrm{H}}(\overline{K}, \overline{P})$ can be made arbitrarily small by choosing R sufficiently large. So we conclude that if R is large and r is small then $\mathrm{dist}_{\mathrm{H}}(O, \mathrm{conv}\, P)$ is small. Finally, in order for O to be C^k in a neighborhood of P, we can extend P along its boundary to a slightly larger strictly convex patch P' which contains P in its interior; and, carry out the above construction for P' instead of P. P' exists by Lemma 3.3.6. $\square$

3.4. **Smoothing of the weak solution.**

Proposition 3.4.1. *Let $O \subset \mathbf{R}^m$ be a C^1 ovaloid, and let $U \subset O$ be a $C^{k \geqslant 2}$ open subset; then, for every closed subset $A \subset U$ there exists a C^k ovaloid $\widetilde{O}$ containing A. Furthermore, $\widetilde{O}$ may be constructed arbitrarily close to O.*

We need the following lemmas:

Lemma 3.4.2. *Let $O \subset \mathbf{R}^m$ be a C^1 ovaloid, and let $U \subset O$ be an open set which is $C^{k \geqslant 2}$ up to its boundary, then U has positive curvature. In particular, $\nu|_U$ is a C^{k-1} embedding, and $h|_U$ is C^{k-1} as well, where ν is the Gauss map of O and h is its support function.*

Proof. The first statement is an elementary consequence of our definition of ovaloid, and the next two statements, as we showed in Section 2.2, are immediate corollaries of the first. $\square$

Recall that by a support function $f\colon \mathbf{R}^m \to \mathbf{R}$ we mean a convex positively homogeneous function. For any subset $A \subset \mathbf{R}^m$, and $f\colon A \to \mathbf{R}$, we define the C^k norm, denoted by $\|\cdot\|_{C^k(A)}$, as the supremum over A of f together with all its partial derivatives up to order k.

Lemma 3.4.3. *Let $h\colon \mathbf{R}^m \to \mathbf{R}$ be a support function and suppose $h|_A$ is C^k, where $A \subset \mathbf{R}^m$ is a compact subset; then, $\|h - \tilde{h}^\epsilon\|_{C^2(A)} \to 0$ as $\epsilon \to 0$ where $\tilde{h}^\epsilon$ denotes the Schneider's convolution of h.*

Proof. This is an immediate consequence of the convolution properties of $\tilde{h}^\epsilon$, see Section 2.3; the details are similar to the corresponding proof of this fact for the ordinary convolution. $\square$

Lemma 3.4.4. *Let $O \subset \mathbf{R}^m$ be a C^1 ovaloid, and let $\widetilde{O}$ be the surface obtained by applying Schneider's convolution to the support function of O; then, $\widetilde{O}$ is a C^∞ ovaloid. In particular the restriction of the support function of $\widetilde{O}$ to any tangent hyperplane to the sphere is strictly convex.*

Proof. See Sections 2.2 and 2.3. □

Proof of Proposition 3.4.1. Since $A \subset U$ is closed, we can replace U by a slightly smaller neighborhood containing A. Thus we can assume, without loss of generality that U is C^k up to its boundary. Let $V \subset U$ be an open set with $\overline{V} \subset U$, and $A \subset V$. Set $V' := \sigma(V)$, and $U' := \sigma(U)$, where $\sigma \colon O \to \mathbf{S}^{m-1}$ is the outward unit normal of O. Then U' and V' are open in $\mathbf{S}^{m-1}$, see Lemma 3.4.2. Let $\overline{\phi} \colon \mathbf{S}^{m-1} \to \mathbf{R}$ be a smooth function with $\operatorname{supp}(\overline{\phi}) \subset U'$, and $\overline{\phi}|_{\overline{V'}} \equiv 1$. Let ϕ be the extension of $\overline{\phi}$ to $\mathbf{R}^m$ given by $\phi(0) := 0$, and $\phi(x) := \overline{\phi}(x/\|x\|)$, when $x \neq 0$. Let h be the support function of O, $\widetilde{h}^\epsilon$ be the Schneider's convolution of h, and define $g \colon \mathbf{R}^m \to \mathbf{R}$ by

$$g(x) := \widetilde{h}^\epsilon(x) + \phi(x)(h(x) - \widetilde{h}^\epsilon(x)).$$

We claim that there exists an $\epsilon > 0$ such that g is a support function, and the boundary of the convex body determined by g is the desired ovaloid. To prove this we have to check: (i) $g(rx) = rg(x)$, for all $r > 0$, (ii) g is C^k on $\mathbf{R}^m - \{0\}$, (iii) $(\operatorname{Hess} g)_p$ is positive semidefinite, for all $p \in \mathbf{R}^m - \{0\}$, (iv) $g|_{V'} = h|_{V'}$, and (v) $(\operatorname{Hess} \overline{g^p})_p$ is positive definite for all $p \in \mathbf{S}^{m-1}$, where $\overline{g^p}$ is the restriction of g to $T_p\mathbf{S}^{m-1}$. (i), (ii), and (iii) show that g is a C^k support function. Thus g determines a convex body with some boundary $\widetilde{O}$. (iv) shows that $V \subset \widetilde{O}$; and, consequently, implies that $A \subset \widetilde{O}$. Finally, (v) implies that $\widetilde{O}$ is a C^k ovaloid. (i) and (iv) are immediate from the definition of g and (ii) follows from the fact that $h|_{U'}$ is C^k, see Lemma 3.4.2. Thus it remains to check (iii) and (v).

(iii) By homogeneity of g, it is enough to check this only for $p \in \mathbf{S}^{m-1}$. Further, since $g|_{\mathbf{S}^m - U'} = \widetilde{h}^\epsilon$, and $\widetilde{h}^\epsilon$ is convex, we need to check (iii) only for $p \in U'$. For every $p \in U'$, let $\{E_p^i\}$, $1 \leqslant i \leqslant m$, be an orthonormal basis for $\mathbf{R}^m$ with $E_p^m = p$, and set $g_{ij} := (\operatorname{Hess} g)_p(E_p^i, E_p^j)$. We have to show that the principal minors of the matrix (g_{ij}) are nonnegative for small ϵ. Since g is homogeneous,

$$g_{im} = 0 = g_{mi},$$

i.e., the last row and column of (g_{ij}) are zero. Thus all the principal minors containing the last row and column of (g_{ij}) are zero. It remains, therefore, to check the principal minors of (g_{ij}) not containing the last row and column.

Let $h_{ij} := (\operatorname{Hess} h)_p(E_p^i, E_p^j)$, and recall that if E_p^i, $1 \leqslant i \leqslant m-1$, are chosen so that they coincide with the principal directions of U at $\sigma^{-1}(p)$, then

$$h_{ij} := \begin{cases} r_i, & 1 \leqslant i = j \leqslant m-1, \\ 0, & \text{otherwise}; \end{cases}$$

where r_i are the principal radii of curvature of U at $\sigma^{-1}(p)$, which are positive. Thus the principal minors of (h_{ij}) not containing the last row and column are positive; therefore, to show that the corresponding principal minors of (g_{ij}) are also nonnegative, it would be sufficient, by continuity of the determinant, to show that $|h_{ij} - g_{ij}| \to 0$ as $\epsilon \to 0$, i.e., we have to check that $\|h - g\|_{C^2(\overline{U'})} \to 0$. To see this note that

$$\|g - h\|_{C^2(\overline{U'})} \leqslant \|g - \widetilde{h}^\epsilon\|_{C^2(\overline{U'})} + \|\widetilde{h}^\epsilon - h\|_{C^2(\overline{U'})} \quad \text{and} \quad \|\widetilde{h}^\epsilon - h\|_{C^2(\overline{U'})} \to 0,$$

by Lemma 3.4.3. Thus, it is enough to show that $\|g - \widetilde{h}^\epsilon\|_{C^2(\overline{U'})} \to 0$. This follows from the following calculations:

$$|g(p) - \widetilde{h}^\epsilon(p)| \leqslant |\phi(p)|\, \|h(p) - \widetilde{h}^\epsilon(p)\|,$$
$$\|Dg_p - D\widetilde{h}^\epsilon{}_p\| \leqslant |h(p) - \widetilde{h}^\epsilon(p)|\, \|D\phi(p)\| + |\phi(p)|\, \|Dh_p - D\widetilde{h}^\epsilon{}_p\|,$$

and

$$\|D^2 g_p - D^2 \widetilde{h}^\epsilon{}_p\| \leqslant$$
$$|h(p) - \widetilde{h}^\epsilon(p)|\, \|D^2\phi_p\| + |\phi(p)|\, \|D^2 h_p - D^2\widetilde{h}^\epsilon{}_p\| + 2\|D\phi_p\|\, \|Dh_p - D\widetilde{h}^\epsilon{}_p\|.$$

Thus, since $\|h - \widetilde{h}^\epsilon\|_{C^2(\overline{U'})} \to 0$, we conclude that $\|g - \widetilde{h}^\epsilon\|_{C^2(\overline{U'})} \to 0$.

(v) $(\operatorname{Hess} g)_p(E^i_p, E^j_p) = (\operatorname{Hess} \overline{g^p})_p(E^i_p, E^j_p)$, for all $1 \leqslant i, j \leqslant m-1$. As we showed in (iii), if $p \in U'$, then the matrix obtained from (g_{ij}) by eliminating the last row and column is positive definite; therefore, $\operatorname{Hess} \overline{g^p}$ is positive definite for all $p \in U'$. If $p \in \mathbf{S}^{m-1} - U'$, then $(\operatorname{Hess} g)_p(E^i_p, E^j_p) = (\operatorname{Hess} \overline{g^p})_p(E^i_p, E^j_p)$ which is positive definite by Lemma 3.4.4. □

Appendix A. A Special Convex Curve

A unit speed parameterization for the curve in Figure 1 is given by

$$\gamma(t) := \left(\sqrt{\pi} F_S\Big(\sqrt{\frac{5}{2}}\sin(t)\Big),\ \sqrt{\pi} F_C\Big(\sqrt{\frac{5}{2}}\sin(t)\Big),\ \sqrt{\frac{5}{2}}\frac{\cos(t)}{4} \right),$$

where

$$F_C(x) := \int_0^x \cos(\pi t^2/2)dt\,, \qquad \text{and} \qquad F_S(x) := \int_0^x \sin(\pi t^2/2)dt$$

are the *Fresnel integrals* studied in optics, and $t \in [0, 2\pi]$. Let Γ be the trace of γ. A computation shows that Γ has nonvanishing curvature. Further, since Γ lies on the boundary of a convex body, it is not difficult to see that its self-linking number must be zero. Thus there are no known obstructions for Γ to bound a positively curved surface; however, no such surface exists:

Theorem A.0.1. *There exists a smooth simple closed curve in $\mathbf{R}^3$ (e.g., as given by the above equation) which (i) does not have any inflection points, and (ii) lies on the boundary of a convex body, but (iii) bounds no surfaces of positive curvatures.*

To prove the above we need the following three lemmas. The first is elementary, so we omit its proof.

Lemma A.0.2. *Let $S \subset \mathbf{R}^3$ be a positively curved surface and $\Gamma \subset S$ be a C^2 curve. Then Γ has no inflection points; thus, for every $p \in \Gamma$, the principal normal $N(p)$ is well-defined. Furthermore $\langle n(p), N(p)\rangle > 0$, where $n(p)$ denotes the inward unit normal of S at p.*

By an "inward" normal of S at a point p we mean the following. If the Gaussian curvature of S at a point p is positive then there exists an open neighborhood of p in S which lies on the boundary of a convex body $K \subset \mathbf{R}^3$. We say $n(p)$ points inward, if it points into the half space of T_pS which contains K.

Lemma A.0.3. *Let M be a 2-manifold (with boundary), $f\colon M \to \mathbf{R}^3$ be an immersion with positive curvature, $H \subset \mathbf{R}^3$ be a plane, and X be a component of $f^{-1}(H)$ which is not a point. Then X is a smoothly embedded one-dimensional submanifold of M.*

Proof. It suffices to show that every $p \in X$ has an open neighborhood U such that $f(U)$ intersects H transversely. Suppose, towards a contradiction, that $f(U)$ is tangent to H at $f(p)$. If X contains more than one point, there exists a point $p' \in U$, $p \neq p'$, such that $f(p') \in H$. Since f is a local embedding, we may assume that $f(p) \neq f(p')$. Let l be the line segment joining $f(p)$ and $f(p')$. Since f has positive curvature, $f(U)$ lies on the boundary of a convex body $K \subset \mathbf{R}^3$, which has to lie on one side of H. Hence l lies in the boundary of K. By the theorem on the invariance of domain, $f(U)$ is open in ∂K, assuming U is small. Consequently $f(U)$ has to contain an open subset of l, which contradicts the assumption that f has positive curvature. □

Lemma A.0.4. *Let M be a compact connected 2-manifold and $f\colon M \to \mathbf{R}^3$ be an immersion with positive curvature. Suppose that f is an embedding on ∂M, and $f(\partial M)$ lies on the boundary of a convex body, then $f(M - \partial M) \cap \mathrm{conv}(f(\partial M)) = \emptyset$.*

Proof. See [1]. □

Proof of Theorem A.0.1. Let Γ be the image of the curve given by the above equation. Suppose that there exists a compact 2-manifold M and a positively curved immersion $f\colon M \to \mathbf{R}^3$ such that $f(\partial M) = \Gamma$, and set $S := f(M)$. Let H denote the xy-plane. H meets Γ at exactly two points, say p and q, see Figure 2.

Let X be the component of $f^{-1}(H)$ which contains $f^{-1}(p)$. Since H meets Γ (and consequently S) transversely, X contains more than one point. Hence, by Lemma A.0.3, X is a smoothly embedded one-dimensional submanifold of M. Thus either X is homeomorphic to $\mathbf{S}^1$ or else is an open curve segment which meets ∂M at distinct points. If X is closed, then it has to be tangent to ∂M, which contradicts the fact that H is transversal to Γ at p. Thus X is an open curve segment. So there exist a connected curve $C := f(X)$ in H with end points at p and q.

Parameterize C by arc length starting at p. Let $t(p)$ be the unit tangent of C, $N(p)$ be the principal normal of Γ, and $n(p)$ be the inward unit normal of S. Since H meets Γ orthogonally, all three vectors lie in H, see Figure 3. Further, Since S has positive curvature, $\langle n(p), N(p)\rangle \neq 0$, by Lemma A.0.3. Thus $t(p)$ and $N(p)$ cannot be parallel; therefore, the lines $T_pS \cap H$ and $T_p\Gamma \cap H$ are distinct, where T_pS denotes

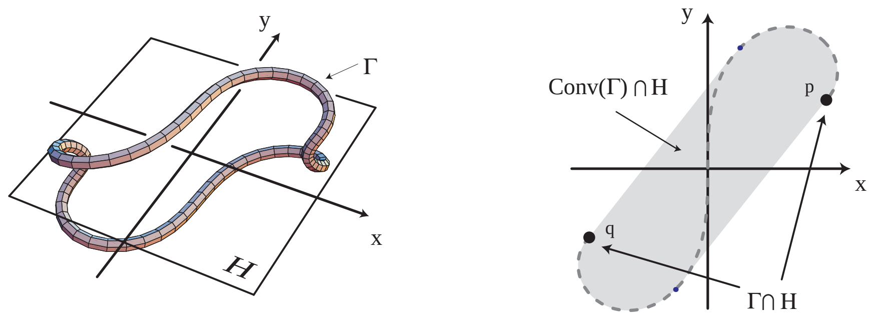

FIGURE 2

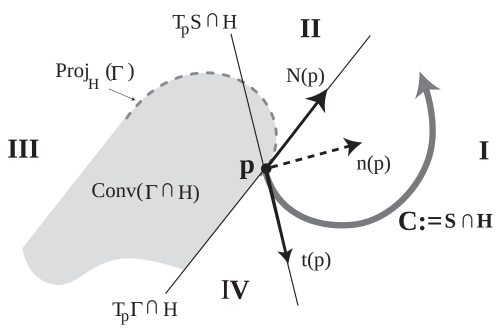

FIGURE 3

the tangent plane to S and $T_p\Gamma$ denotes the tangent plane to Γ which contains $N(p)$. Γ has been constructed so that $T_p\Gamma$ supports Γ. Thus $\operatorname{conv}(\Gamma) \cap H$ lies one side of $T_p\Gamma \cap H$. Further, $t(p)$ points into the half-space, determined by $T_p\Gamma \cap H$, which does not contain $\operatorname{conv}(\Gamma) \cap H$; because, by Lemma A.0.4, $(S - \Gamma) \cap \operatorname{conv}(\Gamma) = \emptyset$, and $T_p\Gamma$ is the unique supporting plane of $\operatorname{conv}(\Gamma)$ passing through p.

Let $T_pS \cap H$ and $T_p\Gamma \cap H$ be oriented by t_p and $N(p)$ respectively, then these two lines determine four quadrants in H, which may be numbered in the standard way. $n(p)$ has to lie in quadrant I or II, because, by Lemma A.0.2, $N(p)$ and $n(p)$ have to lie on the same side of $T_pS \cap H$. Further, $n(p)$ is the principal normal vector to C. Hence, near p, C lies in the first quadrant. Thus t rotates counter-clockwise as it moves away from p. Similarly, we can show that t rotates clockwise as it approaches q, because Γ is invariant under reflection through the origin. So C must have an inflection point, which is a contradiction by Lemma A.0.2. □

APPENDIX B. OPEN PROBLEMS

Here we restate the question [32, Prob. 26] which provided the prime stimulus for this work, and also mention a related problem (this is a truncated version of a list of problems which had appeared in the author's Ph.D. thesis [9]. Since then, one of those problems [9, Conj. E.0.5] has been solved by Stephanie Alexander and the author [1]; another [9, Conj. E.0.4] has been solved by John McCuan [21]; and still another [9, Conj. E.0.6] has been solved by Bo Guan and Joel Spruck [12]).

Question B.0.1 (S.-T. Yau). *Given a metric of positive curvature on the disk what is the condition on a space curve to form the boundary of an isometric embedding of the disk?*

As was mentioned in the introduction, a nontrivial *necessary* condition, has been discovered by H. Rosenberg [25], involving the self-linking number, and the main result of this paper provides a *sufficient* criterion; however, a complete characterization is not yet known.

Question B.0.2 (H. Rosenberg). *Does every curve bounding a surface of positive curvature in 3-space have four vertices, i.e., points where the torsion vanishes?*

V. D. Sedykh [28] has shown that the answer to the above problem is positive provided that the curve lies on a convex body, thus solving a long standing conjecture of P. Scherk.

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