Strictly Convex Submanifolds

And

Hypersurfaces Of Positive Curvature

by

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Abstract

We characterize submanifolds of Euclidean space which lie on closed hypersurfaces of positive curvature, and develop some applications of this result for boundary value problems via Monge-Ampére equations, smoothing of convex polytopes, and an extension of Hadamard's ovaloid theorem to hypersurfaces with boundary.

The main result of this dissertation states that every smooth compact submanifold M of Euclidean space lies embedded in a smooth closed hypersurface of positive curvature if, and only if, M is *strictly convex*, i.e., through every point of M there passes a hyperplane, with contact of order one, with respect to which M lies strictly on one side. As applications of this result we show:

1. Every smooth closed strictly convex submanifold of codimension two bounds a smooth hypersurface of *constant* positive curvature.

2. Let M be a closed strictly convex submanifold of codimension 2; then, if M is $C^{3,1}$, the two hypersurfaces making up the boundary of the convex hull of M are each $C^{1,1}$; this result is optimal.

3. Every polytope P may be approximated arbitrarily closely by a closed hypersurface of nonnegative curvature which coincides with the boundary of P everywhere outside any given open neighborhood of the singular points.

4. Let M be a compact connected hypersurface of positive curvature in Euclidean n-space, $n \geqslant 3$, then M is strictly convex, if, and only if, each boundary component of M lies strictly on one side of the tangent hyperplanes of M at that component.

Furthermore, we discuss some applications for self-linking number of space curves, and umbilic points of ovaloids.

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Dedication

To my parents,

&

for Kate

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1 Introduction

1.1 Statement of the main theorem

We say a C^2 submanifold $M \subset \mathbb{R}^m$ is *strictly convex* if through every point of M there passes a *nonsingular support hyperplane*, i.e., a hyperplane, with contact of order one, with respect to which M lies strictly on one side.

Let $O \subset \mathbb{R}^m$ be an *ovaloid*, i.e., a closed hypersurface of positive curvature; then, every C^2 embedded submanifold of O is strictly convex by Hadamard's theorem. In this paper we prove the converse; thus, obtaining the following characterization:

1.1.1. Main Theorem. *Let $M \subset \mathbb{R}^m$ be a smooth (C^∞) compact embedded submanifold, possibly with boundary; then, M lies embedded in a smooth ovaloid if, and only if, M is strictly convex. Furthermore, if M is strictly convex, then*

1. *Any finite number of nonsingular support hyperplanes at distinct points of M may be extended to a smooth distribution of nonsingular support hyperplanes along M.*

2. *For every smooth distribution of nonsingular support hyperplanes along M there exists a smooth integral ovaloid containing M.*

3. *This ovaloid may be constructed within an arbitrary small distance of the convex hull of M.*

4. *If M is symmetric with respect to some rotation or reflection in $\mathbb{R}^m$, then there exists a smooth ovaloid, containing M, which has the same symmetry.*

Finally, if M is strictly convex, but is only of class C^k, for some $k \geqslant 2$, then there exists an ovaloid, containing M, which is also C^k.

1.2 Applications and connection to other work

Let $M \subset \mathbb{R}^m$ be a closed submanifold of codimension 2. If M is strictly convex, then, by the main theorem, M lies on an ovaloid. In particular, M bounds a hypersurface of positive curvature. Thus a *sufficient* condition for M to bound a hypersurface of positive curvature is that M be strictly convex. This is related to a question posed by S.-T. Yau [Yau1, problem # 26] who asked for conditions for a Jordan curve $\Gamma \subset \mathbb{R}^3$ to bound a disk with a given metric of positive curvature. H. Rosenberg [Ros] has shown that a *necessary* condition is that the self linking number of Γ be zero; however, this condition is not sufficient, as was shown by H. Gluck and L. Pan [GP]. Also we should note that if M is merely convex (i.e. lies on the boundary of its convex hull), but is not *strictly* convex, then it may not bound any surfaces of positive curvature, see Figure 7 (Appendix A).

One motivation for characterizing submanifolds which bound a hypersurface of positive curvature is that such a hypersurface provides a "subsolution", or "barrier", for certain Monge-Ampére equations used to study the existence and regularity of hypersurfaces with prescribed curvature and boundary. For instance, B. Guan and J. Spruck [GS] showed that if M bounds any strictly convex hypersurface, then it bounds a hypersurface of *constant* positive curvature; thus, via this result, an immediate corollary of our main theorem is the following:

1.2.1. Theorem. *Let $M \subset \mathbb{R}^m$ be a smooth strictly convex closed submanifold of codimension 2; then, M bounds a smooth hypersurface of* <u>*constant*</u> *positive curvature. In fact, there exists an $\epsilon > 0$, depending on M, such that for every $0 < K \leqslant \epsilon$, M bounds a hypersurface of constant curvature K.*

The essential idea in [GS] is to apply the *continuity method*, via some a priori estimates, to arrive at a prescribed solution, the existence of which is guaranteed once it is known that a subsolution exists. A specific value for the ϵ in the above theorem may be obtained by taking the minimum of the curvature of any subsolution. We do not know, however, and it would be interesting to find, a sharp estimate for ϵ. From a geometric point of view the result in [GS] was a substantial extension of earlier work in [CNS1], [Krl], and [HRS]; however, it seems that we should be able to go much further. In fact, we conjecture (see Appendix E) that if M bounds any hypersurface of positive curvature (regardless of whether or not the surface is representable by means of a graph), then M bounds a hypersurface of *constant* positive curvature, for any constant less than or equal to the smallest value of the curvature on the initial surface.

The development of the theory of Monge-Ampére equations has a rich history originating from the Minkowski problem, and includes contributions from many well known authors. References may be found in the books by R. Schneider [Sch1] and D. Gilbarg & N. Trudinger [GT]. In particular, see the books by A. Pogorelov [Pog], T. Aubin [Aub], and the survey articles of H. Gluck [Glk] and S.-T. Yau [Yau2].

Another fundamental problem with regard to strictly convex submanifolds, is that of regularity of the two hypersurfaces which form the boundary of the convex hull of M. This question is related to the regularity of certain *degenerate* Monge-Ampére equations over spherical domains. Using this machinery, we prove:

1.2.2. Theorem. *Let $M \subset \mathbb{R}^m$ be a $C^{3,1}$ strictly convex closed submanifold of codimension 2; then, the boundary of the convex hull of M is made up of two $C^{1,1}$ hypersurfaces. (This result is optimal.)*

Earlier L. Caffarelli, L. Nirenberg, and J. Spruck [CNS2] had studied the regularity of degenerate Monge-Ampére equations over convex planar domains. Subject to the existence of a subsolution, B. Guan [Gun] has extended that result to arbitrary planar domains. Here we extend Guan's work to spherical domains via a certain transformation which reduces the problem to the planar case and preserves the subsolution. The optimality of the above theorem follows from a pair of examples in [CNS2], one of which is due to J. Urbas. We should note, however, that as far as $C^{1,1}$ regularity in the *interior* of each cap is concerned, $C^{1,1}$ assumption on the boundary is sufficient [TU]. Note also that if the submanifold is convex (but not *strictly* convex); then, the conclusion of the above theorem may not be true, see Figure 6. We should also mention the work of V. Sedykh [Sed1] who has classified the generic singularities of the convex hull of space curves. For a study of the structure of these objects, from the global point of view, see [RCB].

Another application of the main theorem is that of smoothing convex polytopes. It has been known since H. Minkowski [Min] that the boundary of every convex polytope may be approximated by a smooth closed hypersurface of nonnegative curvature. Here we show that this smoothing may be achieved in an optimal way:

1.2.3. Theorem. *Let $M \subset \mathbb{R}^m$ be the boundary of a convex polytope P; then, there exists a smooth closed hypersurface of nonnegative curvature which coincides with M everywhere outside any given open neighborhood of the singular points of M.*

We prove the above by approximating each facet of the polytope by an ovaloid; turning the edges of each ovaloid inward, using a distance function; and connecting the *rims* via the main theorem. In particular, we will have control over the regions along which the approximation coincides with the boundary of the initial object. If such control is of no concern, then the above theorem may be proved directly by applying a certain convolution, see Section 3.4, to the gauge (a.k.a. distance) function of the given polytope [Sch3]. See [Sch1, pg. 164] for a brief description and references to other work on smooth approximation of convex bodies. For a proof of Minkowski's approximation see the classic book by T. Bonnesen & W. Fenchel [BF].

Next, we mention a result which may be regarded as an extension of Hadamard's theorem, on global convexity of ovaloids, to hypersurfaces with boundary. Hadamard [Had] proved his theorem for closed surfaces of strictly positive curvature in 3-space. Over the years this theorem has been generalized by J. Stoker [Stk], J. van Heijenoort [Hjn], S.-S. Chern & R. Lashof [CL], R. Sacksteder [Sac], and M. do Carmo & E. Lima [dCLi]. See also related papers by P. Hartman [Htm], H.-H. Wu [Wu], and M. do Carmo and H. Lawson [dCLa]. In all these papers, however, it is always assumed that the given submanifold is (geodesically) complete, e.g., without boundary. Here we relinquish this assumption:

1.2.4. Theorem. *Let $M \subset \mathbb{R}^m$, $m \geqslant 3$, be a compact connected hypersurface with positive curvature; then, M is strictly convex if, and only if, each boundary component of M lies strictly on one side of the tangent hyperplanes of M at that component.*

The hypothesis of the above theorem is not unnatural. It is satisfied, for instance, when each boundary component of M lies in a hyperplane. Also note that, much like Hadamard's theorem, we do not a priori assume that M is embedded (i.e. without self intersection). Rather, embeddedness is obtained at the end as a bonus. An outline of the proof is as follows. We will show that if the hypothesis of the theorem is satisfied, then a *collar* of each boundary component of M is strictly convex. This, via our main theorem, is used to show that M can be extended to a closed hypersurface of positive curvature; therefore, M must be globally strictly convex by Hadamard's theorem. In particular, M will be embedded.

There is another point worth noting with regard to the above theorem: if the hypothesis is satisfied, then each boundary component will be strictly convex; however, the mere condition that each boundary component be strictly convex is not strong enough to guarantee the outcome. In particular, there exists a compact connected hypersurface of positive curvature which is not strictly convex but has a strictly convex boundary, see Figure 15. Still, we conjecture (Appendix E) that the strict convexity of the boundary is strong enough to guarantee the embeddedness of the surface. The corresponding fact for minimal surfaces is well known. See the book of R. Osserman [Osm2, pg 143] for a brief description, and references; specially, the paper by W. Meeks & S.-T. Yau [MY].

In some special cases, other results with regard to global convexity of hypersurfaces with boundary have been obtained by L. Rodríguez, [Rod], and W. Kühnel, [Knl]. Their work involves the notions of *tightness* of N. Kuiper [Kpr], and the *two-piece property* of T. Banchoff [Ban]. See [CR] for an introduction to these concepts.

For a survey of some old and new results on *complete* locally convex hypersurfaces, in Riemannian manifolds, see [AC]. A proof of Hadamard's ovaloid theorem may be found in the lecture notes by H. Hopf [Hpf, chap IV]. Some recent application of the generalizations of Hadamard's theorem include a prove of Efimov's theorem [SX], and a beautiful proof of the sphere theorem of Berger & Klingenberg, by M. Gromov [Ech]. [GW, chap 1.1] gives further indication of the breadth and depth of the general problem of characterizing convex objects, not only in differential geometry, but throughout Mathematics.

In Section 5, we will discuss two more applications of the main theorem of this work. One (Section 5.5) is concerned with the self-linking number of space curves. The other (Section 5.6) is a result on deformation of ovaloids, and has some implications related to a well-known conjecture of C. Carathéodory on umbilic points.

Finally, we should mention a paper of W. Weil [Wl] who showed that given a convex polytope P, it is possible to inscribe a smooth ovaloid inside P which touches the interior of each facet at prescribed points. This is an immediate implication of our main theorem, when M is discrete. In this case, much like the general case, the solution is not unique. It would be interesting, therefore, to study the existence and uniqueness of solutions to this problem under some restrictions on curvature. For instance, we could search for the ovaloid which is most *spherical*, i.e., the variation in its radii of curvature is small, see Appendix D.

1.3 Outline of the proof of the main theorem

The proof of the main theorem employs a blend of concepts and techniques from the theory of convex bodies, submanifold geometry, and Monge-Ampére equations. Given a strictly convex compact $C^{k\geqslant 2}$ submanifold $M \subset \mathbb{R}^m$, we give a constructive proof of the existence of a C^k ovaloid O containing M in four steps (see Figure 2):

Step 1. We will show that, by extending the outward unit normal of a nonsingular support hyperplane to a small neighborhood of the point of contact, it is possible to slide each nonsingular support hyperplane locally. This involves studying the second order behavior of the corresponding height functions. By using a partition of unity we then construct a C^{k-1} nonsingular support, i.e., a unit normal vector field given by a C^{k-1} mapping $\sigma\colon M \to \mathbb{S}^{m-1}$ which generates nonsingular support hyperplanes along M. (The motivation for these techniques comes from considering the special case of curves in $\mathbb{R}^3$, see Appendix B.) Furthermore, through a careful study of the regularity of the distance function versus that of the end point map in small tubular neighborhoods of M, we will show that is is possible to construct

σ so that small perturbations of M along σ are actually C^k; even though, σ is in general only C^{k-1}. When σ has this additional property, we say that σ is *proper*.

Step 2. By perturbing M inward a distance of ϵ along σ (see Figure 3), and then building a tubular hypersurface of radius ϵ around the perturbed submanifold, we will show that there exists a C^k strictly convex patch P, i.e., a compact embedded hypersurface with boundary, which contains M and is tangent to every hyperplane generated by σ. We do this, in one step, by using a variation of the endpoint map, based on σ, to embed a portion of the unit normal bundle of M. For the special case when M is a space curve, it is possible to carry out this construction quite explicitly and obtain formulas for the curvature of the patch, see Appendix C.

Step 3. We will show that every strictly convex patch can be extended to a C^1 *ovaloid* O, i.e., a closed hypersurface whose radii of curvature, in a generalized sense, are bounded above and below by positive constants. We construct O (see Figure 4) by (i) forming the inner parallel hypersurface of P at a small distance ϵ, (ii) taking the intersection of all balls of a sufficiently large radius containing the perturbed hypersurface, and (iii) forming the outer parallel body of the intersection at the distance ϵ. (The second step involves proving an analogue of Blaschke's rolling theorem for hypersurfaces with boundary.)

Step 4. We will show that by applying a certain convolution, due to R. Schneider, to the Minkowski support function of O, and then a gluing with the aid of a *fixed* bump function on the sphere, it is possible to construct a sequence O_i of C^k closed hypersurfaces which contain M and converge to O. We will show that, for every i, O_i has uniformly bounded positive curvature except in a small neighborhood of M with fixed radius; however, it turns out that these small neighborhoods converge to P up to the second order; therefore, this sequence will eventually have positive curvature near M as well; thus, producing the desired ovaloid.

2 Basic Notation and Terminology

2.1 Submanifold geometry

We say $M \subset \mathbb{R}^m$ is a C^k n-dimensional embedded submanifold if every point of M has a neighborhood, under the subspace topology, which is C^k-diffeomorphic to an open subset of the halfspace $\mathbb{H}^n$. We say a vector $X_p \in \mathbb{R}^m$ is *tangent* to M at p if there exists a curve $\gamma\colon (-\epsilon, \epsilon) \to M$, such that $\gamma(0) = p$ and $\gamma'(0) = X_p$. If $f\colon M \to \mathbb{R}$ is a C^1 function, then we define

$$X_p f := (f \circ \gamma)'(0),$$

the derivative of f at p in the direction X_p. The set of all tangent vectors to M at p is denoted by T_pM, the *tangent space* of M at p. Note that in our definition T_pM is a subspace of $\mathbb{R}^m$. The orthogonal complement of T_pM in $\mathbb{R}^m$ is denoted by $T_pM^\perp$, the *normal space* of M at p.

By a *vector field* along M we mean a mapping $X : M \to \mathbb{R}^m$, $p \stackrel{X}{\longmapsto} X_p$. We say X is a tangent vector field if, for all $p \in M$, $X_p \in T_pM$. Similarly, X is a normal vector field if $X_p \in T_pM^\perp$, for all $p \in M$. If $f \in C^1(M)$ then, for every tangent vector field X, we define a new function Xf by setting

$$(Xf)(p) := X_p f.$$

Let $TM := \bigcup_{p \in M} T_pM$, the *tangent bundle* of M. Suppose $F\colon M \to \mathbb{R}^m$, $p \stackrel{F}{\longmapsto} (f_1(p), \dots, f_m(p))$ is a tangent vector field, then the differential of F, $F_*\colon TM \to \mathbb{R}^m$ is defined by

$$F_*(X_p) := (X_p f_1, \dots, X_p f_m).$$

Let $X, Y\colon \mathbb{R}^m \to \mathbb{R}^m$ be a pair of vector fields, then we define a new vector field,

$\overline{\nabla}_X Y$, by setting

$$(\overline{\nabla}_X Y)_p := Y_*(X_p) = \overline{\nabla}_{X_p} Y.$$

$\overline{\nabla}$ is the standard Levi-Civita *connection* on $\mathbb{R}^m$. Now recall that, for all $p \in M$, we can write $\mathbb{R}^m = T_pM \oplus T_pM^\perp$. Let $(\,)^\top : \mathbb{R}^m \to T_pM$, and $(\,)^\perp : \mathbb{R}^m \to T_pM^\perp$ be the corresponding projections, and let X and Y be tangent vector fields on M; then, we set

$$\nabla_{X_p} Y := (\overline{\nabla}_{X_p} Y)^\top, \quad \text{and} \quad a(X_p, Y_p) := (\overline{\nabla}_{X_p} Y)^\perp.$$

∇ is the *induced connection* on M and $a\colon T_pM \times T_pM \to T_pM^\perp$ is known as the *second fundamental form* of M. It can be shown (we will do this shortly) that $(\overline{\nabla}_{X_p} Y)^\perp$ depends only on the value of Y at p; thus, a is well-defined, and yields the following equation

$$(\overline{\nabla}_{X_p} Y) = \nabla_{X_p} Y + a(X_p, Y_p),$$

known as *Gauss' formula.*

An important property of the induced connection ∇ is that it is *Riemannian* which means the following: let X and Y be tangent vector fields on M and $\langle X, Y\rangle\colon M \to \mathbb{R}$ be the mapping defined by $\langle X, Y\rangle(p) = \langle X_p, Y_p\rangle$; then, for every $Z_p \in T_pM$,

$$Z_p\langle X, Y\rangle = \langle \nabla_{Z_p} X, Y_p\rangle + \langle X_p, \nabla_{Z_p} Y\rangle.$$

Let ξ_p be a normal vector to M at p. We define the *shape operator* $A_{\xi_p}\colon T_pM \to T_pM$ by

$$A_{\xi_p}(X_p) := -(\overline{\nabla}_{X_p} \xi)^\top,$$

where ξ is any local extension of ξ_p to a normal vector field on M. Now let Y be any tangent vector field on M, then $\langle Y, \xi \rangle = 0$, and consequently $0 = X_p\langle \xi, Y \rangle = \langle \overline{\nabla}_{X_p}\xi, Y_p \rangle + \langle \xi(p), \overline{\nabla}_{X_p} Y \rangle = \langle (\overline{\nabla}_{X_p}\xi)^\top, Y_p \rangle + \langle \xi(p), (\overline{\nabla}_{X_p} Y)^\perp \rangle$, which yields

$$\langle A_{\xi_p} X, Y_p \rangle = \langle a(X_p, Y_p), \xi_p \rangle.$$

The above formula shows that A_{ξ_p} depends only on the value of ξ at p; and, furthermore, it shows that $a(X_p, Y_p)$ is independent of the extensions X and Y, as we had claimed earlier.

The *Lipschitz-Killing curvature* of M at p in the normal direction ξ_p is defined as

$$K(p, \xi_p) := \det(A_{\xi_p}).$$

If M is a hypersurface, i.e., it has codimension 1, then A_{ξ_p} is known as the *Weingarten map* and $K(p, \xi_p)$ is better known as the *Gauss-Kronecker curvature* of M at p. Furthermore, if M is a two dimensional surface in $\mathbb{R}^3$, then $K(p, \xi_p) = K(p)$ is referred to simply as the *Gaussian curvature* of M at p.

It can be shown that A_{ξ_p} is a self-adjoint operator and therefore orthogonally diagonalizable, i.e., there exists an orthonormal basis $X_p^i \in T_pM$, $1 \leqslant i \leqslant n$, such that

$$A_{\xi_p}(X_p^i) = k_i(p, \xi_p) X_p^i.$$

$k_i(p, \xi_p)$ are known as the *principal curvatures* of M at p with respect to the normal vector ξ_p, and X_p^i are the corresponding *principal directions.*

2.2 Notions of convexity

Let $M \subset \mathbb{R}^m$ be an embedded submanifold. We say a hyperplane H *supports* M at $p \in M$, if $p \in H$ and M lies entirely in one of the closed halfspaces determined by H. By a *strictly supporting* hyperplane for M we mean a support hyperplane which intersects M at only one point. If M is C^2 embedded, then we say H is a *nonsingular support* hyperplane if H is a strictly supporting hyperplane with *contact of order one*, i.e., p is a nondegenerate critical point of the height function of M with respect to H, see Section 3.1.

Let $\operatorname{bd} M$ denote the *extrinsic* boundary of M, i.e., the intersection of the closure of M with the closure of the complement of M in $\mathbb{R}^m$. In particular, note that in general $\operatorname{bd} M \neq \partial M$, where we use ∂M to denote the *intrinsic* boundary of M. Also note that if M is compact, then $\operatorname{bd} M = M$ whenever $\operatorname{codim}(M) \neq 0$, and $\operatorname{bd} M = \partial M$ otherwise.

We say a submanifold $M \subset \mathbb{R}^m$ is *convex*, *weakly strictly convex*, or *strictly convex*; if, through every point $p \in \operatorname{bd} M$ there passes, respectively, a supporting, strictly supporting, or nonsingular supporting hyperplane.

A compact connected submanifold of codimension zero is called a *body*. We denote the space of convex bodies in $\mathbb{R}^m$ by $\mathcal{K}^m$. It is easy to show that $\mathcal{K}^m$ is closed under *Minkowski sum*, and *scalar multiplication* defined by

$$A + B := \{\, x + y \mid x \in A,\ and\, y \in B \,\}, \text{ and}$$
$$rA := \{\, rx \mid x \in A \,\};$$

where A and B are arbitrary subsets of $\mathbb{R}^m$ and $r > 0$. Also, $(\mathcal{K}^m, \operatorname{dist})$ is a locally compact metric space where "dist" denotes the *Hausdorff distance* defined by

$$\operatorname{dist}(A, B) := \inf\{\, r \geqslant 0 \mid A \subset B + rB^m,\ and\, B \subset A + rB^m \,\}.$$

B^m denotes the unit ball in $\mathbb{R}^m$.

2.3 Global analysis

Let $M \subset \mathbb{R}^m$ be an embedded C^k submanifold, then we say $f \in C^l(M)$, $l \leqslant k$, if for every $p \in M$, there is a local chart (U, ϕ), $p \in U$, such that $f \circ \phi^{-1} \colon \mathbb{R}^m \to \mathbb{R}$ is C^l. Since the transition map between local charts is C^k, and $l \leqslant k$, it follows that this definition is independent of local charts.

Suppose $f \in C^1(M)$, then we define the *gradient* of f as the unique tangent vector field characterized by the following property: for every tangent vector field X on M,

$$\langle (\operatorname{grad} f)_p, X_p \rangle = X_p(f).$$

The *Hessian* of f, $\operatorname{Hess} f \colon TM \times TM \to \mathbb{R}$, is defined by

$$\operatorname{Hess} f(X_p, Y_p) := X_p(Yf) - (\nabla_{X_p} Y) f,$$

where ∇ is the induced connection on M.

We say p is a critical point of f if $f_{*_p} = 0$. Now recall that since f is real valued $X_p(f) = f_*(X_p)$ for every $X_p \in T_pM$; therefore, from the definition of gradient above it follows that p is a critical point of f if and only if $\operatorname{grad} f = 0$. Furthermore, since $\nabla_{X_p} Y \in T_pM$, it follows that, if p is a critical point, $\operatorname{Hess} f(X_p, Y_p) = X_p(Yf)$. We say p is a *nondegenerate* critical point if $\operatorname{Hess} f(X_p, X_p) \neq 0$ for all $X_p \neq 0$.

Finally, we mention that the Hessian and gradient are related by the following formula

$$\operatorname{Hess} f(X_p, Y_p) = \langle \nabla_{X_p} \operatorname{grad} f, Y_p \rangle.$$

To see the above, observe that $X_p(Yf) = X_p \langle \operatorname{grad} f, Y \rangle = \langle \nabla_{X_p} \operatorname{grad} f, Y_p \rangle + \langle \operatorname{grad} f_p, \nabla_{X_p} Y \rangle$, and $\langle \operatorname{grad} f_p, \nabla_{X_p} Y \rangle = (\nabla_{X_p} Y) f$.

3 Preliminaries

3.1 Convex submanifolds, height functions, and nonsingular support

Let $M \subset \mathbb{R}^m$ be an embedded submanifold, $p \in M$, and $\sigma_p \in \mathbb{S}^{m-1}$ a unit vector associated with p; then, we define the linear *height function* $l_p\colon M \to \mathbb{R}$ by

$$l_p(x) := \langle x, \sigma_p \rangle. \tag{3.1.1}$$

If M is at least C^2 embedded, we say M is a *strictly convex* submanifold if for every point $p \in M$ there exists a unit vector $\sigma_p \in \mathbb{S}^{m-1}$ such that p is the strict absolute maximum and a nondegenerate critical point of the height function l_p, i.e., we require that

$$l_p(x) < l_p(p), \tag{3.1.2}$$

for all $x \in M - \{p\}$;

$$(\operatorname{grad} l_p)_p = 0, \tag{3.1.3}$$

for all $p \in M$; and

$$(\operatorname{Hess} l_p)_p(X_p, X_p) \neq 0, \tag{3.1.4}$$

for all $X_p \in TM$, $X_p \neq 0$.

Note that the three previous equations together with *Morse's lemma* imply that the Hessian of l_p at p is actually negative definite, i.e.,

$$(\operatorname{Hess} l_p)_p(X_p, X_p) < 0 \tag{3.1.5}$$

for all $X_p \in TM$, $X_p \neq 0$.

If any unit vector σ_p satisfies equations (2), (3), and (4), we say σ_p is a *nonsingular support vector* of M at p. The hyperplane through p with unit normal σ_p is denoted by H_p, and is called a *nonsingular support hyperplane* of M at p. Note that $H_p = \{\, x \in \mathbb{R}^m \mid \langle x - p, \sigma_p\rangle = 0 \,\}$; or, equivalently,

$$H_p = \{\, x \in \mathbb{R}^m \mid l_p(x) = l_p(p) \,\}. \tag{3.1.6}$$

From (2) and (6) it follows that

$$M \cap H_p = \{p\}. \tag{3.1.7}$$

Let H_p^+ denote the closed halfspace determined by H_p which contains the point $p + \sigma(p)$, and let H_p^- denote the other halfspace, then

$$H_p^- = \{\, x \in \mathbb{R}^m \mid l_p(x) \leqslant l_p(p) \,\}. \tag{3.1.8}$$

From (2) and (8) it follows that

$$M \subset H_p^-. \tag{3.1.9}$$

Thus H_p is a supporting hyperplane of M. Moreover, (9) together with (7) shows that H_p is a *strictly* supporting hyperplane.

Since by assumption p is a critical point of l_p, i.e., $(\operatorname{grad} l_p)_p = 0$, then for all $X_p \in TM$, $X_p(l_p) = \langle (\operatorname{grad} l_p)_p, X_p\rangle = 0$. Let $\gamma\colon (-\epsilon, \epsilon) \to M$ be a C^1 curve with $\gamma(0) = p$, and $\gamma'(0) = X_p$, then

$$0 = X_p(l_p) = (l_p \circ \gamma)'(0) = \langle \gamma'(0), \sigma_p)\rangle = \langle X_p, \sigma_p\rangle. \tag{3.1.10}$$

Thus we conclude that σ_p is normal to M at p, i.e.,

$$\sigma_p \in T_pM^\perp, \tag{3.1.11}$$

which also shows that H_p is a tangent hyperplane of M. So the order of contact between H_p and M must be at least one. Also note that since p is a critical point of l_p then for all $X_p, Y_p \in TM$, $(\mathrm{Hess}\, l_p)_p(X_p, Y_p) = X_p(Yl_p) = \langle \overline{\nabla}_{X_p} Y, \sigma_p \rangle$, where $\overline{\nabla}$ denotes the standard Levi-Civita connection on $\mathbb{R}^m$. Recall that by Gauss' formula we have $\overline{\nabla}_{X_p} Y = \nabla_{X_p} Y + a(X_p, Y_p)$ where $\nabla_{X_p} Y = (\overline{\nabla}_{X_p} Y)^\top$ is the induced connection on M and $a(X_p, Y_p) = (\overline{\nabla}_{X_p} Y)^\perp$ is the second fundamental form of M. Since $\nabla_{X_p} Y \in T_pM$, by (11) it follows that $\langle \nabla_{X_p} Y, \sigma_p \rangle = 0$. Thus we have

$$(\mathrm{Hess}\, l_p)_p = \langle a(X_p, Y_p), \sigma_p \rangle = \langle A_{\sigma_p} X_p, Y_p \rangle, \tag{3.1.12}$$

where $A_{\sigma_p} \colon T_pM \to T_pM$ is the shape operator, or the Weingarten map, of M. Recall that $A_{\sigma_p}(X_p) = -(\overline{\nabla}_{X_p} \sigma)^\top$, where σ is any differentiable extension of σ_p to a local unit normal vector field. Note that by (12) the non-degeneracy of $(\mathrm{Hess}\, l_p)_p$ is equivalent to non-singularity of A_{σ_p}. In particular, the *Lipschitz-Killing* curvature of M at p does not vanish in the direction of σ_p, i.e.,

$$K(p, \sigma_p) = \det(A_{\sigma_p}) \neq 0. \tag{3.1.13}$$

The above equation shows that the order of contact between M and H_p is at most one. Finally, recall that since by assumption $(\mathrm{Hess}\, l_p)_p$ is negative definite, the eigenvalues of A_{σ_p}, i.e., the principal curvatures of M at p in the direction σ_p, are negative:

$$k_i(p, \sigma_p) < 0. \tag{3.1.14}$$

Note that the important point here is that all principal curvatures have the same sign. The actual sign, whether it is positive or negative, is a matter of convention. More concretely, if we replace σ_p by $-\sigma_p$ then all k_i's will be positive.

3.2 Tubular neighborhoods, distance functions, and the end point map

Let $M^n \subset \mathbb{R}^m$ be a $C^{k\geqslant 2}$ compact embedded submanifold. As always we assume that M may have a boundary. The *normal bundle* of M is defined by

$$NM := \{\, (x,v) \mid x \in M, v \in T_xM^{\perp} \,\}.$$

It is well known that NM is a manifold of dimension m with a canonical C^{k-1} differentiable structure induced by the differential structure of M. We define the *end point map*, end: $NM \to \mathbb{R}^m$ by

$$\mathrm{end}(x,v) := x + v.$$

Since we consider both M and T_xM as subsets of $\mathbb{R}^m$, x and v are vectors in $\mathbb{R}^m$; thus, the above is well-defined. As is well known, $(x,0)$ is a regular value of end. To see this, let $X \in T_{(x,0)}NM$, and suppose $\gamma\colon (-\epsilon,\epsilon) \to NM$, $t \overset{\gamma}{\longmapsto} (x(t), v(t))$, is a curve with $\gamma(0) = (x,0)$ and $\gamma'(0) = X$. Then $\mathrm{end}_*(X) = (\mathrm{end}\circ\gamma)'(0) = x'(0) + v'(0)$. Now since $v(t) \in T_{x(t)}M^{\perp}$, we have $\langle x'(t), v(t)\rangle = 0$ which yields $\langle x'(t), v'(t)\rangle = -\langle x''(t), v(t)\rangle$; therefore, $\langle x'(0), v'(0)\rangle = -\langle x''(0), 0\rangle = 0$. Thus $x'(0)$ and $v'(0)$ are perpendicular; so, $\mathrm{end}_*(X_p) = 0$ if and only if $x'(0) = v'(0) = 0$; i.e., if and only if $X = 0$; thus $(x,0)$ is a regular value. This together with the inverse function theorem, and the fact that $\mathrm{end}\big|_{M\times\{0\}}$ is one-to-one, yields the *tubular neighborhood theorem* which states that end maps an open neighborhood of $M \times \{0\}$ diffeomorphically into $\mathbb{R}^m$. More precisely, let

$$N_\epsilon M := \{\, (x,v) \in NM \mid \|v\| < \epsilon \,\}, \quad \text{and} \quad Tube_\epsilon M := \mathrm{end}(N_\epsilon M),$$

then there exists an $\epsilon > 0$ such that

$$N_\epsilon M \overset{\mathrm{end}}{\longmapsto} Tube_\epsilon M$$

is a C^{k-1} diffeomorphism. In particular, assuming ϵ is sufficiently small, set

$$B_rM := \{\, (x,v) \in NM \mid \|v\| = r \,\}, \quad \textit{and} \quad S_rM := \mathrm{end}(B_rM),$$

where $0 < r < \epsilon$; then, S_rM is a C^{k-1} embedded hypersurface of $\mathbb{R}^m$, since B_rM is an $m-1$ dimensional submanifold of $N_\epsilon M$. We claim, however, that S_rM possesses a higher degree of regularity. To see this let $d\colon \mathbb{R}^m \to \mathbb{R}$ be the distance function of M, i.e.,

$$d(p) := \mathrm{dist}(p, M).$$

We claim that $S_rM = d^{-1}(r)$, and d, restricted to $Tube_\epsilon M - M$ is a C^k submersion. This would show that S_rM is in fact C^k. To prove these assertions, we first define a pair of mappings x, $v\colon Tube_\epsilon M \to M$ by

$$x(p) := \pi_1(\mathrm{end}^{-1}(p)), \quad \textit{and} \quad v(p) := \pi_2(\mathrm{end}^{-1}(p)),$$

where $NM \ni (x,v) \stackrel{\pi_1}{\longmapsto} x \in M$, and $NM \ni (x,v) \stackrel{\pi_2}{\longmapsto} v \in \mathbb{R}^m$. Clearly, x and v are C^{k-1}. Furthermore, since $x(p) + v(p) = \mathrm{end}(x(p), v(p)) = p$, it follows that

$$v(p) = p - x(p),$$

i.e., $p - x(p)$ is perpendicular to M at $x(p)$. This shows that $x(p)$ is a candidate for being the closest point of M to p, and since for every point p, in $Tube_\epsilon M$, there is only one point $x(p) \in M$ such that $p - x(p)$ is perpendicular to P, we conclude that $x(p)$ is indeed the closest point of M to p, i.e., $d(p) = \|p - x(p)\|$, or equivalently

$$d(p) = \|v(p)\|.$$

The above equation shows that $d\big|_{Tube_\epsilon M - M}$ is C^{k-1}; therefore, the gradient of d is well-defined for all $p \in Tube_\epsilon M - M$. We claim that

$$(\mathrm{grad}\, d)_p = \frac{v(p)}{\|v(p)\|}.$$

The above equation would show that $\operatorname{grad} d$ is C^{k-1}, and, therefore, would imply that d must be C^k.

To prove the above we need the *generalized Gauss' lemma* which states that for every $p \in S_r M$, $v(p)$ is perpendicular to $S_r M$ at p; therefore, since $S_r M$ is a level hypersurface of d, $(\operatorname{grad} d)_p$ must be parallel to $v(p)$. Thus $(\operatorname{grad} d)_p = \langle (\operatorname{grad} d)_p, v(p) \rangle \frac{v(p)}{\|v(p)\|}$. Let $\gamma \colon (-\epsilon, \epsilon) \to \mathbb{R}^m$ be a curve such that $\gamma(0) = p$ and $\gamma'(0) = v(p)$, e.g., let $\gamma(t) := \operatorname{end}(p, tv(p)) = p + tv(p)$, then $\langle \operatorname{grad} d_p, v(p) \rangle = v_p(d) = (d \circ \gamma)'(0)$. Now $d \circ \gamma(t) = \|\gamma(t) - x(\gamma(t))\| = \|\gamma(t) - x(p)\| = \|p + t(p - x(p)) - x(p)\| = (1+t)\|p - x(p)\|$. Thus $d \circ \gamma'(0) = 1$ which yields the above formula. Hence we conclude that

$$d \in C^k(Tube_\epsilon M - M).$$

3.3 Ovaloids, their support functions, and a Monge-Ampére equation

By an *ovaloid* we shall always mean a closed hypersurface with bounded radii of curvature, i.e., we require that there be positive constants r and R such that through every point of $p \in O$ there passes a sphere S of radius R and a sphere s of radius r such that a neighborhood of p lies inside S and outside s. By a generalization of Blaschke's rolling theorem due to J. Brooks & J. Strantzen [BS], if this (local) condition is satisfied, then S and s lie, respectively, entirely outside and entirely inside O. In other words, s rolls freely inside O and O rolls freely inside S. This definition has the advantage that it presupposes no degree of differentiability on the part of O. The following are its immediate consequences.

First, through every point $p \in O$ there passes a strictly supporting hyperplane. More explicitly, let H_p be the hyperplane tangent to the external supporting sphere

S at p, then O, being contained inside S, lies strictly on one side of H_p.

Secondly, at every point $p \in M$, there passes only one support hyperplane, because every supporting hyperplane through p must be tangent to some fixed internal supporting sphere s at p. This implies that every ovaloid is necessarily C^1.

Thirdly, if $p \in O$ has a neighborhood which admits a C^2 parameterization, then it can be shown that the principal radii of curvature of O at p, being well-defined, are pinched between r and R; therefore, a C^2 ovaloid has everywhere positive curvature.

Now note that, conversely, if a closed hypersurface M has positive curvature everywhere then its radii of curvature are bounded above and below by some positive constants R and r. By the theorem of Brooks & Strantzen (or some earlier rolling theorems for the smooth case due to to D. Koutroufiotis [Kft], J. Rauch [Rch], or J. Delgado [Del]) this implies that through every point of M there passes a sphere of radius R containing M and a sphere of radius r contained in M; i.e., M must be an ovaloid.

Thus we conclude that our present definition for ovaloid is consistent with the previous one (i.e. a closed hypersurface of positive curvature) and, at the same time, allows us to talk about a C^1 ovaloid.

Now let $\sigma\colon O \to \mathbb{S}^{m-1}$ be the outward unit normal, also know as the *Gauss map*, of O. It is well known that if O is C^k then σ is a C^{k-1} diffeomorphism. Define the support function $h\colon \mathbb{R}^m - \{0\} \to \mathbb{R}$ of O by

$$h(p) := \langle \sigma^{-1}(\frac{p}{\|p\|}), p \rangle.$$

Intuitively, we may think of $h(p)$ as the signed distance between the support hyperplane $H_{\sigma^{-1}(\frac{p}{\|p\|})}$ and the origin. The above equation shows that h is positively homogeneous of degree one, i.e., for every $\lambda > 0$, $h(\lambda p) = \lambda h(p)$.

By looking at the above equation, and noting that σ is C^{k-1}, it would seem that h is also C^{k-1}; however, it turns out that h is actually C^k. To see this let, $p = (x_1 \dots, x_m)$ and assume $\sigma^{-1}(\frac{p}{\|p\|}) = (y_1, \dots, y_m)$; then, $h(p) = \sum_{i=1}^m x_i y_i$, and consequently $\frac{\partial h}{\partial x_i} = y_i$. Thus

$$(\operatorname{grad} h)_p = \sigma^{-1}(\frac{p}{\|p\|})$$

which shows that $\operatorname{grad} h$ is C^{k-1}; therefore, h must be C^k.

Now, assuming that O is C^2, the Hessian of h is well-defined. We claim that for every $p \in \mathbb{R}^m - \{0\}$ there is an orthonormal basis $\{E_p^1, \dots, E_p^m\}$ with respect to which the matrix representation of the Hessian of h at p is of the form

$$\begin{bmatrix} r_1 & & & \\ & \ddots & \mathbf{0} & \\ & \mathbf{0} & r_{m-1} & \\ & & & 0 \end{bmatrix},$$

where $r_i = r_i(\sigma^{-1}(\frac{p}{\|p\|})) := 1/k_i(\sigma^{-1}(\frac{p}{\|p\|})) > 0$ are the principal radii of curvature of O at $\sigma^{-1}(\frac{p}{\|p\|})$. Thus h is a convex function because its Hessian is positive semidefinite everywhere.

To see the above, let $E_p^m := \frac{p}{\|p\|}$, and, for $1 \leqslant i \leqslant m-1$, let $E_p^i := X^i_{\sigma^{-1}(\frac{p}{\|p\|})}$, the principal directions of O at $\sigma^{-1}(\frac{p}{\|p\|})$. $(\operatorname{Hess} h)_p(E_p^i, E_p^j) = \langle \overline{\nabla}_{E_p^i}(\operatorname{grad} h), E_p^j \rangle = \langle \overline{\nabla}_{E_p^i}(\sigma^{-1} \circ \pi), E_p^j \rangle$, where $\pi\colon \mathbb{R}^m - \{0\} \to \mathbb{S}^{m-1}$ is the standard projection, i.e., $\pi(p) := \frac{p}{\|p\|}$. Moreover, $\overline{\nabla}_{E_p^i}(\sigma^{-1} \circ \pi) = (\sigma^{-1} \circ \pi)_*(E_p^i) = (\sigma_*)^{-1}(\pi_*(E_p^i))$. Now our result follows once we observe that $\pi_*(E_p^m) = 0$, $\pi_*(E_p^i) = E_p^i$ for $1 \leqslant i \leqslant m-1$, and $\sigma_*(E_p^i) = k_i E_p^i$. The latter holds because σ is the Gauss map.

Now for all $p \in \mathbb{S}^{m-1}$ let $\overline{h^p}$ denote the restriction of h to the hyperplane $T_p\mathbb{S}^{m-1}$. Since the second fundamental form of $T_p\mathbb{S}^{m-1}$ vanishes, it is easily shown , via Gauss' formula, that $(\operatorname{Hess} \overline{h^p})_p(X_p, Y_p) = (\operatorname{Hess} h)_p(X_p, Y_p)$ for all $X_p, Y_p \in T_p\mathbb{S}^{m-1}$. In particular, since E_p^i, $1 \leqslant i \leqslant m-1$ form a basis for $T_p\mathbb{S}^{m-1}$, it follows that the matrix representation of $(\mathrm{Hess}\,\overline{h^p})_p$ with respect to this basis is diagonal with the principal radii of curvature as the main entries. Thus we obtain the following Monge-Ampére equation:

$$\det((\mathrm{Hess}\,\overline{h^p})_p) = \frac{1}{K(\sigma^{-1}(p))},$$

where $K = k^1k^2\cdots k^{m-1}$ is the Gauss-Kronecker curvature of O, which by assumption is positive, provided the surface is oriented properly. Thus we conclude that, for every $p \in \mathbb{S}^{m-1}$, $\overline{h^p}$ is *strictly convex*, i.e., its Hessian is positive definite.

To summarize, thus far we have shown that to every $C^{k\geqslant 2}$ ovaloid $O \subset \mathbb{R}^m$ there is associated a C^k function $h\colon \mathbb{R}^m - \{0\} \to \mathbb{R}$, known as the support function of O, which is (i) C^k, (ii) positively homogeneous, and (iii) convex; moreover, (iv) the restriction of h to every hyperplane tangent to the sphere is strictly convex.

Now suppose that we are given a function $h\colon \mathbb{R}^m - \{0\} \to \mathbb{R}$ which satisfies the four properties mentioned above; then, there exists a unique C^k ovaloid with support function h. To see this, define $f\colon \mathbb{S}^{m-1} \to \mathbb{R}^m$ by

$$f(p) := (\mathrm{grad}\,h)_p,$$

and set $O := f(\mathbb{S}^{m-1})$. We are going to show that O is the desired ovaloid.

First, it follows from our assumptions that , for every $p \in \mathbb{S}^{m-1}$ there exists a basis $\{E_p^1, \ldots, E_p^m\}$, with $E_p^m = \frac{p}{\|p\|}$ with respect to which the Hessian matrix of h is diagonal with all the main entries, except the last one, positive; therefore, since, for $1 \leqslant i, j \leqslant m-1$, $e_{ij} := \langle f_*(E_p^i), E_p^j\rangle = \langle (\mathrm{grad}\,h)_*(E_i^p), E_j^p\rangle = \langle \overline{\nabla}_{E_p^i}(\mathrm{grad}\,h), E_p^j\rangle = (\mathrm{Hess}\,h)_p(E_p^i, E_p^j)$, we conclude that $e_{ii} > 0$ and all other terms are zero. Hence for every $X_p \in T_p\mathbb{S}^{m-1}$, $X = \sum_{i=1}^{m-1} c_i E_p^i$, we have $\langle f_*(X_p), X_p\rangle = \sum_{i=1}^{m-1} c_i c_j \langle f_*(X_i), X_j\rangle = \sum_{i=1}^{m-1} c_i^2 e_{ii} \neq 0$. Thus $f_*(X_p) \neq 0$, if $X_p \neq 0$. So f is an immersion and therefore O is a closed immersed hypersurface which so far appears to be of class C^{k-1}.

Next, we are going to show that O is strictly convex and C^k. Let $\sigma\colon O \to \mathbb{S}^{m-1}$ be defined by

$$\sigma(f(p)) := p.$$

We claim that σ is the Gauss map of O. To see this, let $Y_{f(p)} \in T_{f(p)}O$, then $Y_{f(p)} = f_*(X_p)$ for some $X_p \in T_p\mathbb{S}^{m-1}$. Hence $\langle \sigma(f(p)), Y_{f(p)}\rangle = \langle p, f_*(X_p)\rangle = \langle E_p^m, (\operatorname{grad} h)_*(X_p)\rangle = \sum_{i=1}^{m-1} c_i \langle E_p^m, (\operatorname{grad} h)_*(E_p^i)\rangle = \sum_{i=1}^{m-1} c_i \langle E_p^m, \overline{\nabla}_{E_p^i}(\operatorname{grad} h)\rangle = \sum_{i=1}^{m-1} c_i (\operatorname{Hess} h)_p(E_p^m, E_p^i) = 0$. Thus we conclude that

$$\sigma(f(p)) \in T_{f(p)}O^{\perp}.$$

Now since $\sigma_* \circ f_* = id$ and f_* is an immersion, it follows that σ is also an immersion and therefore O must have everywhere positive curvature. Thus, by Hadamard's theorem O must be an ovaloid. Furthermore, since the Gauss map of O is C^{k-1}, by construction, it follows that O is C^k.

3.4 Convex bodies, generalized curvature, and Schneider's transform

Let $\mathcal{K}^m$ denote the space of *convex bodies* is $\mathbb{R}^m$, i.e., compact convex subsets with nonempty interior. $\mathcal{K}^m$ is closed under Minkowski sum and scalar product. Furthermore, $(\mathcal{K}^m, \operatorname{dist})$ is a complete locally compact metric space, where "dist" denotes the Hausdorff distance. To every $K \in \mathcal{K}^m$, there is associated a *support function* $h_K\colon \mathbb{R}^m \to \mathbb{R}$ defined by

$$h_K(p) := \sup\{\langle p, x\rangle \mid x \in K\}.$$

For every $p \in \mathbb{S}^{m-1}$, $h_K(p)$ is the signed distance from the support hyperplane of K, with outward unit normal p, to the origin. It is easily shown that $h_{(K+K')} = h_K + h_{K'}$ and $h_{(rK)} = r h_K$ for all $K, K' \in \mathcal{K}^m$, and $r > 0$.

We say a convex body K *rolls freely* inside a convex body L, if for every $p \in \operatorname{bd} L$, there is a rigid motion $g : \mathbb{R}^m \to \mathbb{R}^m$, ($g \in SO(m) \times \mathbb{R}^m$) such that $g(K) \subset L$ and $p \in g(K)$.

We say the radii of curvature of K are *bounded below* by r if there exists a ball of radius r rolling freely inside K. Similarly, we say the radii of curvature of K are *bounded above* by R if K rolls freely inside a ball of radius R. If the radii of curvature of K are bounded above and below then we say that the boundary of K has positive curvature (in a weak sense).

Next we are going to describe an operation for smoothing convex bodies. This is due to R. Schneider, see [Sch1] and [Sch2]. Let $\theta_\epsilon \colon [0, \infty) \to [0, \infty)$ be a smooth function with $supp(\theta_\epsilon) \subset [\epsilon/2, \epsilon]$, and $\int_{\mathbb{R}^m} \theta_\epsilon(\|x\|)dx = 1$. Then the *Schneider convolution* of h is defined by

$$\widetilde{h}^\epsilon(p) := \int_{\mathbb{R}^m} h(p + \|p\|x)\theta_\epsilon(\|x\|)dx.$$

Note that the above is just the standard convolution of h, with a kernel function θ_ϵ, plus a minor adjustment to ensure that the outcome is homogeneous. It is well known that the convolution of a convex function is convex, thus $\widetilde{h}^\epsilon$ is a support function. This convolution defines a transformation $T_\epsilon : \mathcal{K}^m \to \mathcal{K}^m$ by

$$h_{T_\epsilon(K)} := \widetilde{h}^\epsilon{}_K.$$

This transformation has the property that $\operatorname{dist}(K, T_\epsilon(K)) \leqslant \epsilon$. Also it turns out that $T_\epsilon(K + K') = T_\epsilon(K) + T_\epsilon(K')$, $T_\epsilon(\lambda K) = \lambda T_\epsilon(K)$ when $\lambda > 0$, $T_\epsilon(g(K)) = g(T_\epsilon(K))$ for every rigid motion g, and if $K \subset L$, then $T_\epsilon(K) \subset T_\epsilon(L)$. From these properties if follows that if the radii of curvature of K are pinched between r and R then the radii of curvature of $T_\epsilon(K)$ must have the same bounds. To see this, observe that $T_\epsilon(rB^m) = rT_\epsilon(B^m) = rB^m$, where B^m denotes the unit ball in $\mathbb{R}^m$; also, note that if K rolls freely in L then $T_\epsilon(K)$ rolls freely in $T_\epsilon(L)$.

4 Proof of the Main Theorem

Our principal aim here is to prove the first sentence in the statement of the main theorem, i.e., that there exists a smooth solution containing the given data. The proof is divided into four propositions, corresponding to the steps outlined in Section 1.3. These are proved in the next four subsections. Each proposition uses a number of lemmas whose statements and proofs appear at the end of the proof of the corresponding proposition.

The additional information given in items 1 to 3 of the statement of the main theorem, is primarily a by product of our method of construction, and is obtained without too much extra effort. It is only the last sentence in the statement of the main theorem whose proof requires considerable additional work; in order to ensure that the solution has the same degree of regularity as the given data (C^k), we need to construct a C^{k-1} nonsingular support which is *proper* (c.f. step 1 in Section 1.3). Without this additional requirement, the solution will be C^{k-1}.

Item 4, in the statement of the main theorem, i.e., the existence of a solution which inherits the symmetries of the given data, is easy to show once we establish the existence of any solution. The argument is as follows. Let $M \subset \mathbb{R}^m$ be a strictly convex submanifold. Suppose there exists an ovaloid O containing M. Choose the origin of the coordinate system inside O. Suppose M is symmetric with respect to some orthogonal transformation $g \in \mathcal{O}(m)$, i.e., $g(M) = M$. We wish to show that there exists an ovaloid $\overline{O}$, containing M, such that $g(\overline{O}) = \overline{O}$. To this end, let $\rho \colon \mathbb{S}^{m-1} \to \mathbb{R}$ be the function whose (radial) graph is O, i.e., suppose $O = \{\rho(x)x \mid x \in \mathbb{S}^{m-1}\}$. Set $\overline{\rho} := (\rho + \rho \circ g^{-1})/2$. Then the graph of $\overline{\rho}$ is the desired ovaloid. This follows from observing that the Hessian of a spherical function is positive definite if, and only if, the corresponding radial graph has positive curvature.

4.1 Construction of a C^{k-1} proper nonsingular support

Here we prove:

4.1.1. Proposition. *Every compact $C^{k\geqslant 2}$ embedded strictly convex submanifold $M \subset \mathbb{R}^m$ admits a C^{k-1} proper nonsingular support.*

Proof. By assumption, for every $p \in M$ there exists a nonsingular support vector $\xi_p \in \mathbb{S}^{m-1}$. Recall that this means that p is the strict absolute maximum and a nondegenerate critical point of the height function $M \ni x \mapsto \langle x, \xi_p \rangle \in \mathbb{R}$.

We want to show that given any finite number of distinct points $x_j \in M$, $1 \leqslant j \leqslant n$, there exists a mapping $\sigma : M \to \mathbb{S}^{m-1}$ such that:

(i) σ is C^{k-1},

(ii) $\sigma(p)$ is a nonsingular support vector, for every $p \in M$,

(iii) there exists an $\epsilon_M > 0$ such that $M_\epsilon := \{p + \epsilon\sigma(p) \mid p \in M\}$ is a C^k submanifold for every $|\epsilon| < \epsilon_M$, and, finally,

(iv) $\sigma(x_j) = \xi_{x_j}$, for all $1 \leqslant j \leqslant n$.

The proof consists of two parts. In part I, we show that the above requirements follow from a set of local conditions. In part II we prove the local conditions.

Part I. We claim that to prove $\{$(i)$\cdots$(iv)$\}$, it is sufficient to show that for every nonsingular support vector $\xi_p \in \mathbb{S}^{m-1}$, there exists a pair (U^p, σ^p) where U^p is an open neighborhood of p and $\sigma^p : U^p \to \mathbb{S}^{m-1}$ is a mapping satisfying the following properties:

(i)' σ^p is C^{k-1},

(ii)' $\sigma^p(q)$ is a nonsingular support vector, for every $q \in U^p$, (iii)' For every U^p there exists an $\epsilon_{U^p} > 0$ such that $U^p_\epsilon := \{q + \epsilon\sigma(q) \mid q \in U^p\}$ is a C^k submanifold for every $|\epsilon| < \epsilon_{U^p}$, and, finally,

(iv)' $\sigma^p(p) = \xi_p$.

Assume that {(i)',... ,(iv)'} hold. We are now going to construct a mapping satisfying {(i),... ,(iv)}. First, observe that $\{U^p\}_{p\in M}$ forms an open covering for M, and, since M is compact, there exists a finite subcover $\mathcal{U} = \{U^{p_i}\}$, $1 \leqslant i \leqslant N$. Furthermore, it is easy to see that we can choose $\mathcal{U}$ so that $U^{x_j} \in \mathcal{U}$, for every $1 \leqslant j \leqslant n$, and, also, we can require that U^{x_j} be the only neighborhood in $\mathcal{U}$ containing x_j. Now let $\{\phi_i\}$ be a C^k partition of unity subordinate to $\mathcal{U}$, and set

$$\sigma(p) := \frac{\sum_{i=1}^N \phi_i(p)\sigma^{p_i}(p)}{\|\sum_{i=1}^N \phi_i(p)\sigma^{p_i}(p)\|}.$$

We claim that the mapping $\sigma\colon M \to \mathbb{S}^{m-1}$, $p \overset{\sigma}{\mapsto} \sigma(p)$, is well-defined and satisfies (i), (ii), (iv), and, with some care, (iii) as well. $\sigma(p)$ is well-defined and is in fact a nonsingular support vector by Lemma 4.1.2; thus, (ii) is satisfied. (i) is an immediate consequence of (i)', via the assumption that $\phi_i's$ are C^k. (iv) follows from (iv)' and the specific way we constructed the subcover $\mathcal{U}$, i.e., the assumption that each x_i is covered only once. Thus, it remains to show that σ satisfies (iii).

First, let $f\colon M \to \mathbb{R}^m$ be given by

$$f(p) := p + \epsilon\sigma(p),$$

then f is clearly C^{k-1}. Furthermore, we claim that f is an embedding. To see this, observe that $f(p) = \mathrm{end}(p, \epsilon\sigma(p))$, where $\mathrm{end}\colon NM \to \mathbb{R}^m$, $(p, v) \overset{\mathrm{end}}{\longmapsto} p + v$, is the end-point map. By the tubular neighborhood theorem, since M is compact, end is an embedding for small ϵ. Furthermore, by Lemma 4.1.3, σ is also an embedding which implies that so is the mapping $p \mapsto (p, \epsilon\sigma(p))$, and consequently f. So we conclude that, since $M_\epsilon = f(M)$, M_ϵ is at least a C^{k-1} embedded submanifold.

Now, suppose $p \in M$ is covered only by a single neighborhood $U^{p_k} \in \mathcal{U}$, then, by (iii)', it follows that M_ϵ is C^k in a neighborhood of $f(p)$, because $f(p) \subset U_\epsilon^{p_k} \subset M_\epsilon$. On the other hand, if p is covered by more than one neighborhood $U^{p_{k_i}} \in \mathcal{U}$, then $f(p)$ is contained in a neighborhood which is a blending of C^k manifolds $U_\epsilon^{p_{k_i}}$. This blending occurs in $Tube_\epsilon M$ which is also C^k by Lemma 4.1.8, and is completely determined by the choice of the partition of unity $\{\phi_i\}$. We claim, therefore, that it should be possible to choose $\{\phi_i\}$ so that the blending of overlapping neighborhoods, and consequently M_ϵ, is C^k everywhere. This can be done, for instance, by an appropriate smoothing of f. We show this explicitly only for the local case (iii)' in part II.2 of this proof.

Part II. In this part we show that for every $p \in M$ there exists an open neighborhood U^p and a mapping $\sigma^p \colon U^p \to \mathbb{S}^{m-1}$ which satisfies {(i)'$\cdots$(iv)'}. We do this in two steps: in II.1 we prove that, for every $p \in M$, there exists a pair (U^p, σ^p) which satisfies all the required properties except the third; and, in II.2 we use the pair constructed in II.1 to construct another pair which satisfies all four properties.

II.1. Let ξ_p be a nonsingular support vector for M at p. By Lemma 4.1.4, ξ_p may be extended locally to a C^{k-1} unit normal vector field. More explicitly, there exists an open neighborhood V^p of p and a mapping $\sigma^p \colon V^p \to \mathbb{S}^{m-1}$, such that $\sigma^p(p) = \xi_p$, $\sigma^p(q) \in T_qM^\perp$ for every $q \in V^p$, and σ^p is C^{k-1}. We claim that there exists an open neighborhood $U^p \subset V^p$, $p \in U^p$, such that (U^p, σ^p) satisfies {(i)',(ii)',(iv)'}. Of course, (i)' and (iv)' are immediate, so it remains only to show (ii)'.

To prove (ii)', let $l_q \colon M \to \mathbb{R}$ be the height function given by

$$l_q(x) = \langle x, \sigma^p(q) \rangle,$$

and recall that we have to show three things: (1) $l_q(x) < l_q(q)$ for all $x \in M - \{q\}$, (2) $(\operatorname{grad} l_q)_q = 0$, and (3) $(\operatorname{Hess} l_q)_q \neq 0$. (2) follows because, by construction, $\sigma^p(q) \in T_qM^{\perp}$, and (3) follows by continuity of σ^p and the fact that $(\operatorname{Hess} l_p)_p < 0$, see Lemma 4.1.5. Thus it remains to show (1).

To see (1), let $L\colon V^p \times M \to \mathbb{R}$ be defined by

$$L(q,x) := l_q(x) - l_q(q).$$

We have to show that there exists an open neighborhood $U^p \subset V^p$, $U^p \ni p$, such that $L\big|_A < 0$, where $A := U^p \times M - \triangle(U^p \times M)$; or, equivalently

$$A = \{\,(q,x) \mid q \in U^p,\ x \in M,\ \&\, q \neq x\,\}.$$

We do this by partitioning A into a pair of subsets:

$$B_r := \{\,(q,x) \in A \mid \operatorname{dist}(q,x) < r\,\}, \quad \textit{and} \quad C_r := \{\,(q,x) \in A \mid \operatorname{dist}(q,x) \geqslant r\,\},$$

where "dist" denotes here the standard intrinsic distance in M, i.e., the one arising from the metric induced by the ambient space $\mathbb{R}^m$. First, we show that there exists an $r > 0$ such that $L\big|_{B_r} < 0$. This follows by means of a Taylor expansion, and continuity of σ^p, see Lemma 4.1.6.

So it remains to show that $L\big|_{C_r} < 0$. To see this, let

$$D := \{\,(p,x) \mid x \in M,\ \&\ \operatorname{dist}(x,p) \geqslant \frac{r}{2}\,\}.$$

Since $\sigma^p(p) = \xi_p$ which is a nonsingular support vector by assumption, we have $l_p(x) < l_p(p)$ for all $x \in M - \{p\}$. In particular, $L\big|_D < 0$; therefore, by compactness of D and continuity of L, L must be negative over some open neighborhood of D with some radius $\delta < \frac{r}{2}$. Now suppose that U^p is small enough so that its radius is less than δ, i.e.,

$$U_p := \{\, x \in M \mid \operatorname{dist}(x,p) < \delta \,\}.$$

We are going to show that $C_r \subset U_\delta(D)$, which is all we need.

To see that $C_r \subset U_\delta(D)$, let $(q_0, x_0) \in C_r$, then $\operatorname{dist}(q_0, x_0) > r$, by the definition of C_r. Moreover, $\operatorname{dist}(q_0, p) < \delta$, by definition of C_r, and because the radius of U^p is less that δ, by assumption. Now $\operatorname{dist}(x_0, p) \geqslant \operatorname{dist}(q_0, x_0) - \operatorname{dist}(p, q_0) \geqslant r - \delta > \frac{r}{2}$. Thus, $(p, x_0) \in D$; and, therefore,

$$\overline{\operatorname{dist}}((q_0, x_0), D) \leqslant \overline{\operatorname{dist}}((q_0, x_0), (p, x_0)) = \operatorname{dist}(q_0, p) < \delta,$$

where $\overline{\operatorname{dist}}\colon M \times M \to \mathbb{R}$ is given by by $\overline{\operatorname{dist}}((p,q),(p',q')) := (\operatorname{dist}(p,p') + \operatorname{dist}(q,q'))^{\frac{1}{2}}$. So we conclude that $(q_0, x_0) \in U_\delta(D)$, which implies $C_r \subset U_\delta(D)$, as we had claimed earlier.

II.2 We showed in II.1 that there exists a pair (U^p, σ^p) which satisfies (i)', (ii)', and (iv)'. Here we prove that there exists a pair $(\widetilde{U}^p, \widetilde{\sigma}^p)$ which satisfies all four properties {(i)'$\cdots$(iv)'}.

First, define $\overline{\sigma}^p\colon U^p \to BM$ by

$$q \overset{\overline{\sigma}^p}{\longmapsto} (q, \sigma^p(q)),$$

where BM denotes the unit normal bundle of M. Also, define $\operatorname{end}_\epsilon\colon BM \to Tube_\epsilon M$ by

$$(q, v) \overset{\operatorname{end}_\epsilon}{\longmapsto} q + \epsilon v,$$

where $|\epsilon|$ is small. Now set $f^p := \operatorname{end}_\epsilon \circ \overline{\sigma}^p$, then $f^p\colon U^p \to Tube_\epsilon M$, and we have

$$q \overset{f_p}{\longmapsto} q + \epsilon \sigma^p(q).$$

Note that by the tubular neighborhood theorem, and Lemma 4.1.3, f^p is a C^{k-1} embedding. Also note that, by Lemma 4.1.8, $Tube_\epsilon M$ is C^k; therefore, by Lemma

4.1.7, if U^p is sufficiently small, then there exists a C^k embedding $\widetilde{f}^p\colon U^p \to Tube_\epsilon M$ such that

$$\widetilde{f}^p(p) = f^p(p), \quad and \quad \widetilde{f}^p_{*_p} = f^p_{*_p}.$$

Let $\pi_1\colon BM \to M$, be the standard projection map, i.e., $(q,v) \overset{\pi_1}{\longmapsto} q$, and set

$$x := \pi_1 \circ \mathrm{end}_\epsilon^{-1},$$

then $x\colon Tub_\epsilon M \to M$. Now let $\widetilde{x}$ denote the restriction of x to $\widetilde{f}^p(U^p)$, i.e.,

$$\widetilde{x} := x\Big|_{\widetilde{f}^p(U^p)},$$

and let $\widetilde{U}^p \subset M$ be given by $\widetilde{U}^p := x \circ \widetilde{f}(U^p)$, then $\widetilde{x}\colon \widetilde{f}^p(U^p) \to \widetilde{U}^p$. Finally, set

$$\widetilde{\sigma}^p(q) := \frac{1}{\epsilon}[\widetilde{x}^{-1}(q) - q].$$

We claim that, assuming U^p is sufficiently small, $\widetilde{\sigma}^p(q)$ is well-defined for all $q \in \widetilde{U}^p$. Furthermore, we claim that $\widetilde{\sigma}^p\colon \widetilde{U}^p \to \mathbb{S}^{m-1}$, and, finally, we show that $(\widetilde{\sigma}^p, \widetilde{U}^p)$ satisfies all four properties {(i)',$\cdots$,(iv)'}.

First, we show that $\widetilde{\sigma}^p(q)$ is well-defined for all $q \in \widetilde{U}^p$. To this end, we have to prove that $\widetilde{x}\colon \widetilde{f}^p(U^p) \to \widetilde{U}^p$ is invertible. To see this, let p_1, $p_2 \in \widetilde{f}^p(U^p)$, and suppose that $\widetilde{x}(p_1) = \widetilde{x}(p_2)$. We are going to show that $p_1 = p_2$. Now note that $p_1 = \widetilde{f}^p(q_1)$ and $p_2 = \widetilde{f}^p(q_2)$ for some q_1, $q_2 \in U^p$; therefore, since $\widetilde{x}(p_1) = \widetilde{x}(p_2)$, we have $\widetilde{x} \circ \widetilde{f}^p(q_1) = \widetilde{x} \circ \widetilde{f}^p(q_2)$, which implies that $x \circ \widetilde{f}^p(q_1) = x \circ \widetilde{f}^p(q_2)$. Thus, it is enough to show that $x \circ \widetilde{f}^p\colon U^p \to \widetilde{U}^p$ is one-to-one. We do this, via the inverse function theorem, by showing that $(x \circ \widetilde{f}^p)_{*_p}$ is nonsingular:

$$\begin{aligned}
(x \circ \widetilde{f}^p)_{*_p} &= x_{*_{\widetilde{f}^p(p)}} \circ \widetilde{f}^p_{*_p} = x_{*_{f^p(p)}} \circ f^p_{*_p} \\
&= (x \circ f^p)_{*_p} = (\pi_1 \circ \mathrm{end}_\epsilon^{-1} \circ \mathrm{end}_\epsilon \circ \overline{\sigma}^p)_{*_p} \\
&= (\pi_1 \circ \overline{\sigma}^p)_{*_p} = (id)_{*_p} = I,
\end{aligned}$$

where id denotes the identity map and I the identity matrix. Thus, we conclude that $\det((x \circ \widetilde{f}^p)_{*_p}) = 1 \neq 0$; therefore, $x \circ \widetilde{f}^p \colon U^p \to \widetilde{U}^p$ is one-to-one, assuming U^p is small. In particular $\widetilde{x}^{-1} \colon \widetilde{U}^p \to \widetilde{f}^p(U^p)$ is well-defined.

Secondly, we show that $\widetilde{\sigma}^p \colon \widetilde{U}^p \to \mathbb{S}^{m-1}$, i.e., $\|\widetilde{\sigma}^p(q)\| = 1$ for all $q \in \widetilde{U}^p$. To see this, note that, by definition, $\|\widetilde{\sigma}^p(q)\| = \frac{1}{\epsilon}\|\widetilde{x}^{-1}(q) - q\|$; thus, it is sufficient to show that $\widetilde{x}^{-1}(q) = q + \epsilon v$ for some unit vector v. To this end, recall that, by definition, $\widetilde{x}^{-1}(q) \in Tube_\epsilon M := \mathrm{end}_\epsilon(BM)$; therefore, $\widetilde{x}^{-1}(q) = \mathrm{end}_\epsilon(q', v_{q'})$ for some $q' \in \widetilde{U}^p$, and $v_{q'} \in T_{q'}M^\perp$. This implies that $q = x \circ \mathrm{end}_\epsilon(q', v_{q'}) = \pi_1 \circ \mathrm{end}_\epsilon^{-1} \circ \mathrm{end}_\epsilon(q', v_{q'}) = q'$; therefore, for every $q \in \widetilde{U}^p$, there exists a unit vector $v_q \in T_qM^\perp$ such that

$$\widetilde{x}^{-1}(q) = \mathrm{end}_\epsilon(q, v_q) = q + \epsilon v_q.$$

In particular, $\|\widetilde{\sigma}^p(q)\| = \|v_q\| = 1$.

Now that $\widetilde{\sigma}^p \colon \widetilde{U}^p \to \mathbb{S}^{m-1}$ is well-defined, we show that $(\widetilde{\sigma}^p, \widetilde{U}^p)$ satisfies the four properties {(i)'$\cdots$(iv)'}:

(i)'. From the definition of $\widetilde{\sigma}^p(q)$ it follows that $\epsilon\widetilde{\sigma}^p(q) + q = \widetilde{x}^{-1}(q)$; therefore, $\widetilde{\sigma}^p$ must be C^{k-1}, because, as we showed earlier, x is a C^{k-1} embedding.

(ii)'. As we showed earlier, $\widetilde{x}^{-1}(q) = q + \epsilon v_q$ for some unit vector $v_q \in T_qM$; therefore, from the definition of $\widetilde{\sigma}^p$, it follows that $\widetilde{\sigma}^p = v_q$. In particular, $\widetilde{\sigma}$ is a local unit normal vector field. Thus, as we showed in II.1, to show that $\widetilde{\sigma}^p(q)$ is a nonsingular support vector, it suffices to show that $\widetilde{\sigma}^p(p)$ is a nonsingular support vector. This will be shown in part (iv)'.

(iii)'. $\widetilde{U}^p_\epsilon = \widetilde{f}(U^p)$, and $\widetilde{f}^p$ is a C^k embedding by construction; therefore, $\widetilde{U}^p_\epsilon$ is a C^k embedded submanifold.

(iv)'. By construction $\sigma^p(p) = \xi_p$, see II.1; thus it suffices to show that $\widetilde{\sigma}^p(p) = \sigma^p(p)$. First, note that by definition of f^p, $\sigma^p(p) = \frac{1}{\epsilon}(f^p(p) - p) = \frac{1}{\epsilon}(x^{-1} \circ x \circ f^p(p) - p)$. As we showed earlier, $x \circ f^p(p) = \pi_1 \circ \overline{\sigma}^p(p) = p$; therefore, $\sigma^p(p) =$

$\frac{1}{\epsilon}(\widetilde{x}^{-1}(p) - p) = \widetilde{\sigma}^p(p)$. ■

4.1.2. Lemma. *Let $\sigma_1 \ldots \sigma_N$ be nonsingular support vectors of M at a fixed point, then any normalized convex combination of $\sigma_1 \ldots \sigma_N$ is well-defined and is also a nonsingular support vector.*

Proof. Let $1 \leqslant i \leqslant N$, and set $l_i(x) = \langle x, \sigma_i \rangle$. By assumption, there exists a point $p \in M$ such that p is the strict absolute maximum and a nondegenerate critical point of the height function l_i. Let $\sigma := \sum_{i=1}^N c_i \sigma_i$, where $c_i \geqslant 0$ and $\sum_{i=1}^N c_i = 1$. We have to show that $\hat{\sigma} := \sigma / \|\sigma\|$ is well-defined and is a nonsingular support vector, i.e., p is the strict absolute maximum and a nondegenerate critical point of the height function $\hat{l}(x) := \langle x, \hat{\sigma} \rangle$.

First, we show that $\hat{\sigma}$ is well-defined, i.e., $\sigma \neq 0$. To see this let $l(x) := \langle x, \sigma \rangle$. It is enough to show that $l \not\equiv 0$. To see this let $x \in M - \{p\}$, then

$$\begin{aligned} l(x) &= \sum_{i=1}^N c_i \langle x, \sigma_i \rangle = \sum_{i=1}^N c_i l_i(x) \\ &< \sum_{i=1}^N c_i l_i(p) = \sum_{i=1}^N c_i \langle p, \sigma_i \rangle \\ &= \langle p, \sigma \rangle = l(p). \end{aligned}$$

So we conclude that $l \not\equiv 0$, which shows that $\hat{\sigma}$, and consequently $\hat{l}$, is well-defined.

Next we show that p is the strict absolute maximum of $\hat{l}$. To see this, let $x \in M - \{p\}$, then

$$\hat{l}(x) = \frac{1}{\|\sigma\|} l(x) < \frac{1}{\|\sigma\|} l(p) = \hat{l}(p).$$

So it only remains to show that p is a nondegenerate critical point of $\hat{l}$. This is an immediate consequence of the fact that $\hat{l} = \frac{1}{\|\sigma\|} \sum_{i=1}^N c_i l_i$, $c_i > 0$, and the assumption that p is a nondegenerate critical point of l_i. More explicitly, since, by assumption, $(\operatorname{grad} l_i)_p = 0$ and $(\operatorname{Hess} l_i)_p \neq 0$, it follows, by linearity of the operators grad and Hess, that $(\operatorname{grad} \hat{l})_p = 0$ and $(\operatorname{Hess} \hat{l})_p \neq 0$. ■

4.1.3. Lemma. *Let $M \subset \mathbb{R}^m$ be a compact embedded $C^{k\geqslant 2}$ submanifold, and suppose $\sigma\colon M \to \mathbb{S}^{m-1}$ is a C^l nonsingular support, $0 \leqslant l \leqslant k$, then σ is a C^l embedding.*

Proof. Since M is compact, it suffices to show that σ is a one-to-one immersion.

To see that σ is one-to-one, recall that, by assumption, $l_p(x) < l_p(p)$ for all $p \in M$ and $x \in M - \{p\}$, where $l_p(x) := \langle x, \sigma(p)\rangle$. Now let p, $q \in M$, and assume that $p \neq q$; then,

$$\begin{aligned}\langle q-p, \sigma(p)-\sigma(q)\rangle &= \langle q-p, \sigma(p)\rangle + \langle p-q, \sigma(q)\rangle \\ &= (l_p(q) - l_p(p)) + (l_q(p) - l_q(q)) \\ &< 0,\end{aligned}$$

which implies $\sigma(p) \neq \sigma(q)$.

So it remains to show that σ is an immersion. Let $p \in M$. We are going to prove that $\sigma_*(X_p) \neq 0$ for all $X_p \in T_pM - \{0\}$. To see this, let X_p^i be the principal directions of M at p with respect to $\sigma(p)$, and recall that

$$\begin{aligned}\langle \sigma_*(X_p^i), X_p^j\rangle &= \langle \overline{\nabla}_{X_p^i}\sigma, X_j\rangle = \langle (\overline{\nabla}_{X_p^i}\sigma)^\top, X_j\rangle \\ &= -\langle A_{\sigma(p)}(X_i^p), X_j^p\rangle = \begin{cases} -k_i(p,\sigma(p)) & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}\end{aligned}$$

Now if $X_p \in T_pM$, then $X_p = \sum_{i=1}^n c_i X_i$, for some $c_i \in \mathbb{R}$. Consequently, it follows that

$$\langle \sigma_*(X_p), X_p\rangle = \sum_{i=1}^n \sum_{j=1}^n c_i c_j \langle \sigma_*(X_p^i), X_p^j\rangle = -\sum_{i=1}^n c_i^2 k_i(p,\sigma(p)).$$

Thus we conclude that $\sigma_*(X_p) \neq 0$, whenever $X_p \neq 0$; because, as we showed in the preliminaries, the assumption that $\sigma(p)$ is a nonsingular support vector implies that $k_i(p, \sigma(p)) < 0$. ■

4.1.4. Lemma. *Let $M \subset \mathbb{R}^m$ be a $C^{k\geqslant 1}$ embedded submanifold, then any unit vector normal to M may be extended locally to a C^{k-1} unit normal vector field.*

Proof. Let $\xi_p \in \mathbb{S}^{m-1}$ be a unit vector which is perpendicular to M at p, i.e., $\sigma_p \in T_pM^{\perp}$. We are going to show that there exists an open neighborhood $U \subset M$, $p \in U$, and a C^{k-1} function $\sigma\colon U \to \mathbb{S}^{m-1}$ such that $\sigma(p) = \xi_p$, and $\sigma(q) \in T_qM$ for all $q \in U$.

Let $f\colon U \to \mathbb{R}^m$ be defined by

$$f(q) := \mathrm{Proj}_{T_qM^{\perp}}(\xi_p),$$

i.e., the projection of ξ_p into the normal space of M at q, and set

$$\sigma(q) := \frac{f(q)}{\|f(q)\|}.$$

We claim that the mapping $q \overset{\sigma}{\mapsto} \sigma(q)$ is the desired vector field provided U is sufficiently small.

First note that, clearly, $\sigma(p) = \xi_p$, and, if σ is well-defined, $\sigma(q) \in T_qM$. Thus, it remains to check that σ is well-defined and C^{k-1}. To this end, it is enough to show that $f(q) \neq 0$ for all $q \in U$, and f is C^{k-1}.

Secondly recall that, since M is C^k, there exists a C^{k-1} linearly independent local vector frame $q \overset{X^i}{\longmapsto} X^i_q$ defined on U. More explicitly, let (U, ϕ) be a C^k local chart for M centered at p; let e_i be the standard frame field on $\mathbb{R}^m$; and, define $X^i\colon U \to \mathbb{R}^m$ by

$$X^i(q) := (\phi^{-1})_{*_{\phi(q)}}(e_i(\phi(p))) = \frac{\partial(\phi^{-1})}{\partial x_i}(\phi(q)),$$

then X^i are linearly independent and C^{k-1}, because $\phi\colon U \to \phi(U) \subset \mathbb{R}^m$ is a C^k diffeomorphism by assumption.

Now a calculation shows that

$$f(q) = \xi_p - \sum_{i=1}^{n} \left(\sum_{j=1}^{n} g^{ij}(q) \langle \xi_p, X_q^j \rangle \right) X_q^i,$$

where g^{ij} are the entries of the inverse of the matrix (g_{ij}), and $g_{ij}(p) := \langle X_p^i, X_p^j \rangle$ are the coefficients of the metric tensor of M at p. Clearly, f is C^{k-1}. In particular, f is continuous and, therefore, f does not vanish in a small neighborhood U of p, because $f(p) = \xi_p \neq 0$. ■

4.1.5. Lemma. *Let $M \subset \mathbb{R}^m$ be a C^2 embedded submanifold, and suppose that H is a nonsingular support hyperplane of M at a point p; then, any continuous local extension of H to a distribution of tangent hyperplanes has contact of order one with M at a neighborhood of p.*

Proof. Let ξ_p be the outward unit normal of H, and, for every $x \in M$, set $l_p(x) := \langle x, \xi_p \rangle$. By assumption, p is a nondegenerate critical point and a local maximum of $l_p \colon M \to \mathbb{R}$. Recall that this implies, via Morse's lemma, that

$$(\operatorname{Hess} l_p)_p < 0,$$

i.e., the Hessian of l_p at p is negative definite. Let U be a neighborhood of p in M, and suppose we are given a continuous map $\sigma \colon U \to \mathbb{S}^{m-1}$ with $\sigma(p) := \xi_p$ and $\sigma(q) \in T_qM^{\perp}$, for all $q \in U$. We want to show that if U is small, then q is a nondegenerate critical point for the height function l_q, i.e., $(\operatorname{grad} l_q)_q = 0$, and $(\operatorname{Hess} l_q)_q \neq 0$ for all $q \in U$.

The fact that $(\operatorname{grad} l_q)_q = 0$, is an easy consequence of the assumption that $\sigma(q) \in T_qM^{\perp}$. So it remains to show that the Hessian has the required property. To this end note that since $q \mapsto \sigma(q)$ is continuous, by assumption, it follows that the mappings

$$U \ni q \longmapsto k_i(q, \sigma(q)) \in \mathbb{R}$$

are continuous as well, where $k_i(q,\sigma(q))$ are the principal curvatures of M at q with respect to the direction $\sigma(q)$. Thus, recalling that $k_i(q,\sigma(q))$ are the eigenvalues of $(\operatorname{Hess} l_q)_q$ and $k_i(p,\sigma(p)) < 0$ by assumption, we conclude that $k_i(q,\sigma(q)) < 0$ for all $q \in U$, assuming U is small. Consequently, $(\operatorname{Hess} l_q)_q$ is negative definite for all $q \in U$. In particular, $(\operatorname{Hess} l_q)_q \neq 0$. ■

4.1.6. Lemma. *Let $M \subset \mathbb{R}^m$ be a compact C^2 embedded submanifold; then, any continuous distribution of locally supporting hyperplanes along M uniformly locally strictly support M, if each hyperplane has contact of order one with M.*

Proof. By assumption, there exists a continuous unit normal vector field $\sigma\colon M \to \mathbb{S}^{m-1}$ such that for every $p \in M$, there exists a $\delta_p > 0$ such that

$$l_p(q) < l_p(p), \quad \forall q \in U_{\delta_p}(p),$$

where $l_p(q) := \langle p, \sigma(q)\rangle$, and $U_{\delta_p}(p) \subset M$ is an open neighborhood of p with radius δ_p. Furthermore, we have $(\operatorname{grad} l_p)_p = 0$ and $(\operatorname{Hess} l_p)_p \neq 0$ for all $p \in M$. Recall that these equations, via Morse's lemma, imply that $(\operatorname{Hess} l_p)_p < 0$ which in turn implies that the principal curvatures

$$k_i(p,\sigma(p)) < 0,$$

for all $p \in M$. We want to show that there exists a $\delta > 0$, independent of p, such that $l_p(q) < l_p(p)$ for all $q \in U_\delta(p)$.

To obtain this uniform estimate δ, identify a neighborhood of $p \in M$ with Euclidean n-space via normal coordinates; then, by Taylor's theorem we have

$$l_p(q) - l_p(p) = \frac{1}{2}(\operatorname{Hess} l_p)_p(p-q, p-q) + o_p(\|p-q\|^2).$$

First note that, since we are using normal coordinates, $\|p-q\| = \operatorname{dist}_M(p,q)$, so we can think of o_p as a function defined on M. Secondly, the mapping $p \mapsto (\operatorname{Hess} l_p)_p$

is continuous; because, the eigenvalues of $(\mathrm{Hess}\, l_p)_p$ are the principal curvatures $k_i(p, \sigma(p))$ which are continuous functions of p, by continuity of σ. We conclude, therefore, that the mapping $M \ni p \mapsto o_p \in C^0(M)$ is continuous. Now let

$$k := \sup_{p \in M}\{k_1(p, \sigma(p)), \dots , k_n(p, \sigma(p))\},$$

and note that, since M is compact, $k < 0$. Using this we can write

$$l_p(q) - l_p(p) \leqslant \frac{1}{2} k\|p - q\|^2 + o_p(\|p - q\|^2).$$

As we showed earlier, $p \mapsto o_p$ is continuous; therefore, there exists a $\delta > 0$, such that for all $p \in M$,

$$\frac{o_p(\|p - q\|^2)}{\|p - q\|^2} < \frac{-k}{2},$$

whenever $q \in U_\delta(p)$. So we conclude that $l_p(q) - l_p(p) < 0$ for all $p \in M$, and $q \in U_\delta(p)$. ■

4.1.7. Lemma. *Let M and N be C^k manifolds, $p \in M$, $U \subset M$ a small neighborhood of p, and $f\colon M \to N$ a C^{k-1} map. Then there exists a C^k map $g : U \to N$ such that $g(p) = f(p)$ and $f_{*_p} = g_{*_p}$.*

Proof. This follows easily by writing the first order Taylor expansion of f in local coordinates. ■

4.1.8. Lemma. *Let $M \subset \mathbb{R}^m$ be a compact C^k embedded submanifold, then the tubular hypersurface of M at a small distance is also C^k.*

Proof. This is an immediate consequence of the fact that the the distance function of a C^k embedded submanifold is C^k at points near M, see Section 3.2. ■

4.2 Construction of a C^k strictly convex patch

Here we prove:

4.2.1. Proposition. *Let $M \subset \mathbb{R}^m$ be a compact $C^{k\geqslant 2}$ embedded submanifold, then for every C^{k-1} nonsingular support σ for M there exists a C^{k-1} strictly convex patch P containing M and tangent to all the hyperplanes generated by σ. Furthermore, if σ is proper, then we can construct P so that it is C^k.*

Proof. Let BM denote the unit normal bundle of M, $r > 0$, and define $f \colon BM \to \mathbb{R}^m$ by

$$f(p, v) := p + r(v - \sigma(p)),$$

where $\sigma \colon M \to \mathbb{S}^{m-1}$ is a given C^{k-1} nonsingular support. Let $\overline{\sigma} \colon M \to BM$ be given by $\overline{\sigma}(p) := (p, \sigma(p))$, and let $U \subset BM$ be an open neighborhood of $\overline{\sigma}(M)$. We claim that there exist sufficiently small U and r such that

$$P := f(\overline{U})$$

is the desired patch, where $\overline{U}$ denotes the closure of U.

To prove this assertion, we need to check the following: (i) $M \subset P$, (ii) P is embedded, (iii) P is tangent to all the hyperplanes generated by σ, (iv) P has everywhere positive curvature, (v) P is strictly supported by all its tangent hyperplanes, (vi) P is at least C^{k-1}; and, if σ is proper, then P is actually C^k.

(i) $f(p, \sigma(p)) = p$, thus $f(\overline{\sigma}(M)) = M$. Furthermore, $\overline{\sigma}(M) \subset U$, by assumption, so we conclude that $M \subset f(\overline{U}) = P$.

(ii) Here we show that P is embedded, provided U is sufficiently small. To this end, we have to prove that $f \colon \overline{U} \to \mathbb{R}^m$ is an embedding. Since $\overline{U}$ is compact and U is assumed to be a small neighborhood of $\overline{\sigma}(M)$, it suffices to show that f is a one-to-one immersion in a neighborhood of $\overline{\sigma}(M)$. Now since, $f(p, \sigma(p)) = p$, it follows that $f\big|_{\overline{\sigma}(M)}$ is one-to-one; therefore, by Lemma 4.2.3, it suffices to show that f is an immersion in a neighborhood of $\overline{\sigma}(M)$. Finally, since $\sigma(M)$ is compact, it suffices to show that every $\overline{\sigma}(p) \in \overline{\sigma}(M)$ is a regular value of f.

Let $Z \in T_{\overline{\sigma}(p)}BM - \{0\}$. We are going to show that $f_*(Z) \neq 0$. Let $\gamma\colon (-\epsilon, \epsilon) \to BM$ be a curve with $\gamma(0) = \overline{\sigma}(p)$, and $\gamma'(0) = Z$. We have to show that $(f\circ\gamma)'(0) \neq 0$. Before starting the computations, note that $\gamma(t) = \overline{\sigma}(q(t)) = (q(t), \sigma(q(t))) = (q(t), v(t))$ where $q(t)$ is a curve in M and $v(t) := \sigma(q(t))$ is a curve in $\mathbb{S}^{m-1}$ with $q(0) = p$ and $v(0) = \sigma(q(0)) = \sigma(p)$. Also note that $Z = (q'(0), v'(0)) := (X, V)$, where $X \in T_pM$ and $V \in T_{\sigma(p)}\mathbb{S}^{m-1}$. Now we can write

$$\begin{aligned} f_*(Z) &= (f\circ\gamma)'(0) \\ &= q'(0) + r(v'(0) - (\sigma\circ q)'(0)) \\ &= X + r(V - \sigma_*(X)). \end{aligned}$$

Since we have assumed that $Z \neq 0$, X and V cannot vanish simultaneously; therefore, if $X = 0$, then $f_*(Z) = rV \neq 0$ and we are done. So suppose that $X \neq 0$, we are going to show that, in this case, $\langle f_*(Z), X\rangle \neq 0$, which would imply that $f_*(Z) \neq 0$. First, note that from the above calculation it follows that

$$\langle f_*(Z), X\rangle = \|X\|^2 + r(\langle V, X\rangle - \langle \sigma_*(X), X\rangle).$$

Next, recall that $v(t) = v(q(t)) \in T_{q(t)}M^{\perp}$; therefore, $\langle v(t), x'(t)\rangle = 0$. So it follows that $\langle v'(t), q'(t)\rangle = -\langle v(t), q''(t)\rangle$, which yields

$$\begin{aligned} \langle V, X\rangle &= \langle v'(0), q'(0)\rangle = -\langle v(0), q''(0)\rangle \\ &= -\frac{d^2}{dt^2}\langle \sigma(p), q(t)\rangle\Big|_{t=0} \\ &= -(\operatorname{Hess} l_p)_p(X, X) \\ &= -\langle A_{\sigma(p)}(X), X\rangle, \end{aligned}$$

where $l_p(x) := \langle \sigma(p), x \rangle$. Also, recall that

$$\begin{aligned}\langle \sigma_*(X), X \rangle &= \langle \overline{\nabla}_X \sigma, X \rangle \\ &= \langle (\overline{\nabla}_X \sigma)^\top, X \rangle \\ &= -\langle A_{\sigma(p)}(X), X \rangle.\end{aligned}$$

From the three previous calculations it follows that

$$\langle f_*(Z), X \rangle = \|X\|^2 \neq 0.$$

So we conclude that $f_*(Z) \neq 0$, which completes part (ii).

(iii) Here we show that P is tangent to all the hyperplanes generated by σ, i.e., $\sigma(p) \in T_p(P)$, for all $p \in M$. We are going to prove this by showing that $\langle W, \sigma(p) \rangle = 0$, for all $W \in T_pP$. To see this, let $\gamma\colon (-\epsilon, \epsilon) \to P$ be a curve with $\gamma(0) = p$ and $\gamma'(0) = W$, then

$$\langle \sigma(p), W \rangle = \langle \sigma(p), \gamma'(0) \rangle = \frac{d}{dt} \langle \sigma(p), \gamma(t) \rangle \Big|_{t=0}.$$

Now note that $\gamma(t) = f(q(t), v(t)) = q(t) + r(v(t) - \sigma(q(t)))$. Thus

$$\begin{aligned}\langle \sigma(p), W \rangle &= \langle \sigma(p), q'(0) \rangle + r(\langle \sigma(p), v'(0) \rangle - \langle \sigma(p), \sigma_*(q'(0)) \rangle) \\ &= \langle \sigma(p), X \rangle + r(\langle \sigma(p), V \rangle - \langle \sigma(p), \sigma_*(X) \rangle),\end{aligned}$$

where $X \in T_pM$ and $V \in T_{\sigma(p)}\mathbb{S}^{m-1}$. Now recall that $\sigma(p) \in T_{\sigma(p)}M^\perp$, thus $\langle \sigma(p), x \rangle = 0$. Also $\sigma(p) \in (T_{\sigma(p)}\mathbb{S}^{m-1})^\perp$, thus $\langle \sigma(p), X \rangle = 0$. Finally, $\sigma_*(X) \in T_{\sigma(p)}\mathbb{S}^{m-1}$, so $\langle \sigma(p), \sigma_*(X) \rangle = 0$ as well. So we conclude that

$$\langle \sigma(p), W \rangle = 0,$$

for all $p \in M$ and $W \in T_pP$, which completes this step.

(iv) Here we show that the (sectional) curvature of P is positive along M, if r is sufficiently small; and, conclude that P has positive curvature everywhere, provided U is small as well.

Let $p \in M$ and let $l_p \colon P \to \mathbb{R}$ be the height function determined by $\sigma(p)$, i.e., $l_p(x) := \langle x, \sigma(p) \rangle$. It is sufficient to show that $(\mathrm{Hess}\, l_p)_p$ is negative definite; because, this would imply that all the principal curvatures have the same sign which in turn shows that the sectional curvatures are all positive. Let $W \in T_pP$, and recall that, since p is a critical point of l_p,

$$(\mathrm{Hess}\, l_p)_p(W, W) = (l \circ \gamma)''(0),$$

where γ is a curve on P as in step (iii), i.e., $\gamma(t) = f(q(t), v(t))$ where $x(t)$ and $v(t)$ are curves on M and $\mathbb{S}^{m-1}$ respectively. We begin our calculations by writing

$$\begin{aligned}
(l \circ \gamma)''(0) &= (l(f(q(t), v(t))))''(0) \\
&= \frac{d^2}{dt^2}\langle \sigma(p), q(t) + r(v(t) - \sigma(p(t)))\rangle\big|_{t=0} \\
&= \langle \sigma(p), q''(0)\rangle + r\langle \sigma(p), v''(0)\rangle - r\langle \sigma(p), (\sigma \circ q)''(0)\rangle.
\end{aligned}$$

Now we perform three calculations corresponding to each of three terms in the last sentence above. First,

$$\begin{aligned}
\langle \sigma(p), q''(0)\rangle &= \frac{d^2}{dt^2}\langle \sigma(p), q(t)\rangle\big|_{t=0} \\
&= \langle A_{\sigma(p)}(q'(0)), q'(0)\rangle \\
&= \langle A_{\sigma(p)}(X), X\rangle,
\end{aligned}$$

where $A_{\sigma(p)}$ is the shape operator of M which is by assumption negative definite. Secondly, $\langle v(t), v(t)\rangle = 1$; therefore, $\langle v(t), v'(t)\rangle = 0$, and thus

$$\langle \sigma(p), v''(0)\rangle = \langle v(0), v''(0)\rangle$$

$$
\begin{aligned}
&= -\langle v'(0), v'(0)\rangle \\
&= -\|V\|^2.
\end{aligned}
$$

Thirdly, $\langle \sigma \circ q(t), \sigma \circ q(t)\rangle = 1$; therefore, $\langle (\sigma \circ q)'(t), \sigma \circ q(t)\rangle = 0$. So we have

$$
\begin{aligned}
\langle \sigma(p), (\sigma \circ q)''(0)\rangle &= \langle \sigma \circ q(0), (\sigma \circ q)''(0)\rangle \\
&= -\langle (\sigma \circ q)'(0), (\sigma \circ q)'(0)\rangle \\
&= -\|\sigma_*(X)\|^2.
\end{aligned}
$$

From the five preceding calculations it follows that

$$
(\operatorname{Hess} l_p)_p(W, W) = \langle A_{\sigma(p)}(X), X\rangle - r\|V\|^2 + r\|\sigma_*(X)\|^2.
$$

Now let $k := \sup_{p \in M}\{k_1(p, \sigma(p)), \dots, k_N(p, \sigma(p))\}$, then $\langle A_{\sigma(p)}(X), X\rangle \leqslant k\|X\|^2$. Also let $\lambda(p)$ be the norm of the linear operator σ_{*_p}, and set $\lambda := \sup_{p \in M}\lambda(p)$, then $\|\sigma_*(X)\| \leqslant \lambda\|X\|$. So, assuming $W \neq 0$, we have

$$
(\operatorname{Hess} l_p)_p(W, W) < (k + r\lambda^2)\|X\|^2.
$$

Thus if we set $r < -k/\lambda^2$, then $(\operatorname{Hess} l_p)_p(W, W) < 0$ and we are done.

(v) Here we show that P lies strictly on one side of all its tangent hyperplanes provided that P is sufficiently *thin* with respect to M, i.e., provided U is sufficiently small. Let us say a point $p \in P$ is *exposed*, if the tangent hyperplane at p strictly supports P, then we have to show that all points of P are exposed. Now since P has positive curvature, and $M \subset P$ is compact, it is sufficient to show, by Lemma 4.2.2, that every point $p \in M$ is an exposed point, assuming that P is sufficiently thin.

Let $p \in M$, we showed, in (iii), that $\sigma(p) \in T_pP^{\perp}$. Thus the hyperplane H_p generated by $\sigma(p)$ at p is indeed the tangent hyperplane of P at p. Let $l_p\colon P \to \mathbb{R}$

be given by $l_p(q) := \langle p, q\rangle$. Recall that to prove that H_p is a strictly supporting hyperplane for P, it is sufficient to show that $l_p(q) < l_p(p)$ for all $q \in P - p$. Let $L(p,q) := l_p(x) - l_p(p)$. We have to show that $L\big|_A < 0$, where $A := \{(p,q) \mid p \in M, q \in P, \,\&\, q \neq p\}$. The rest of the proof is now very similar to part II.1 of the proof of Proposition 4.1.1. Let $B_r := \{(p,q) \in A \mid \mathrm{dist}(p,q) < r\}$, and $C_r := \{(p,q) \in A \mid \mathrm{dist}(p,q) \geqslant r\}$. Now if r is sufficiently small then $L\big|_{B_r} < 0$ by local convexity of P, see Lemma 4.1.6. Also, it can be shown that $L\big|_{C_r} < 0$ by letting $D := \{(p,q) \in M \mid \mathrm{dist}(p,q) \geqslant r/2\}$, and observing that $L\big|_D < 0$ because σ supports M by assumption. It follows, therefore, that L must be negative throughout an open neighborhood of D. Provided P is sufficiently thin, it can be shown that this open neighborhood contains C_r. Hence $L\big|_{C_r} < 0$. The explicit details, as we mentioned before, are very similar to part II.1 of Proposition 4.1.1; therefore, we suppress them at this point.

(vi) P must be at least C^{k-1} because f is C^{k-1} and an embedding by (i). Furthermore, if σ is proper, then the perturbations $M_\epsilon := \{p - \epsilon\sigma(p) \mid p \in M\}$ are C^k for small ϵ; therefore, since P is a segment of the tube around the perturbed submanifold, i.e., $P \subset Tube_\epsilon M_\epsilon$, P will be C^k as well ; because, as we showed in the preliminaries, the distance function of a C^k submanifold is C^k everywhere except at the focal points. Since f is an embedding, P does not contain any of the focal points of the perturbed submanifold M_ϵ; therefore, the distance function of M_ϵ is a C^k submersion when restricted to a neighborhood of P. Hence, P is a C^k submanifold; because, locally, P is a level hypersurface of the distance function of M_ϵ. ■

4.2.2. Lemma. *Let $S \subset \mathbb{R}^m$ be a compact C^2 embedded hypersurface with positive curvature, then the set of exposed points of S is open.*

Proof. Let $p \in S$ be an exposed point, then p is a nondegenerate critical point and the strict absolute maximum of the height function l_p, where $l_p(x) := \langle x, \sigma(p)\rangle$, where $\sigma(p)$ is the outward unit normal of S at p. We have to show that there exists an open neighborhood U of p such that, for all $q \in U_p$ the height function l_q has a strict absolute maximum and a nondegenerate critical point at q. To this end, recall that we have to check the following: (1) $l_q(x) < l_q(q)$, for all $q \in M - \{x\}$, (2) $(\operatorname{grad} l_q)_q = 0$, and (3) $(\operatorname{Hess} l_q)_q \neq 0$. (2) follows from the fact that $\sigma(q) \in T_q(M)$, and (3) follows by continuity of σ and the assumption that $(\operatorname{Hess} l_p)_p < 0$, see Lemma 4.1.5. Thus it remains to show (1). To see (1) let $L(q,x) := l_q(x) - l_q(q)$. We have to show that $L\big|_A < 0$, where $A := \{(q,x) | q \in U, x \in M, \;\&\; x \neq q\}$. This can be done very explicitly, much like part II.1 of the proof of Proposition 4.1.1 and the proof of part (v) of Proposition 4.2.1, by partitioning A into a pair of subsets. One consisting of all pairs (q,x) where $\operatorname{dist}(q,x) < r$, and the other its complement. The fact that L is negative for the nearby points follows from Lemma 4.1.6. To show that L is positive for far away points let $D := \{(p,x) \mid \operatorname{dist}(q,x) \geqslant r/2\}$ and proceed as in part II.1 of the proof of Proposition 4.2.1. ■

4.2.3. Lemma. *Let $f\colon M \to N$ be an immersion, and $A \subset M$ a compact subset. If f restricted to A is one-to-one, then f is one-to-one in an open neighborhood of A. In particular, there exists an open neighborhood U of A such that f restricted to U is an embedding.*

Proof. Let $\{U_i\}$ be a sequence of open sets in M such that $\overline{U}_{i+1} \subset U_i$ and $\cap U_i = A$. This sequence can be constructed, for instance, by putting a metric on M and letting $U_i := \{p \in M \mid \operatorname{dist}(p,A) < \frac{1}{i}\}$. If $f\big|_{\overline{U}_i}$ is one-to-one for some i then we are done, so suppose not. Then, for every i, there exists u_i, $v_i \in U_i$ such that $u_i \neq v_i$, but $f(u_i) = f(v_i)$. Since M is compact, $\{u_i\}$ has a converging subsequence, say $\{u_{ij}\}$, which must converge to a point $a_u \in A$, because A is closed. Now consider the sequence $\{v_{ij}\}$. This sequence also contains a converging subsequence, say $\{v_{i_{j_k}}\}$, which necessarily converges to a point $a_v \in A$. Let $\tilde{u}_k := u_{i_{j_k}}$ and $\tilde{v}_k := v_{i_{j_k}}$, then $\tilde{u}_k \to a_u$, and $\tilde{v}_k \to a_v$, but $\tilde{u}_k \neq \tilde{v}_k$.

Now, since f is continuous, $f(\tilde{u}_k) \to f(a_u)$, and $f(\tilde{v}_k) \to f(a_v)$. This implies that $f(a_u) = f(a_v)$, and since, by assumption, $f\big|_A$ is one-to-one, it follows that $a_u = a_v$. Thus f is not one-to-one in any neighborhood of a_u. This is a contradiction, because f, being an immersion, must be locally one-to-one. ■

4.3 Extending the patch to a C^1 ovaloid

4.3.1. Proposition. *Every C^k strictly convex patch P may be extended to a C^1 ovaloid O. Moreover, we can construct O so that it is C^k in an open neighborhood of P, and, arbitrarily close to the convex hull of P.*

Proof. Let $\overline{P}$ be the inner parallel hypersurface of P at the distance of r. By Lemma 4.3.2, $\overline{P}$ is a strictly convex hypersurface once r is sufficiently small; therefore, by Lemma 4.3.3, through every point $\overline{p} \in \overline{P}$ there passes a ball $B_{\overline{p}}$, of some fixed radius R, such that $\overline{p} \in \operatorname{bd} B_{\overline{p}}$ and $\overline{P} \subset B_{\overline{p}}$. Let

$$\overline{K} := \bigcap_{\overline{p} \in \overline{P}} B_{\overline{p}}, \quad K := \overline{K} + rB^m,$$

where B^m denotes the unit ball, and set

$$O := \operatorname{bd} K.$$

Since, by Lemma 4.3.4, $\overline{K}$ is a convex body, it follows that K, being the Minkowski sum of two convex bodies, is also a convex body; therefore, as is well known, O must be a closed hypersurface. We claim that O is in fact the desired ovaloid; i.e., we have to check the following: (i) $P \subset O$, (ii) O is a C^1 ovaloid, and (iii) for every $\epsilon > 0$ there exist r and R such that $\operatorname{dist}(O, \operatorname{conv} P) < \epsilon$.

(i) We have to prove that $P \subset \operatorname{bd} K$. Since K is compact, $\operatorname{bd} K = K - \operatorname{int} K$. So it is sufficient to show that $P \subset K$ but $P \not\subset \operatorname{int} K$. First we show that $P \subset K$. To this end, it is sufficient to prove that $\operatorname{dist}(P, \overline{K}) \leqslant r$. This follows easily from the triangular inequality:

$$\operatorname{dist}(P, \overline{K}) \leqslant \operatorname{dist}(P, \overline{P}) + \operatorname{dist}(\overline{P}, \overline{K}) = r + 0 = r.$$

So it remains to show that $P \not\subset \operatorname{int} K$. Pick a point $p \in P$, and let $\overline{p} \subset \overline{P}$ be a point with $\operatorname{dist}(p, \overline{p}) = r$. By assumption, there exists a ball $B_{\overline{p}}$ such that $\overline{p} \in \operatorname{bd} B_{\overline{p}}$ and $\overline{K} \subset B_{\overline{p}}$. Let $B_p := B_{\overline{p}} + rB^m$; then, $p \in \operatorname{bd} B_p$ and $K \subset B_p$. Hence $p \notin \operatorname{int} K$.

(ii) Here we prove that O is a C^1 ovaloid. Recall that we have to show that there exists a ball which rolls freely inside O and that O rolls freely inside some ball. First, since $K := \overline{K} + rB^m$, it is clear that a ball of radius r rolls freely inside O. Secondly, $\overline{K}$ rolls freely inside a ball of radius R, by Lemma 4.3.4; therefore, K must rolls freely inside a ball of radius $R + r$.

(iii) First note that, since K is convex, $\operatorname{dist}(O, \operatorname{conv} P) = \operatorname{dist}(K, \operatorname{conv} P)$. Secondly, since $P \subset \operatorname{conv} P$, $\operatorname{dist}(K, \operatorname{conv} P) \leqslant \operatorname{dist}(K, P)$. Finally, by the triangular inequality,

$$\operatorname{dist}(K, P) \leqslant \operatorname{dist}(K, \overline{K}) + \operatorname{dist}(\overline{K}, \overline{P}) + \operatorname{dist}(\overline{P}, P) = r + \operatorname{dist}(\overline{K}, \overline{P}) + r.$$

By by Lemma 4.3.5, $\operatorname{dist}(\overline{K}, \overline{P})$ can be made arbitrarily small by choosing R sufficiently large. So we conclude that if R is large and r is small then $\operatorname{dist}(O, \operatorname{conv} P)$ is small.

Finally, in order for O to be C^k in a neighborhood of P, we can extend P along its boundary to a slightly larger strictly convex patch P' which contains P in its interior; and, carry out the above construction for P' instead of P. The fact that P' exists follows from Lemma 4.3.6. ■

4.3.2. Lemma. *Let $P \subset \mathbb{R}^m$ be a C^k strictly convex patch, then there exists a $\delta > 0$ such that, for all $r < \delta$, the inner parallel hypersurface of P at the distance r is a C^k strictly convex patch.*

Proof. First note that P, being a strictly convex hypersurface, has a unique C^{k-1} nonsingular strict support σ which is just its outward unit normal vector field. We define the inner parallel hypersurface of P at the distance r by

$$\overline{P} := \{\, x - r\sigma(x) \mid x \in P \,\}.$$

We have to show that there exists a $\delta > 0$ such that, for all $r < \delta$, (i) $\overline{P}$ is a C^k embedded hypersurface, (ii) $\overline{P}$ has everywhere positive curvature, and (iii) $\overline{P}$ is strictly supported by all of its tangent hyperplanes. Form (ii) and (iii) it follows that through every point of P there passes a nonsingular support hyperplane, which shows that $\overline{P}$ is strictly convex.

(i) Define $f\colon P \to \mathbb{R}^m$ by

$$f(p) := p - r\sigma(p),$$

and note that $\overline{P} = f(P)$. We are going to show that there exists a $\delta_1 < 0$ such that, for every $r < \delta_1$, f is an embedding. This would show that P is a C^{k-1} embedded submanifold. Once we show this, we can use the distance function to show that P is actually C^k; because, P is a C^k submanifold by assumption and the distance function of a C^k embedded submanifold is C^k in a small tubular neighborhood of the submanifold, as we showed in the preliminaries. In particular, if f is an embedding, then $\overline{P}$ does not contain any of the focal points of P. Therefore, it can be shown that the distance function of P restricted to a small neighborhood of $\overline{P}$ is a C^k submersion; from which it follows that $\overline{P}$ is a C^k submanifold; because, locally, $\overline{P}$ is a level hypersurface of the distance function of P. This argument is very similar to the one we gave in part (v) of the proof of the Proposition 4.2.1.

First, we show that f is an embedding. Since P is compact, it is sufficient to show that f is a one-to-one immersion. To see that f is an immersion let $p \in P$, and let X_p^i be the principal directions of P at p. Recall that $\sigma_*(X_p^i) = -A_{\sigma(p)}(X_p^i) = -k^i(p)X_p^i$, where $k^i(p) := k^i(p, \sigma(p))$ are the principal curvatures of P at p in the direction $\sigma(p)$. Also recall that, by assumption, $\sigma(p)$ is a nonsingular support vector; therefore, $k^i(p) < 0$ for all $p \in P$. Now we are going to compute $f_*(X_p^i)$. Let $\gamma\colon (-\epsilon, \epsilon) \to P$ be a curve with $\gamma(0) = p$ and $\gamma'(0) = X_p^i$, then

$$\begin{aligned} f_*(X_p^i) &= (f \circ \gamma)'(0) = \gamma'(0) - r(\sigma \circ \gamma)'(0) \\ &= X_p^i - r\sigma_*(X_p^i) = (1 + rk^i(p))X^i(p). \end{aligned}$$

Thus $f_*(X_p^i) \neq 0$ if $r \neq \frac{-1}{k^i(p)}$. In particular, if $0 \leqslant r < \lambda$, where

$$\lambda := \inf_{p \in M} \left\{ \frac{1}{|k^1(p)|}, \dots, \frac{1}{|k^n(p)|} \right\},$$

then f is an immersion. To see this let $X_p \in T_pP$, and suppose that $f_*(X_p) = 0$; then $0 = \sum_{i=1}^n c_i f_*(X_p^i) = \sum_{i=1}^n c_i(1 + rk_i(p))X^i(p)$. This is possible only when all $c_i's$ are zero, because $\{X_p^i\}$ is linearly independent. Hence $ker(f_{*_p}) = 0$ for all $p \in P$, i.e., f is an immersion.

Next we show that f is one-to-one. To see this let $F : P \times \mathbb{R} \to \mathbb{R}^m$ be defined by $F(p, r) := p - r\sigma(p)$. From what we have shown so far it follows that $F\big|_{P \times (-\lambda, \lambda)}$ is an immersion. Thus, by Lemma 4.2.3, $F\big|_{P \times (-\epsilon, \epsilon)}$, must be an embedding for some small $\epsilon > 0$; because, $F\big|_{P \times \{0\}}$ is one-to-one. In particular, if

$$\delta_1 < min\{\epsilon, \lambda\},$$

then f is a C^{k-1} embedding for every $0 \leqslant r \leqslant \delta_1$.

(ii) Let δ_1 be as in part (i), then, as we showed above, $\overline{P}$ is a C^k embedded submanifold, if $0 < r < \delta_1$. Assuming that $0 < r < \delta_1$, we now show that $\overline{P}$ has everywhere positive curvature. To see this let $\overline{\sigma}\colon \overline{P} \to \mathbb{S}^{m-1}$ be defined by

$$\overline{\sigma}(f(p)) := \sigma(p).$$

We claim that $\overline{\sigma}$ is the Gauss map of $\overline{p}$, and use this map to calculate the curvature of $\overline{P}$. First we show that $\overline{\sigma}$ is the Gauss map of $\overline{P}$, i.e., $\overline{\sigma}(f(p)) \in T_{f(p)}\overline{P}^{\perp}$. To see this, let X_p^i be the principal directions of P at p, and note that $\{f_*(X_p^i)\}$ forms a basis for $T_{f(p)}\overline{P}$; because, f_* is an immersion. Thus all we need is to show that $\langle \overline{\sigma}(f(p)), f_*(X_p^i)\rangle = 0$. This is done in the following calculation:

$$\begin{aligned} \langle \overline{\sigma}(f(p)), f_*(X_p^i)\rangle &= \langle \sigma(p), (1 + rk^i(p))X_p^i\rangle \\ &= (1 + rk^i(p))\langle \sigma(p), X_p^i\rangle = 0. \end{aligned}$$

Now we show that $\overline{P}$ has positive curvature, i.e., at every point $f(p) \in \overline{P}$ all principal curvatures are nonzero and have the same sign. Since $\overline{\sigma}$ is the Gauss map of $\overline{P}$, and recalling that the principal curvatures are simply the eigenvalues of the differential of the Gauss map, it suffices to show that the eigenvalues of $\overline{\sigma}_{*_{f(p)}}$ are positive. This is shown in the following calculation:

$$\begin{aligned} (1 + rk^i(p))\overline{\sigma}_*(X_p^i) &= \overline{\sigma}_*((1 + rk^i(p))X_p^i) = \overline{\sigma}_*(f_*(X_p^i)) \\ &= \sigma_*(X_p^i) = -k^i(p)X_p^i, \end{aligned}$$

from which it follows

$$\overline{\sigma_*}(X_p^i) = \frac{-k^i(p)}{1 + rk^i(p)}X_p^i.$$

Note that, since $\overline{\sigma}(f(p)) = \sigma(p)$, T_pP and $T_{f(p)}\overline{P}$ are parallel; and, consequently, $X_p \in T_{f(p)}\overline{P}$. Hence, the above calculation shows that the principal directions of $\overline{P}$ at $f(p)$ are the X_p^i with corresponding principal curvatures $\frac{-k^i(p)}{1+rk^i(p)}$. These curvatures are well-defined, because r is sufficiently small, i.e., $r < \delta_1$; and all have the same sign, because all k^i have the same sign by assumption. Thus we say that $\overline{P}$ has positive curvature.

(iii) Assume $0 < r < \delta_1$, then, as we showed above, $\overline{P}$, is embedded and has everywhere positive curvature. Here we show that there exists a $0 < \delta \leqslant \delta_1$ such that, if $0 < r < \delta$ then $\overline{P}$ is strictly supported by all of its tangent hyperplanes.

For every $\overline{p} \in \overline{P}$ let $\bar{l}_{\overline{p}} \colon \overline{P} \to \mathbb{R}$ be defined by

$$\bar{l}_{\overline{p}}(\overline{q}) = \langle \overline{q}, \overline{\sigma}(\overline{p}) \rangle.$$

We have to show that $\bar{l}_{\overline{p}}(\overline{q}) < \bar{l}_{\overline{p}}(\overline{p})$ for all $\overline{q} \in \overline{P} - \{\overline{p}\}$ and $\overline{p} \in \overline{P}$. To see this, let $\overline{L}(\overline{p}, \overline{q}) := \bar{l}_{\overline{p}}(\overline{q}) - \bar{l}_{\overline{p}}(\overline{p})$; then, we have to show that $\overline{L}\big|_A < 0$, where $\overline{A} := \{(\overline{p}, \overline{q}) \mid \overline{p} \in \overline{P}, \overline{q} \in \overline{P}, \ \& \ \overline{p} \neq \overline{q}\}$. Now note that $\overline{A} = \overline{B}_\alpha \cup \overline{C}_\alpha$, where $\overline{B}_\alpha := \{(\overline{p}, \overline{q}) \in A \mid \mathrm{dist}_{\overline{P}}(\overline{p}, \overline{q}) < \alpha\}$, and $\overline{C}_\alpha$ is the complement of B_α in A. Since $\overline{P}$ is compact and , by Lemma 4.1.6, has positive curvature, $\overline{P}$ is uniformly locally strictly convex. In particular, there exists an $\alpha > 0$ such that $\overline{L}\big|_{B_\alpha} < 0$. Thus it remains to show that $\overline{L}\big|_{C_\alpha} < 0$. This follows from the following calculation.

First, note that $\overline{q} = f(q) = q - r\sigma(q)$ for some $q \in P$. Also, note that $\overline{\sigma}(\overline{p}) = \overline{\sigma}(f(p)) = \sigma(p)$, by definition of $\overline{\sigma}$. Thus we can write

$$\begin{aligned}
\bar{l}_{\overline{p}}(\overline{q}) - \bar{l}_{\overline{p}}(\overline{p}) &= \langle \overline{q}, \overline{\sigma}(\overline{p}) \rangle - \langle \overline{p}, \overline{\sigma}(\overline{p}) \rangle \\
&= \langle q - r\sigma(q), \sigma(p) \rangle - \langle p - r\sigma(p), \sigma(p) \rangle \\
&= \langle q, \sigma(p) \rangle - r\langle \sigma(q), \sigma(p) \rangle - \langle p, \sigma(p) \rangle + r \\
&= l_p(q) - l_p(p) + r(1 - \langle \sigma(q), \sigma(p) \rangle) \\
&\leqslant l_p(q) - l_p(p) + 2r,
\end{aligned}$$

where l_p is the height function of P. Now, recalling that $L(p,q) := l_p(q) - l_p(p)$, we can write

$$\overline{L}(\overline{p},\overline{q}) \leqslant L(p,q) + 2r.$$

Note that if $(\overline{p},\overline{q}) \in \overline{C}_\alpha$, then $(p,q) \in C_{\alpha'}$, where $C_{\alpha'} := \{(p,q) \mid p \in P, q \in p, \ \&\ \mathrm{dist}_P(p,q) \geqslant \alpha'\}$. Let

$$\eta := \sup\{\, L(p,q) \mid (p,q) \in C_{\alpha'} \}.$$

Recall that, since P is strictly convex, $\eta < 0$. Now it is clear that if $r < -\eta/2$, then $\overline{L}|_{\overline{C}_\alpha} < 0$. Hence we conclude that if $r < \delta$, where $\delta := min\{\delta_1, -\eta/2\}$, then $\overline{P}$ is strictly convex. ■

4.3.3. Lemma. *For every strictly convex patch P there exists a $\Delta > 0$ such that, if $R > \Delta$, through every point of P there passes a sphere of radius R containing P.*

Proof. Let $\sigma\colon P \to \mathbb{S}^{m-1}$ be the outward unit normal map of P and set

$$B^{p,r} := B_r(p - r\sigma(p)),$$

i.e., $B^{p,r}$ is a ball of radius r with center at $p - r\sigma(p)$. Note that $p \in \mathrm{bd}\, B^{p,r}$, because $\|p - (p - r\sigma(p))\| = r$. We claim that there exists a $0 < \Delta < \infty$ such that $P \subset B^{p,r}$ for all $p \in P$, if $r > \Delta$. To see this let

$$\lambda := \sup_{p \in P}\{\, r_1(p), \ldots, r_n(p)\,\},$$

where $r_i(p) := 1/|k^i(p)|$ are the principal curvatures of P at p. If $r > \lambda$, then it is easily shown that every $p \in P$ has an open neighborhood $U_{\delta_p}(p) \subset P$ such that $U_{\delta_p}(p) \subset B^{p,r}$. Furthermore, since P is compact, it can be shown that there is a δ, independent of p such that

$$U_\delta(p) \subset B^{p,r},$$

for all $p \in P$. Now define a function $f\colon P \to \mathbb{R}$ by setting

$$f(p) := \sup\left\{ \frac{\|q-p\|^2}{-2\langle q-p, \sigma(p)\rangle} \;\middle|\; q \in P - U_\delta(p) \right\}.$$

Note that $|\langle q-p, \sigma(p)\rangle| = \text{dist}(T_pP, q) > 0$, for all $q \in P - \{p\}$; because, P is strictly convex by assumption. Hence f is well-defined. Furthermore, it is not hard to see that f must be continuous as well. Let

$$\eta := \sup\, f.$$

Now if $r > \eta$, then for all $q \in U_\delta(p)$ we have

$$\|q - (p - r\sigma(p))\| = (\|q-p\|^2 + 2r\langle q-p, \sigma(p)\rangle + r^2)^{\frac{1}{2}} \leqslant r,$$

which implies

$$P - U_\delta(p) \subset B^{p,r}.$$

Thus if we set $\Delta := max\{\lambda, \eta\}$; then, for every $R > \Delta$, $P \subset B^{p,R}$ for all $p \in P$. ■

4.3.4. Lemma. *Let K be an arbitrary intersection of balls of fixed radius R, then through every point in the boundary of K there passes a sphere of radius R containing K. In particular if K contains more than one point, then K is a convex body.*

Proof. Let $\{B_i\}_{i\in I}$ be an arbitrary collection of balls of fixed radius R, and set $K := \bigcap\limits_{i\in I} B_i$. We have to show that for every $x \in \text{bd}\, K$, there is a ball B of radius R such that $K \subset B$ and $x \in \text{bd}\, B$.

To see this, note that if $x \in bdB_i$ for some $i \in I$, then we are done. So suppose that $x \in \text{int}\, B_i$ for every $i \in I$. Then for every $\epsilon > 0$, there is an $i \in I$ such that $U_\epsilon(x) \cap \text{bd}\, B_i \neq \emptyset$; because, otherwise, x would be an interior point of K. In particular, for every $\epsilon = \frac{1}{n}$, $n \in \mathbb{N}$, there is a ball, say B_n, such that $U_{1/n} \cap B_n \neq \emptyset$. If

$K \neq \emptyset$ then the sequence $\{B_n\}_{n\in\mathbb{N}}$ is bounded. So by Blaschke's selection principal there is a subsequence $B_{n'}$ converging to some body B, i.e.,

$$\lim_{n'\to\infty} \operatorname{dist}(B, B_{n'}) = 0,$$

where dist is the Hausdorff distance. We claim that B is the desired ball. To show this, we have to check the following: (i) B is a ball of radius R, (ii) $K \subset B$, and (iii) $x \in \operatorname{bd} B$. Furthermore, we claim that (iv) if K contains more than one point then $\int K \neq \emptyset$. This ensures that, unless K is trivial, K is indeed a convex body.

(i) Clearly B, being the limit of balls of radius R, is itself a ball of radius R.

(ii) Since $K \subset b_{n'}$, for every n', it follows that $K \subset B$; otherwise, there would be a $k \in K$, with $k \neq B$, so we would have $\operatorname{dist}(k, B) > \epsilon > 0$. But for n' sufficiently large, $\operatorname{dist}(B, B_{n'}) < \epsilon$ which would yield $k \notin B_{n'}$; because,

$$\operatorname{dist}(k, B_{n'}) \geqslant \operatorname{dist}(k, B) - \operatorname{dist}(B, B_{n'}) > \epsilon - \epsilon = 0.$$

Of course this is a contradiction; because, by assumption, $K \subset B_{n'}$. So we conclude that $K \subset B$.

(iii) Since by construction $U_{\frac{1}{n'}}(x) \cap \operatorname{bd} B_{n'} \neq \emptyset$, it follows that $x \in \operatorname{bd} B$; otherwise, we would have $x \in \operatorname{int} B$ which would yield $\operatorname{dist}(x, \operatorname{bd} B) = \epsilon > 0$. In that case, we can choose n' large enough so that

$$\operatorname{dist}(x, \operatorname{bd} B_{n'}) \geqslant \operatorname{dist}(x, \operatorname{bd} B) - \operatorname{dist}(\operatorname{bd} B, \operatorname{bd} B_{n'}) = \epsilon - \epsilon/2 = \epsilon/2.$$

So $U_{\epsilon/2}(x) \subset B_{n'}$, which means $x \in \operatorname{int} B_{n'}$, a contradiction.

(iv) Now we prove that if K contains more than one point, then the interior of K is not empty. Suppose x, $y \in K$, and pick a $z \in (x, y)$; then, $z = (1-\lambda)x + \lambda y$ for some $\lambda \in (0, 1)$. We are going to show that that $z \in \operatorname{int} K$. This follows from the following calculation. If o_i is the center of B_i, then for every $i \in I$ we have:

$$|z - o_i| \;=\; |(1-\lambda)(x - o_i) + \lambda(y - o_i)|$$

$$\begin{aligned} &= [(1-\lambda)^2|x-o_i|^2+\lambda^2|y-o_i|^2+2\lambda(1-\lambda)\langle x-o_i, y-o_i\rangle]^{\frac{1}{2}} \\ &= [(1-\lambda)|x-o_i|^2+\lambda|y-o_i|^2-\lambda(1-\lambda)|x-y|^2]^{\frac{1}{2}} \\ &\leqslant (R^2-\lambda(1-\lambda)|x-y|^2)^{\frac{1}{2}}. \end{aligned}$$

Now let $0 < r \leqslant R - (R^2-\lambda(1-\lambda)|x-y|^2)^{\frac{1}{2}}$, and $w \in B(r,z)$, then

$$\begin{aligned} |w-o_i| &\leqslant |w-z|+|z-o_i| \\ &\leqslant r+(R^2-\lambda(1-\lambda)|x-y|^2)^{\frac{1}{2}} \leqslant R. \end{aligned}$$

Hence $B(r,z) \subset B(R,o_i)$ for every i which implies $z \in \operatorname{int} K$. ■

4.3.5. Lemma. *Let $A \subset \mathbb{R}^m$ be a compact subset, and K_R be the intersection of all balls of radius R containing A. Then for every $\epsilon > 0$, there exists an $R < \infty$ such that* $\operatorname{dist}(K_R, \operatorname{conv} A) < \epsilon$.

Proof. Fix an $\epsilon > 0$, and suppose $\operatorname{dist}(p, \operatorname{conv} A) \geqslant \epsilon$ for some point $p \in \mathbb{R}^m$. It is enough to show that for every such point p, there exists a ball B of radius R, depending only on ϵ, such that $\operatorname{conv} A \subset B$, but $p \notin B$. We are going to show that this is possible and we will derive the following estimate for R:

$$R \geqslant \frac{\epsilon}{4} + \frac{\delta^2}{\epsilon},$$

where $\delta := \operatorname{diam}(A)$, the largest distance between pairs of points in A. To see this let p' be the (unique) point of $\operatorname{conv} A$ which is closest to p, and let l be the line determined by p, and p'. Let H be the hyperplane which contains p' and is perpendicular to l, and let H^+ be the half space which does not contain p. Then $\operatorname{conv} A \subset H^+$. Let $B^+ := B(p', \delta) \cap H^+$. Then $\operatorname{conv} A \subset B^+$, because $\operatorname{diam}(\operatorname{conv} A) = \operatorname{diam}(A) = \delta$. Next, choose a point o on l which lies in H^+, and suppose $\operatorname{dist}(o, p') + \epsilon/2 = R$. Then $p \notin B(o, R)$. Furthermore, for the $\operatorname{conv} A$

to be contained in $B(o, R)$, we must have $\sup_{x \in \text{conv } A} \{\text{dist}(x, \text{conv } A)\} \leqslant R$. Since $convA \subset B^+$, we have

$$\sup_{x \in \text{conv } A} \{\text{dist}(x, \text{conv } A)\} \leqslant \sup_{x \in B^+} \{\text{dist}(x, B^+)\} = ((R - \frac{\epsilon}{2})^2 + \delta^2)^{\frac{1}{2}}.$$

The last equality follows because the farthest point of B^+ with respect to p lies on the *equator*, i.e., the boundary of the intersection of B^+ with H. Now if we set the right hand side of the above to be less than or equal to R, then we obtain the desired estimate. ■

4.3.6. Lemma. *Every C^k strictly convex patch P may be extended to a C^k strictly convex hypersurface without boundary. In particular, P is contained in the interior of a C^k strictly convex patch.*

Proof. Using the notion of the *double* of a manifold, see [Mun], it can be shown that every compact embedded hypersurface with boundary may be extended along its boundaries. Thus there exists an embedded hypersurface S without boundary such that $P \subset S$. Now since P is compact and has positive curvature, then there exist an open neighborhood $U \subset S$, $P \subset U$, such that U has positive curvature. We claim that if U is sufficiently small, then U is strictly convex. Since U has positive curvature, it suffices to show that U lies on one side of all its tangent hyperplanes. To see this, suppose U is small enough so that it has compact closure $\overline{U}$. Then U is uniformly locally strictly convex by Lemma 4.1.6, i.e., every $q \in U$ has a neighborhood of radius $\delta > 0$ which lies on one side of the tangent hyperplane at q. We claim that there exists an $0 < \epsilon < \delta$ such that if the radius of U with respect to P is less than ϵ, then U is strictly convex. It is not very difficult to write down the details explicitly. They are somewhat similar to the proof of Lemma 4.3.4. ■

4.4 Smoothing the ovaloid

4.4.1. Proposition. *Let $O \subset \mathbb{R}^m$ be a C^1 ovaloid, and let $U \subset O$ be a $C^{k \geqslant 2}$ open subset; then, for every closed subset $A \subset U$ there exists a C^k ovaloid $\widetilde{O}$ containing A. Furthermore, $\widetilde{O}$ may be constructed arbitrarily close to O.*

Proof. First note that since $A \subset U$ is closed, we can replace U by a slightly smaller neighborhood containing A. Thus we can assume, without loss of generality that U is C^k up to its boundary. Let $V \subset U$ be an open set with $\overline{V} \subset U$, and $A \subset V$. Set $V' := \sigma(V)$, and $U' := \sigma(U)$, where $\sigma \colon O \to \mathbb{S}^{m-1}$ is the outward unit normal map of O. Then U' and V' are open in $\mathbb{S}^{m-1}$, see Lemma 4.4.2. Let $\overline{\phi} \colon \mathbb{S}^{m-1} \to \mathbb{R}$ be a smooth function with $supp(\overline{\phi}) \subset U'$, and $\overline{\phi}\big|_{V}' \equiv 1$. Let ϕ be the homogeneous extension of $\overline{p}$ to $\mathbb{R}^m$, i.e., set $\phi(0) := 0$, and $\phi(x) := \|x\|\phi(\frac{x}{\|x\|})$, when $x \neq 0$. Now let h be the support function of O and let $\widetilde{h^\epsilon}$ be the Schneider convolution of h. Finally, define $g \colon \mathbb{R}^m \to \mathbb{R}$ by

$$g(x) := \widetilde{h^\epsilon}(x) + \phi(x)(h(x) - \widetilde{h^\epsilon}(x)).$$

We claim that, there exists an $\epsilon > 0$, such that g is a support function, and the boundary of the convex body determined by g is the desired ovaloid.

In order to prove the above assertion, we have to check the following: (i)$g(rx) = rg(x)$, for all $r > 0$, (ii) g is C^k on $\mathbb{R}^m - \{0\}$, (iii) $(\text{Hess}\, g)_p$ is positive semidefinite, for all $p \in \mathbb{R}^m - \{0\}$, (iv) $g\big|_{V'} = h\big|_{V'}$, and (v) $(\text{Hess}\, \overline{g^p})_p$ is positive definite for all $p \in \mathbb{S}^{m-1}$, where $\overline{g^p}$ is the restriction of g to the tangent hyperplane to $\mathbb{S}^{m-1}$ at p.

(i), (ii), and (iii) show that g is a C^k support function. Thus g determines a convex body with some boundary $\widetilde{O}$. (iv) shows that $V \subset \widetilde{O}$; and, consequently, implies that $A \subset \widetilde{O}$. Finally, (v) implies that $\widetilde{O}$ is a C^k ovaloid. (i) and (iv) are immediate from the definition of g and (ii) follows from the fact that $h\big|_{U'}$ is C^k, see Lemma 4.4.2. Thus it remains to check only (iii) and (v).

(iii) First note that, by homogeneity of g, it is enough to check this only for $p \in S^{m-1}$. Secondly, since $g\big|_{\mathbb{S}^m - U'} = \widetilde{h}^\epsilon$, and $\widetilde{h}^\epsilon$ is convex, it follows that we need to check (iii) only for $p \in U'$.

For every $p \in U'$, let $\{E_p^i\}$, $1 \leqslant i \leqslant m$, be an orthonormal basis for $\mathbb{R}^m$ with $E_p^m = p$, and set $g_{ij} := (\operatorname{Hess} g)_p(E_p^i, E_p^j)$. We have to show that the principal minors of the matrix (g_{ij}) are nonnegative for small ϵ. First note that, since g is homogeneous,

$$g_{i,m} = 0 = g_{m,i};$$

i.e., the last row and column of (g_{ij}) are zero. Thus all the principal minors containing the last row and column of (g_{ij}) are zero. It remains, therefore, to check the principal minors of (g_{ij}) not containing the last row and column.

Let $h_{ij} := (\operatorname{Hess} h)_p(E_p^i, E_p^j)$, and recall that if E_p^i, $1 \leqslant i \leqslant m-1$, are chosen so that they coincide with the principal directions of U at $\sigma^{-1}(p)$, then

$$h_{ij} := \begin{cases} 0 & i \neq j, \\ r_i & 1 \leqslant i = j \leqslant m-1, \\ 0 & i = j = m; \end{cases}$$

where r_i are the principal radii of curvature of U at $\sigma^{-1}(p)$, which are positive by assumption. Thus the principal minors of (h_{ij}) not containing the last row and column are positive; therefore, to show that the corresponding principal minors of (g_{ij}) are also nonnegative, it would be sufficient, by continuity of the determinant, to show that $|h_{ij} - g_{ij}| \to 0$ as $\epsilon \to 0$, i.e., we have to show that $\|h - g\|_{C^2(\overline{U'})} \to 0$. To see this note that

$$\|g - h\|_{C^2(\overline{U'})} \leqslant \|g - \widetilde{h}^\epsilon\|_{C^2(\overline{U'})} + \|\widetilde{h}^\epsilon - h\|_{C^2(\overline{U'})} \quad \text{and} \quad \|\widetilde{h}^\epsilon - h\|_{C^2(\overline{U'})} \to 0,$$

by Lemma 4.4.3. Thus, it is enough to show that $\|g - \widetilde{h}^\epsilon\|_{C^2(\overline{U'})} \to 0$. This follows from the following calculations:

$$|g(p) - \widetilde{h}^\epsilon(p)| \leqslant |\phi(p)|\,\|h(p) - \widetilde{h}^\epsilon(p)\|,$$

$$\|Dg_p - D\widetilde{h}^\epsilon{}_p\| \leqslant |h(p) - \widetilde{h}^\epsilon(p)|\,\|D\phi(p)\| + |\phi(p)|\,\|Dh_p - D\widetilde{h}^\epsilon{}_p\|,$$

and

$$\|D^2 g_p - D^2\widetilde{h}^\epsilon{}_p\| \leqslant$$
$$|h(p) - \widetilde{h}^\epsilon(p)|\,\|D^2\phi_p\| + |\phi(p)|\,\|D^2 h_p - D^2\widetilde{h}^\epsilon{}_p\| + 2\|D\phi_p\|\,\|Dh_p - D\widetilde{h}^\epsilon{}_p\|.$$

Thus, since $\|h - \widetilde{h}^\epsilon\|_{C^2(\overline{U'})} \to 0$, we conclude that $\|g - \widetilde{h}^\epsilon\|_{C^2(\overline{U'})} \to 0$.

(v) $(\mathrm{Hess}\, g)_p(E_p^i, E_p^j) = (\mathrm{Hess}\,\overline{g^p})_p(E_p^i, E_p^j)$, for all $1 \leqslant i, j \leqslant m-1$. As we showed in (iii), if $p \in U'$, then the matrix obtained from (g_{ij}) by eliminating the last row and column is positive definite; therefore, $\mathrm{Hess}\,\overline{g^p}$ is positive definite for all $p \in U'$. If $p \in \mathbb{S}^{m-1} - U'$, then $(\mathrm{Hess}\, g)_p(E_p^i, E_p^j) = (\mathrm{Hess}\,\overline{g^p})_p(E_p^i, E_p^j)$ which is positive definite by Lemma 4.4.4. ■

4.4.2. Lemma. *Let $O \subset \mathbb{R}^m$ be a C^1 ovaloid, and let $U \subset O$ be an open set which is $C^{k \geqslant 2}$ up to its boundary, then U has positive curvature. In particular, $\nu\big|_U$ is a C^{k-1} embedding, and $h\big|_U$ is C^{k-1} as well, where ν is the Gauss map of O and h is its support function.*

Proof. The first statement is an elementary consequence of the definition of a C^1 ovaloid. The next two statements, as we showed in Section 3.3., are immediate corollaries of the first. ■

4.4.3. Lemma. *Let $h\colon \mathbb{R} \to \mathbb{R}^m$ be a support function and suppose $h\big|_A$ is C^k, where $A \subset \mathbb{R}^m$ is a compact subset; then, $\|h - \tilde{h}^\epsilon\|_{C^2(A)} \to 0$ as $\epsilon \to 0$ where $\tilde{h}^\epsilon$ denotes the Schneider's convolution of h.*

Proof. This is an immediate consequence of the convolution properties of $\tilde{h}^\epsilon$. The details are similar to the corresponding proof of this fact for the ordinary convolution. ∎

4.4.4. Lemma. *Let $O \subset \mathbb{R}^m$ be a C^1 ovaloid, and let $\widetilde{O}$ be the surface obtained by applying Schneider's convolution to the support function of O; then, $\widetilde{O}$ is a C^∞ ovaloid. In particular the restriction of the support function of $\widetilde{O}$ to any tangent hyperplane to the sphere is strictly convex.*

Proof. We proved these statements in Sections 3.3 and 3.4. ∎

5 Applications

In the first four subsections we give condensed proofs of theorems mentioned in 1.2. In 5.5 and 5.6, we prove, respectively, results on the self-linking number of space curves, and umbilic points of ovaloids.

5.1 Hypersurfaces of constant positive curvature

As we explained in Section 1.2, Theorem 1.2.1 is an immediate corollary of the main theorem via a result of Guan and Spruck [GS].

5.2 Regularity of the convex hull of strictly convex submanifolds

Proof (Theorem 1.2.2). By our main theorem we know that there exists a C^2 ovaloid O containing M. If we choose the origin o of the coordinate system in $\mathbb{R}^n$ so that

$$o \in \mathrm{conv}(O) - \mathrm{conv}(M),$$

then M projects radially injectively into a submanifold contained in a hemisphere of the unit sphere $\mathbb{S}^{m-1}$. Choose the direction of the m^{th} axis in $\mathbb{R}^m$ so that the projection of M is contained in the upper hemisphere $(\mathbb{S}^{m-1})^+$; then, M projects into the boundary, ∂U, of some open set U with closure $\overline{U} \subset \mathrm{int}((\mathbb{S}^{m-1})^+)$. Note that ∂U is a $C^{3,1}$ embedded submanifold of $(\mathbb{S}^{m-1})^+$, and there exists a function $\phi \in C^{3,1}(\partial U)$ such that

$$M = \{\, \phi(x)x \mid x \in \partial U \,\},$$

i.e., we can represent M as a radial graph over the boundary of U. Now for every $\rho \in C^2(\overline{U})$ define $S_\rho := \{\, \rho(x)x \mid x \in \overline{U} \,\}$. We say ρ is strictly convex, if S_ρ is strictly convex. Now set

$$A := \{\, \rho \in C^2(\overline{U}) \mid \rho\, is\, strictly\, convex,\ and\ \rho\big|_{\partial U} = \phi \,\},$$

and let $\underline{\rho} = \inf A$; then, $S_{\underline{\rho}}$ is the upper half of the boundary of the convex hull of M. Thus, it is sufficient to show that $\underline{\rho} \in C^{1,1}(\overline{U})$. We do this by transforming this situation to a planar problem and applying a result of B. Guan, [Gun] who extended a theorem in [CNS2].

Let $\pi\colon (\mathbb{S}^{m-1})^+ \to T_{n.p.}\mathbb{S}^{m-1}$ be the stereographic projection, where $n.p. := (0, 0, \dots, 1)$ denotes the north pole. Let $\overline{\Omega} := \pi(\overline{U})$, and set $u_\rho := h \circ \pi^{-1}$, where $h := 1/\rho$; and set $\tilde{u}_\rho(x) := \sqrt{1 + x^2}\, u(x)$; then, $u_\rho \in C^2(\overline{\Omega})$. Now it turns out that the mapping $C^2(\overline{U}) \ni \rho \mapsto \tilde{u}_\rho \in C^2(\overline{\Omega})$ preserves the positiveness of the curvature of the corresponding surfaces. Set

$$B := \{\, \tilde{u}_\rho \mid \rho \in A \,\},$$

and let $\underline{\tilde{u}} := \inf B$. Now it follows from a theorem in [Gun] that $\underline{\tilde{u}}$ is $C^{1,1}$; therefore, $\underline{\rho}$ is $C^{1,1}$, because, $\underline{\tilde{u}} = \tilde{u}_{\underline{\rho}}$. ■

5.3 Optimal smoothing for convex polytopes

Proof (Theorem 1.2.3). First recall that the boundary of a convex polytope is made up from a collection of n facets F_i. We may think of each F_i as a compact embedded hypersurface with boundary. By Minkowski's approximation theorem, we can approximate each F_i by a compact embedded hypersurface $\widetilde{F}_i$ such that $\widetilde{F}_i \subset F_i$, and $\operatorname{bd} \widetilde{F}_i$ is a smooth ovaloid in F_i. It is sufficient to show that each $\widetilde{F}_i$ may be extended to a smooth hypersurface $Plate_i$ with a compact annular collar C_i such that $C := \bigcup_{i=1}^{n} C_i$ is a strictly convex submanifold of R^m. To see this recall that, by the main theorem, C may be extended to a smooth ovaloid O. Now since, by assumption, each C_i is homeomorphic to an $n-1$ dimensional annulus, then, by the generalized Jordan curve theorem, C_i divides O into a pair of regions, say O_i^+ and O_i^-. Let O_i^+ be the region neighboring the outside boundary of C_i, where by the outside boundary we mean the boundary component which coincides with the boundary of $Plate_i$. Finally, set

$$O' := (\bigcup_{i=1}^{n} Plate_i) \cup (\bigcap_{i=1}^{n} O_i^+),$$

then O' is a closed hypersurface with nonnegative curvature which contains each $\widetilde{F}_i$.

Thus it remains to show that we can construct the collection $Plate_i$ so that C is strictly convex. To construct $Plate_i$, suppose $F_i \subset R^{m-1} \times \{0\}$ and the positive direction of the m^{th} axis points into the interior of P. Next, define $f\colon F_i \to \mathbb{R}$ by

$$f(x) := \begin{cases} 0, & \text{if } x \in \widetilde{F}_i; \\ \exp\left(\frac{-1}{\operatorname{dist}^2(x,\widetilde{F}_i)}\right), & \text{otherwise.} \end{cases}$$

And let $Plate_i$ be the graph of f over $U_\delta(\widetilde{F}_i)$. We claim that there exists a $\delta > 0$ such that $\{Plate_i\}$ has the desired property, i.e., their collective collar C is strictly convex. ■

5.4 Global convexity of hypersurfaces with boundary

Proof (Theorem 1.2.4). It is enough to show that a collar of each boundary component of M is strictly convex. To see this, let Γ_i, $1 \leqslant i \leqslant N$, denote the boundary components of M (since M is compact there are finitely many components). Suppose a collar U_i of Γ_i is strictly convex. We can assume that $\overline{U}_i$ is a strictly convex submanifold; therefore, by the main theorem, $\overline{U}_i$ may be extended to an ovaloid O_i.

Now note that Γ_i separates O_i into a pair of domains, say O_i^+ and O_i^-. Let O_i^+ be the region which contains $\overline{U}_i$. Finally, set

$$\widetilde{M} := M \bigcup_{i=1}^{N} O_i^-.$$

Then $\widetilde{M}$ is a closed hypersurface, and it has everywhere positive curvature; therefore, by Hadamard's theorem, $\widetilde{M}$ is strictly convex. So M is strictly convex, and, in particular, embedded.

Thus it remains to show that each component Γ_i has a collar U_i which is strictly convex. To see this, let $p \in \Gamma_i$. Then $T_pM \cap \Gamma_i = \{p\}$, by assumption. Since M has positive curvature, there is a neighborhood $V_{\delta(p)}(p)$ such that $V_{\delta(p)}(p) \cap T_pM = \{p\}$. Furthermore, since Γ_i is compact, there exists a δ independent of p. Thus $V_\delta(p) \cap T_pM = \{p\}$ for every $p \in \Gamma_i$. Let $U_i := \{x \in M \mid \text{dist}(x, \Gamma_i) < \delta\}$. Then it can be shown that $U_i \cap T_pM = \{p\}$ for all $p \in \Gamma_i$, i.e., p is an exposed point of U_i. Now recall that, by Lemma 4.2.2, the set of exposed points of U_i is open. This set is an open neighborhood of Γ_i, and it contains U_i provided δ is small. ■

5.5 Self-linking number of space curves

Here we prove:

5.5.1. Theorem. *Let $\Gamma \subset \mathbb{R}^3$ be a simple closed C^2 curve with non-vanishing curvature. Suppose through every point of Γ there passes a hyperplane which intersects Γ at no other point. Then the self-linking number of Γ is zero.*

Proof. If Γ satisfies the hypothesis of the above theorem, then it is *weakly* strictly convex (c.f. Section 2.2). Since, in addition, Γ has non-vanishing curvature, it can be shown that Γ is strictly convex (in the strong sense), see Appendix B. Now, by the main theorem, Γ lies on a C^2 ovaloid. In particular, Γ bounds a surface of positive curvature, which is embedded and is homeomorphic to a disk. According to H. Rosenberg [Ros], this can happen only when the self-linking number of Γ is zero. ■

5.5.2. Note. This result should remain true under a weakened hypothesis. See Conjecture E.0.3.

5.6 Deforming ovaloids; umbilic points

Here we prove:

5.6.1. Theorem. *Let $O \subset \mathbb{R}^m$ be a smooth ovaloid, and $p \in O$. Let S be a smooth surface of positive curvature, and $p' \in S$. Then for every open subset $U \subset O$, $p \in U$, there exists a smooth one-parameter family of smooth ovaloids, $t \mapsto O_t$, such that:*

1. *$O_0 = O$,*
2. *$M - U \subset O_t$, $p \in O_t$, and $T_pO_t = T_pO$, for all $t \in [0, 1]$,*
3. *A neighborhood of p in O_1 coincides with a neighborhood of p' in S.*

Proof. By a rigid motion, if necessary, we can assume that $p' = p$, $T_pS = T_pO$, and S lies on the same side of T_pO as does O. Since O is strictly convex, and $p \in U$, $T_pS \cap (O - U) = \emptyset$. Since $O - U$ is compact, by a continuity argument it follows that $T_qS \cap (O - U) = \emptyset$ for all $q \in V$, where $V \subset S$ is some open neighborhood of p. Furthermore, by choosing V sufficiently small, we can ensure that $T_qO \cap V = \emptyset$ for all $q \in O - U$. Hence, we can assume that $(O - U) \cup V$ is strictly convex. Now, by the main theorem, $(O - U) \cup V$ lies on some ovaloid, say O_1. Suppose the origin of the coordinate system is inside O and O_1. Let ρ and $\rho_1\colon \mathbb{S}^{m-1} \to \mathbb{R}$ be a pair of functions whose radial graphs coincide with O and O_1 respectively. Set $\rho_t := (1 - t)\rho + t\rho_1$, and let O_t be the graph of ρ_t. ■

By letting S be a sphere, we obtain the following:

5.6.2. Corollary. *In the space of ovaloids, with respect to the Hausdorff metric, the set of ovaloids with infinitely many umbilic points is dense.*

There is a well known conjecture [BG, pg 418], attributed to C. Carathéodory: every ovaloid in $\mathbb{R}^3$ has at least two umbilic points. This conjecture is known to be true in the analytic case, see the paper by T. Klotz [Klz]. Also see the paper by E. Feldman [Fel], where he obtains related results in terms of generic properties of immersions; more specifically, he proves that, in the space of immersions of the sphere into the Euclidean space, there exists an open and dense subset which has an even number of isolated umbilic points. We should also mention a local, but more general, conjecture due to C. Loewner. This concerns the "index" of an isolated umbilic point, and implies Carathéodory's conjecture via Hopf's formula. In the analytic case, there is a proof of Loewner's conjecture due to C. Titus [Tit], and there has been some recent work on extending this result to the smooth case, see the paper by G. Carlos, M. Francesco, and S.-B. Federico [CFF].

A Pictures

All pictures in this work have been created by the author using *Adobe Illustrator* and *Mathematica*. Some of the graphs were generated with the aid of the packages written by A. Gray [Gry].

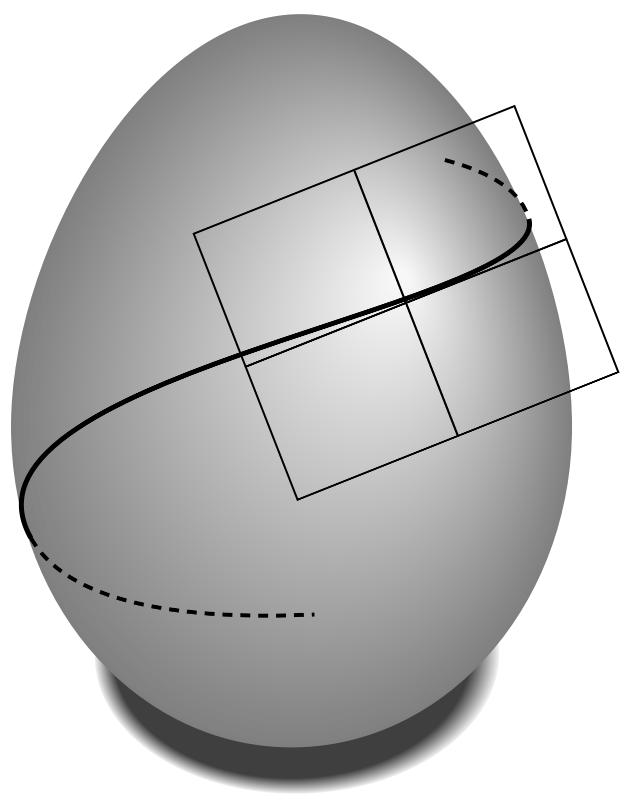

Figure 1: Every embedded submanifold of an ovaloid is strictly convex.

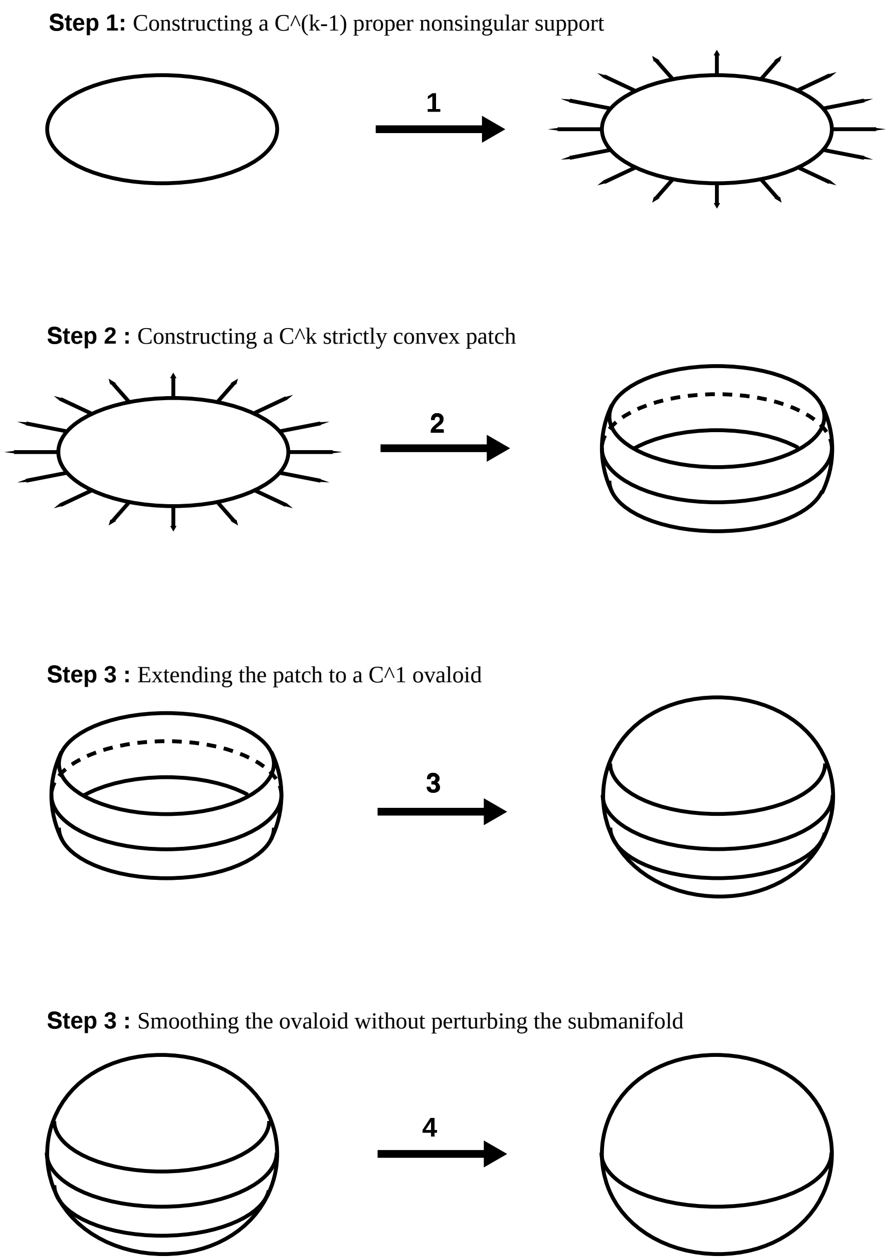

Figure 2: The four steps involved in proving the main theorem.

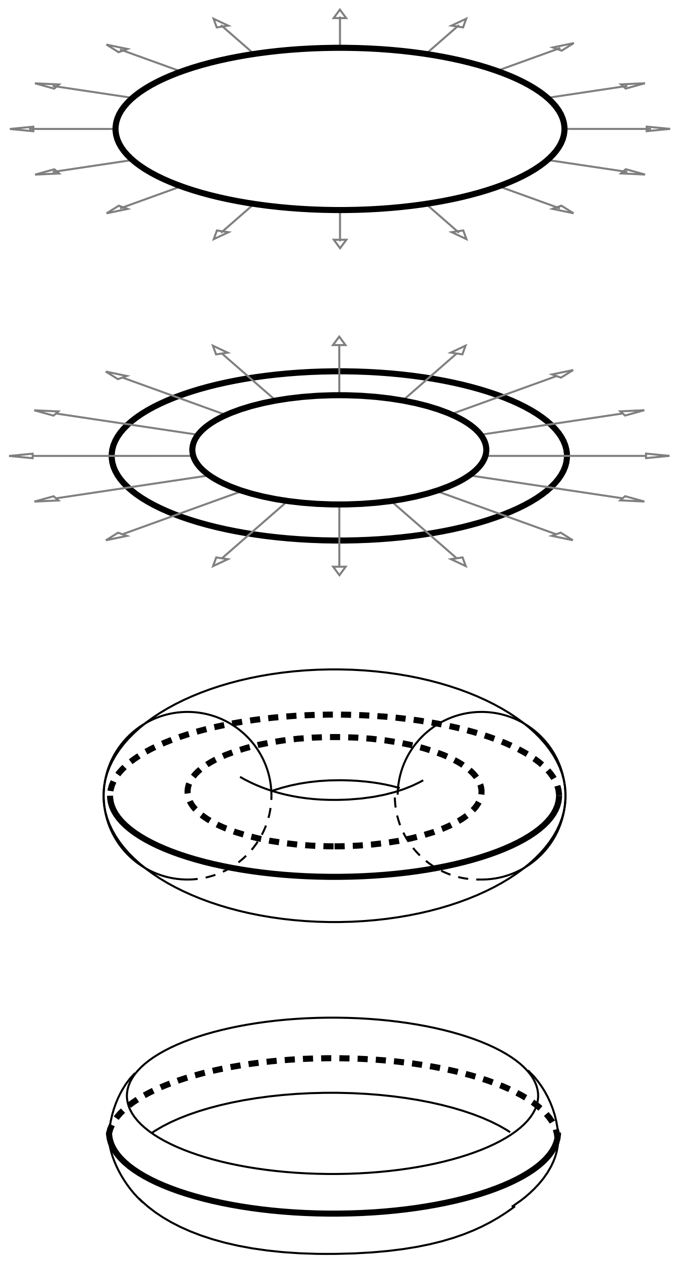

Figure 3: The details for the step 2 in the proof of the main theorem.

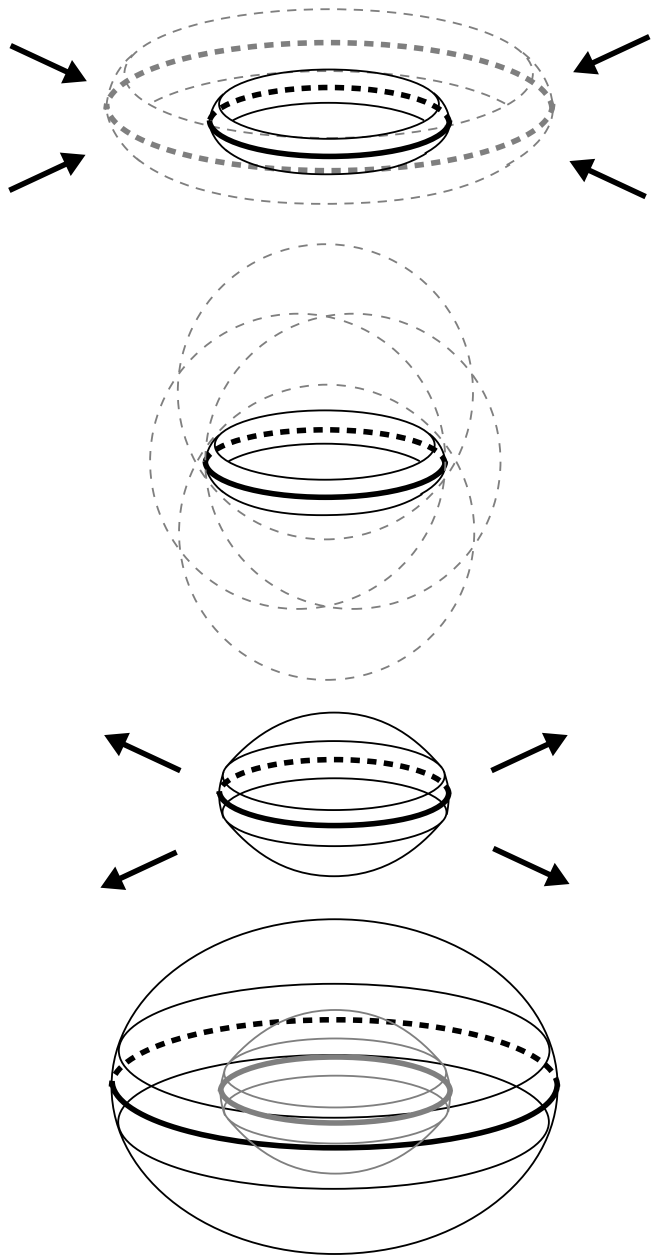

Figure 4: The details for the step 3 in the proof of the main theorem.

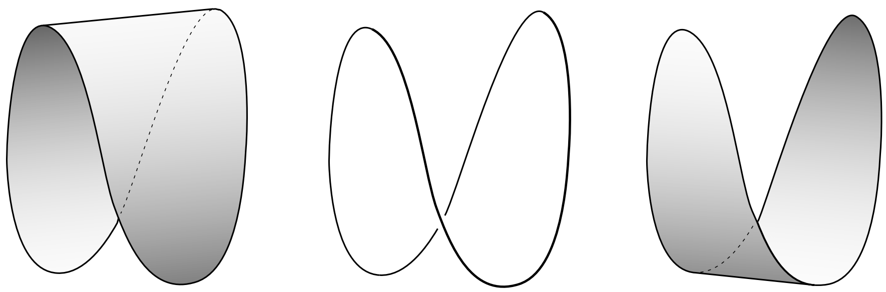

Figure 5: A closed strictly convex submanifold of codimension 2, and the two hypersurfaces forming the boundary of its convex hull.

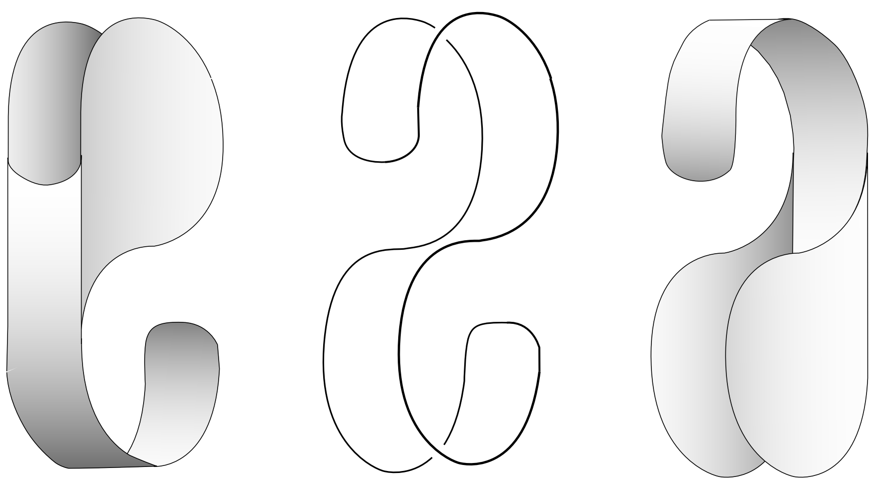

Figure 6: If the submanifold is not *strictly* convex, then the boundary of its convex hull may have singularities within the interior of each cap.

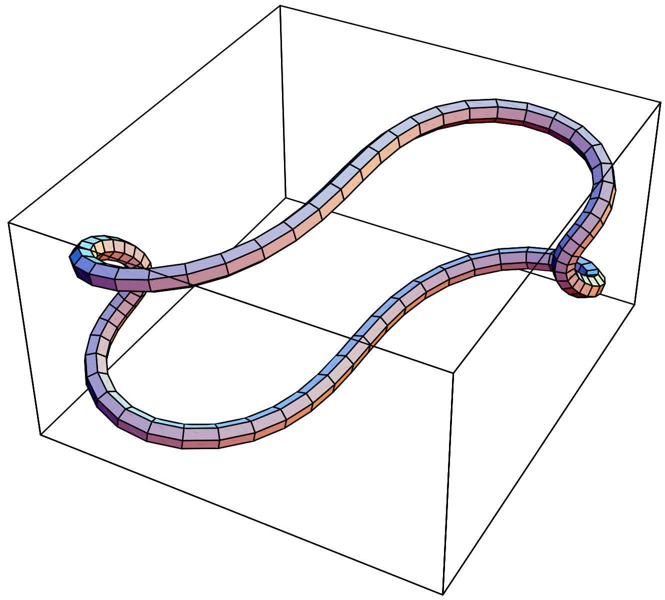

Figure 7: Double-S: a convex space curve which does not bound any surfaces of positive curvature.

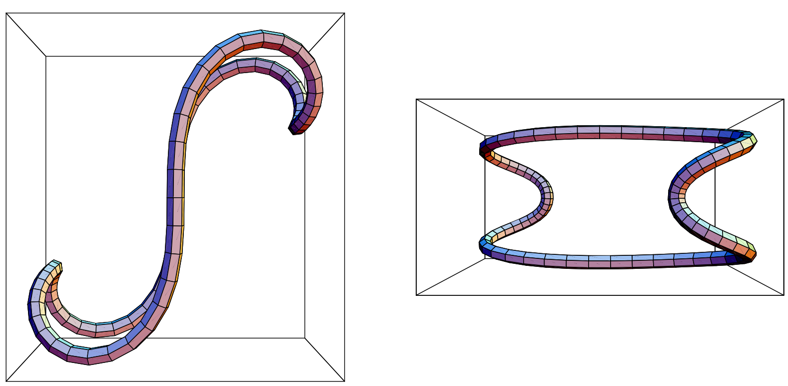

Figure 8: Double-S from other view points.

The formula for double-S is given by

$$\gamma(t) = \left(\sqrt{\pi} FresnelS\Big(\sqrt{\frac{5}{2}} sin(t)\Big),\ \sqrt{\pi} FresnelC\Big(\sqrt{\frac{5}{2}} sin(t)\Big),\ \sqrt{\frac{5}{2}} \frac{cos(t)}{4} \right),$$

where

$$FresnelC(x) = \int_0^x cos(\pi t^2/2)dt,$$
$$FresnelS(x) = \int_0^x sin(\pi t^2/2)dt,$$

and $t \in [0, 2\pi]$.

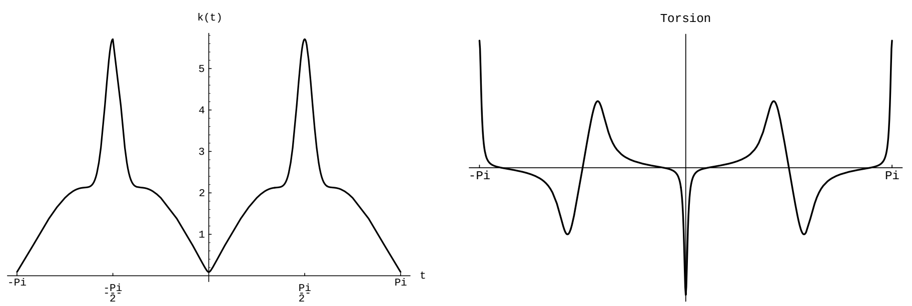

Figure 9: Curvature and torsion of double-S.

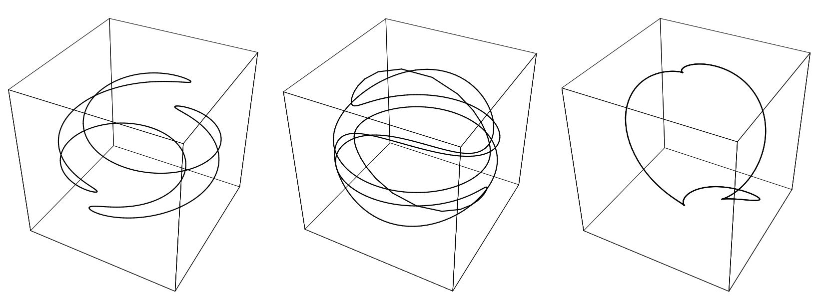

Figure 10: The Tangent, Normal, and Binormal spherical images of double-S.

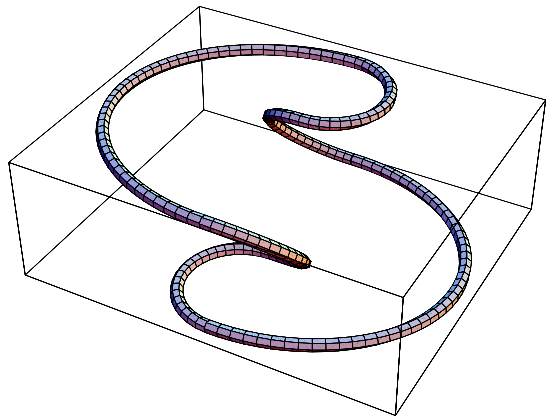

Figure 11: A curve by H. Gluck & L. Pan. Even though this curve has self-linking number zero, it does not bound any surfaces of positive curvature.

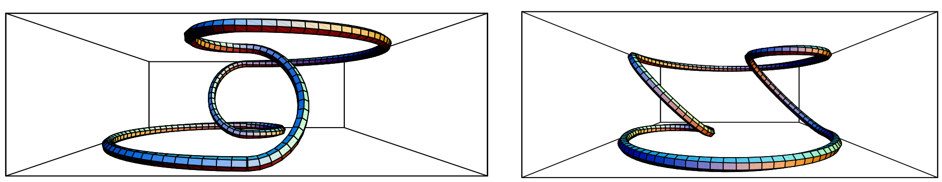

Figure 12: Side views of the curve by Gluck & Pan.

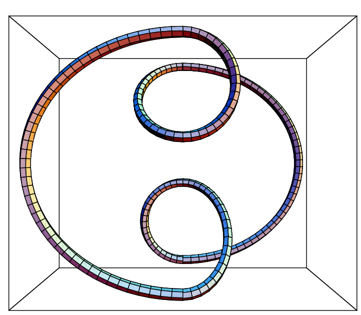

Figure 13: Top view of the curve by Gluck & Pan.

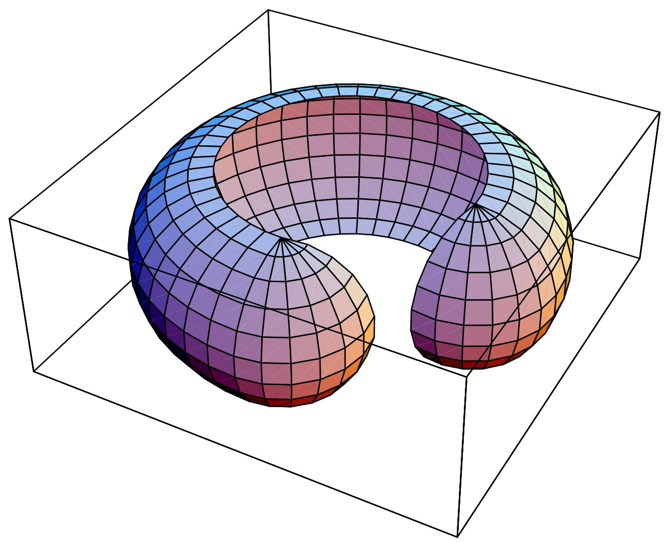

Figure 14: A surface of positive curvature which is not strictly convex but has strictly convex boundary.

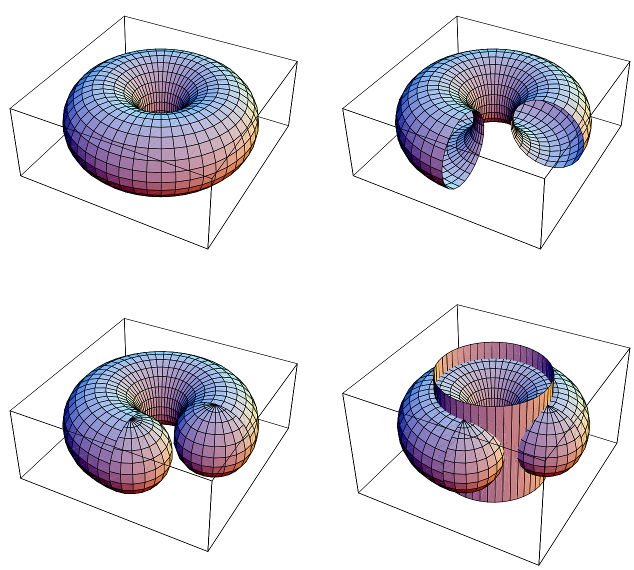

Figure 15: The steps for constructing the above surface.

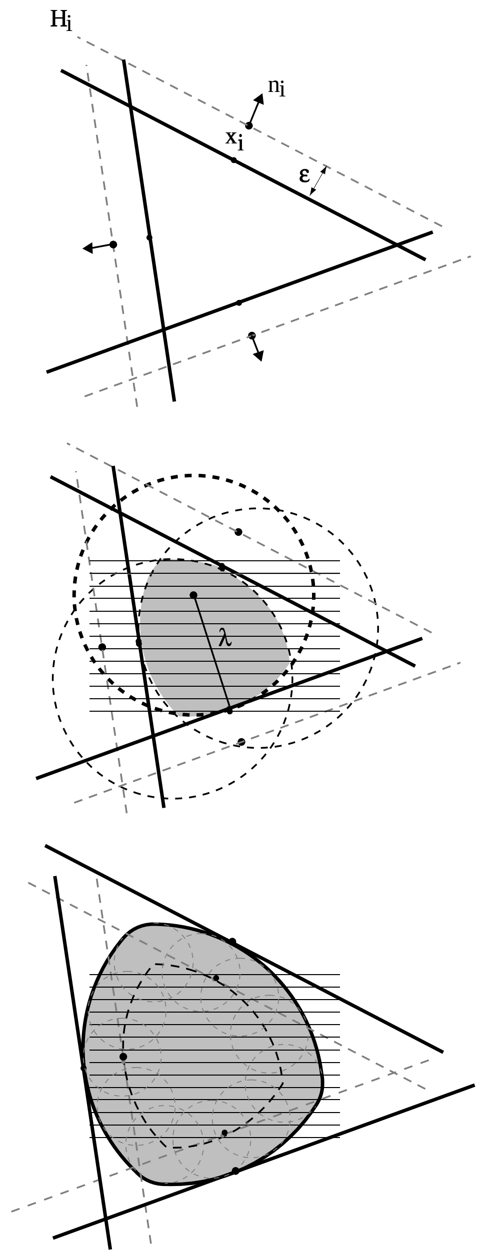

Figure 16: The steps for an elementary construction of a C^1 ovaloid in $\mathbb{R}^n$ for a discrete prescription of points and strictly supporting hyperplanes. The radii of the curvature of the solution are bounded above and below by $\lambda + \epsilon$ and ϵ respectively.

B Support Properties of Space Curves

In step 1 of the proof of the main theorem, we showed that every smooth compact strictly convex submanifold $M \subset \mathbb{R}^m$ admits a smooth nonsingular support. Here we give an explicit proof of this fact for the special case where $\dim(M) = 1$, and $m = 3$. Besides providing some motivation for the general proof, the propositions in this section explore natural properties of space curves, and, therefore, may be of independent interest.

The central result of this appendix is as follows. Let $\Gamma \subset \mathbb{R}^3$ be a weakly strictly convex Jordan curve without inflection points. Then Γ admits a C^{k-2} nonsingular support. Thus, in particular, if Γ is strictly convex in the weak sense and has no inflection points, then Γ is strictly convex in the strong sense (see Section 2.2 for a review of the terminology). The proof is divided into three steps carried out in Sections B.2, B.3, and B.4.

In B.1 we develop some preliminary notation and terminology; specifically, the concept of *exposure*.

In B.2 we show that through every point of Γ there passes a *nonsingular* support plane, i.e., one whose unit normal is not perpendicular to the principal normal of Γ at the point of contact.

In B.3 we show that it is possible to slide each nonsingular support plane along a small neighborhood of the point of contact, and, thus, construct a local one-parameter family of nonsingular support planes in a neighborhood of each point of Γ.

In B.4 we complete the construction of a nonsingular support by using a partition of unity to glue together the local nonsingular supports constructed in B.3.

B.1 Notation and preliminaries

We begin with a definition:

B.1.1. Definition. Let $X \subset \mathbb{R}^m$ be an arbitrary subset, then for every $x_0 \in \mathbb{R}^m$ we define the *exposure* of X at x_0 by

$$E_{x_0}(X) = \{\, n \in \mathbb{S}^{m-1} \mid \langle x - x_0, n\rangle < 0, \quad \forall x \in X - \{x_0\}\,\}. \tag{B.1.1}$$

B.1.2. Note. $E_{x_0}(X)$ is the set of outer unit normals to strictly supporting hyperplanes of $X \cup \{x_0\}$ passing through x_0. In particular, if x is an interior point of X, then $E_{x_0}(X) = \emptyset$; and, if x is a boundary point of X and X is a strictly convex body, then $E_{x_0}(X) \neq \emptyset$.

There are four elementary propositions on the properties of general support cones which we need to know:

B.1.3. Proposition. *For every $X, Y \subset \mathbb{R}^m$, and $z \in \mathbb{R}^m$, $E_z(X \cup Y) = E_z(X) \cap E_z(Y)$.*

Proof. This is an immediate consequence of the definition of E, and follows easily from manipulating formula (B.1.1). ∎

B.1.4. Proposition. *Let $X \subset \mathbb{R}^m$, $x_1, x_2 \in X$, and suppose $x_1 \neq x_2$, then $E_{x_1}(X) \cap E_{x_2}(X) = \emptyset$.*

Proof. Let $n_1 \in E_{x_1}(X)$ and $n_2 \in E_{x_2}(X)$, then from (B.1.1) it follows that

$$\langle x_2 - x_1, n_1 - n_2\rangle = \langle x_2 - x_1, n_1\rangle + \langle x_1 - x_2, n_2\rangle < 0.$$

Hence $n_1 - n_2 \neq 0$. ∎

B.1.5. Proposition. *Let $X \subset \mathbb{R}^m$ be compact and $x_0 \in \mathbb{R}^m - X$, then $E_{x_0}(X)$ is open (in $\mathbb{S}^{m-1}$).*

Proof. Define $f : \mathbb{R}^m \times \mathbb{S}^{n-1} \to \mathbb{R}$ by $f(x,n) = \langle x - x_0, n\rangle$. Suppose $n \in E(X, x_0)$, then $f(x,n) < 0$ for all $x \in X$. Since X is compact, $f(x,n) \leqslant \epsilon < 0$. Let $A = f^{-1}(-\infty, \epsilon/2)$, then $(x,n) \in A$ for all $x \in X$, and, since A is open, it follows that $(x,n) \in U_x \times V_{x,n} \subset A$ where U_x is an open neighborhoods of x in X, and $V_{x,n}$ is an open neighborhood of n in $\mathbb{S}^{n-1}$. $\{U_x\}_{x\in X}$ is an open covering for X and therefore must have a finite sub-covering, say $\{U_{x_i}\}_{1\leqslant i\leqslant L}$. Let $V_n = \bigcap_{i=1}^{L} V_{n,x_i}$, then $X \times V_n \subset A$, that is for each $x \in X$, $f(x,n) = \langle x - x_0, n\rangle < 0$ for all $n \in V_n$. Hence $V_n \subset E(X, x_0)$. ■

B.1.6. Proposition. *$E_{x_0}(X)$ is path connected for all $X \subset \mathbb{R}^m$ and $x_0 \in \mathbb{R}^m$.*

Proof. Let $n_0, n_1 \in E_{x_0}(X)$ and suppose $n_0 \neq -n_1$, then, for all $\lambda \in [0,1]$, $(1-\lambda)n_0 + \lambda n_1 \neq 0$ and therefore $n(\lambda) := \frac{(1-\lambda)n_0+\lambda n_1}{|(1-\lambda)n_0+\lambda n_1|}$ is well-defined. It is easy to verify that $\langle x - x_0, n_\lambda\rangle < 0$ for all $x \in X$, $x \neq x_0$, and $\lambda \in [0,1]$. Thus $n(\lambda) \in E_{x_0}(X)$. So n_0 and n_1 are connected by a path in $E_{x_0}(X)$.

Now suppose $n_0 = -n_1$, then for all $x \in X$, $x \neq x_0$, we have $\langle x - x_0, n_0\rangle < 0$ and $-\langle x - x_0, n_0\rangle = \langle x - x_0, n_1\rangle < 0$. This is possible only when $X = \{x_0\}$, in which case $E_{x_0}(X) = \mathbb{S}^{n-1}$. Hence $E_{x_0}(X)$ is path connected. ■

Next we fix some notation and prove a basic proposition on support properties of space curves.

B.1.7. Notation. Throughout this appendix $\Gamma \subset \mathbb{R}^3$ denotes a weakly strictly convex $C^{k\geqslant 2}$ simple closed oriented curve without inflection points; γ_0 is an arbitrary point of Γ; $T(\gamma_0)$, $N(\gamma_0)$, $B(\gamma_0)$, and $\kappa(\gamma_0)$ denote, respectively, the tangent, principal normal, binormal, and curvature of Γ at γ_0; $(\mathbb{R}, \gamma)$ denotes a unit speed periodic parameterization of Γ consistent with the given orientation, i.e., $\gamma\colon \mathbb{R} \to \mathbb{R}^3$ is a $C^{k\geqslant 2}$ mapping of period L, $\gamma(\mathbb{R}) = \Gamma$, $|\gamma'(t)| = 1$, $\gamma(t_0) = \gamma_0$, and $\gamma'(t_0) = T(\gamma_0)$;

$\Gamma_{\epsilon,\gamma_0}$ denotes an open neighborhood of Γ of radius $\epsilon > 0$ centered around γ_0, more explicitly,

$$\Gamma_{\epsilon,\gamma_0} := \{\, \gamma(t) \mid |t - t_0| < \epsilon \,\};$$

$n_\theta(\gamma_0)$, $0 \leqslant \theta < 2\pi$, denotes a unit normal vector to Γ at γ_0 defined by

$$n_\theta(\gamma_0) := \cos(\theta)N(\gamma_0) + \sin(\theta)B(\gamma_0).$$

We say a unit normal n to Γ at γ_0 makes an *angle* $\theta \in [0, 2\pi)$ with the principal normal of Γ at γ_0, $N(\gamma_0)$, if and only if $n = n_\theta(\gamma_0)$.

B.1.8. Proposition. *If* $n_0 \in E_{\gamma_0}(\Gamma)$*, then* $\langle n_0, T(\gamma_0)\rangle = 0$ *and* $\langle N(\gamma_0), n_0\rangle \leqslant 0$*. In particular,* $n = n_\theta(\gamma_0)$ *for some* $\theta \in [\pi/2, 3\pi/2]$*.*

Proof. Let (I, γ) be a regular parameterization by arclength with $\gamma(t_0) = \gamma_0$. Let $f(t) = \langle \gamma(t) - \gamma(t_0), n_0\rangle$. By (B.1.1) $f(t)$ has a maximum at $t = t_0$. Hence $\langle \gamma'(t_0), n_0\rangle = f'(t_0) = 0$ and $\langle N(\gamma_0), n_0\rangle = \langle \gamma''(t_0)/\kappa(\gamma_0), n_0\rangle = f''(t_0)/\kappa(\gamma_0) \leqslant 0$. This proves the first statement. To see the second, note that $\{T(\gamma_0), N(\gamma_0), B(\gamma_0)\}$ forms an orthonormal basis for $\mathbb{R}^3$, so we can write $n_0 = aT(\gamma_0) + bN(\gamma_0) + cB(\gamma_0)$ where $a^2 + b^2 + c^2 = 1$. Now it is clear that $a = 0$ and $b \leqslant 0$; therefore, we can set $b = cos(\theta)$ and $c = sin(\theta)$ where $\theta \in [\pi/2, 3\pi/2]$. ■

B.2 Rotating the support planes

In this section we show that every strictly supporting plane of Γ can be rotated around the tangent line at the point of contact by a small angle without coming into contact with Γ at a new point. In particular, at every point of Γ there exists a strictly supporting plane whose outer unit normal is not perpendicular to the principal normal of Γ at the point of contact. So through every point there passes a nonsingular support hyperplane, and, therefore, Γ is strictly convex (in the strong sense).

First, we show that *locally* Γ has many support planes at every point. In fact, every unit normal to Γ at γ_0 whose angle with the principal normal of Γ at γ_0 is strictly between $\pi/2$ and $3\pi/2$ generates a plane which locally strictly supports Γ:

B.2.1. Proposition. *For every* $n_0 \in \mathbb{S}^2$ *such that* $\langle n_0, T(\gamma_0)\rangle = 0$ *and* $\langle n_0, N(\gamma_0)\rangle < 0$ *there is an* $\epsilon > 0$ *such that* $n_0 \in E_{\gamma_0}(\Gamma_{\epsilon,\gamma_0})$.

Proof. Recall that $\gamma''(t) = \kappa(\gamma(t))N(\gamma(t))$. By Taylor's Theorem we have:

$$\gamma(t) - \gamma(t_0) = T(t_0)(t - t_0) + k(\gamma(s))N(\gamma(s))\frac{(t - t_0)^2}{2}, \qquad \text{(B.2.1)}$$

where s is between t and t_0. Let n_0 be as in the statement of the proposition, then

$$\langle \gamma(t) - \gamma(t_0), n_0\rangle = k(\gamma(s))\langle N(\gamma(s)), n_0\rangle \frac{(t - t_0)^2}{2}.$$

As $t \to t_0$, $\langle N(\gamma(s)), n_0\rangle \to \langle N(\gamma(t_0)), n_0\rangle < 0$. It follows, then, that there exists an $\epsilon > 0$ such that $\langle \gamma(t) - \gamma(t_0), n_0\rangle < 0$ for all $0 < |t - t_0| < \epsilon$. Thus the plane with outer unit normal n_0 passing through γ_0 strictly supports $\Gamma_{\epsilon,\gamma_0}$, i.e., $n_0 \in E_{\gamma_0}(\Gamma_{\epsilon,\gamma_0})$. ■

Next, we show that if Γ has a unique strictly supporting plane H_0 passing through γ_0, then H_0 must be locally unique as well:

B.2.2. Proposition. *If* $E_{\gamma_0}(\Gamma) = \{n_0\}$, *then* $E_{\gamma_0}(\Gamma_{\epsilon,\gamma_0}) = \{n_0\}$ *for every* $\epsilon > 0$.

Proof. Let $A := E_{\gamma_0}(\Gamma_{\epsilon,\gamma_0})$ and $B := E_{\gamma_0}(\Gamma - \Gamma_{\epsilon,\gamma_0})$, then, from Proposition B.1.3 it follows that

$$A \cap B = E_{\gamma_0}(\Gamma_{\epsilon,\gamma_0} \cup (\Gamma - \Gamma_{\epsilon,\gamma_0})) = E_{\gamma_0}(\Gamma) = \{n_0\};$$

therefore, $A \cap B$ is closed in A. By Proposition B.1.5, B is open in $\mathbb{S}^2$; therefore, $A \cap B$ is open in A. Thus $\{n_0\}$ is an open and closed subset of A which is connected by Proposition B.1.6. Hence $A = \{n_0\}$. ■

From these two propositions it follows that the strictly supporting planes of Γ are not unique at any point:

B.2.3. Corollary. *For all $\gamma_0 \in \Gamma$, $E_{\gamma_0}(\Gamma)$ contains more than one point.*

Note that if the assumption of non-vanishing of curvature is removed, then the above is false, as the following example shows:

B.2.4. Example. Let $\gamma : [-1, 1] \to \mathbb{R}^3$ be defined by $\gamma(t) = (t, t^3, t^6)$. First note that the $xy - plane$ strictly supports γ at the origin because $\gamma_3(t) = t^6 \geqslant 0$, and $\gamma(t) = 0$ if and only if $t = 0$. Furthermore, note that the $xy - plane$ is the *unique* strictly supporting hyperplane for Γ at the origin. To see this, let H be a strictly supporting hyperplane for γ at $t = 0$ with outer unit normal $n = (n_1, n_2, n_3)$, then $f(t) := \langle \gamma(t), n \rangle = \langle \gamma(t) - \gamma(0), n \rangle < 0$, for all $t \in [-1, 0) \cup (0, 1]$. Hence $f'(0) = \langle \gamma'(0), n \rangle = n_1 = 0$. Consequently, $f(t) = n_2 t^3 + n_3 t^6$. For f(t) to be negative for all $t \in [-1, 0) \cup (0, 1]$, we have to have $n_2 = 0$. So we conclude that H is the $xy - plane$.

Now we show that if Γ admits two distinct strictly supporting planes at γ_0, then each one of these planes may be rotated, in the appropriate direction, to coincide with the other. More precisely, let H_0, H_1 be strictly supporting planes for Γ at γ_0 with outer unit normals n_0, n_1 making angles θ_0, θ_1 with the principal normal of Γ at γ_0, $N(\gamma_0)$; then, every unit normal n to Γ at γ_0 whose angle with $N(\gamma_0)$ is between θ_0 and θ_1 generates a strictly supporting plane H for Γ at γ_0. This is proved in the following proposition:

B.2.5. Proposition. *Suppose* $n_{\theta_0}(\gamma_0)$, $n_{\theta_1}(\gamma_0) \in E_{\gamma_0}(\Gamma)$, $\theta_0 \leqslant \theta_1$, *then* $n_\theta(\gamma_0) \in E_{\gamma_0}(\Gamma)$ *for all* $\theta \in [\theta_0, \theta_1]$.

Proof. By Proposition B.1.6, there is a continuous map $n : [0,1] \to E_{\gamma_0}(\gamma)$ such that $n(0) = n_{\theta_0}(\gamma_0)$ and $n(1) = n_{\theta_1}(\gamma_0)$.

Define $\Theta : E_{\gamma_0}(\Gamma) \to [\pi/2, 3\pi/2]$ by $\Theta(n_\theta(\gamma_0)) = \theta$. By Proposition B.1.8, Θ is well-defined; moreover, it is clear that Θ is continuous. Thus $\Theta \circ n$ is a path joining θ_0 and θ_1.

Now suppose $\theta \in [\theta_0, \theta_1]$, then, for some $\lambda \in [0,1]$, $\Theta \circ n(\lambda) = \theta$. Θ is one-to-one; therefore, $n_\theta(\lambda_0) = \Theta^{-1}(\theta) = n(\lambda)$. Thus we conclude $n_\theta(\gamma_0) \in E_{\gamma_0}(\Gamma)$. ■

Using the above proposition and the previous corollary we can now show that at every point of Γ there passes a strictly supporting plane whose outer unit normal is not perpendicular to the principal normal of Γ at the point of contact:

B.2.6. Lemma. *For every* γ_0 *in* Γ *there exists a* $\theta_0 \in (\pi/2, 3\pi/2)$ *such that* $n_{\theta_0}(\gamma_0) \in E_{\gamma_0}(\Gamma)$.

Proof. By Corollary B.2.3, there are n_0, $n_1 \in E_{\gamma_0}(\Gamma)$ such that $n_0 \neq n_1$. By Proposition B.1.8, we have $n_0 = n_{\theta_0}(\gamma_0)$, and $n_1 = n_{\theta_1}(\gamma_0)$ for some θ_0, $\theta_1 \in [\pi/2, 3\pi/2]$. By Proposition B.2.5, $n_\theta(\gamma_0) \in E_{\gamma_0}(\Gamma)$, for every $\theta \in [\theta_0, \theta_1]$.

Now, since $n_0 \neq n_1$, we have $\theta_0 \neq \theta_1$. Consequently $(\theta_0, \theta_1) \neq \emptyset$. Let $\theta \in (\theta_0, \theta_1)$, then $\theta \in (\pi/2, 3\pi/2)$. ■

B.3 Sliding the support planes

Here we show that every point of Γ has an open neighborhood where we can construct a local one-parameter family of strictly supporting hyperplanes.

First we are going to show that for every $\theta \in (\pi/2, 3\pi/2)$, there is an $\epsilon > 0$ such that for all $\gamma_0 \in \Gamma$ the unit normal to Γ at γ_0 making an angle of θ with the principal normal of Γ at γ_0 generates a plane which strictly supports an open neighborhood of Γ of radius ϵ centered around γ_0:

B.3.1. Proposition. *For every $\theta \in (\pi/2, 3\pi/2)$, there is an $\epsilon > 0$ such that $n_\theta(\gamma_0) \in E_{\gamma_0}(\Gamma_{\epsilon,\gamma_0})$ for all $\gamma_0 \in \Gamma$.*

Proof. Recall that $(\mathbb{R}, \gamma)$ is a unit speed parameterization of Γ by arclength with period L. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(t, t_0) := \langle \gamma(t) - \gamma(t_0), n_\theta(\gamma(t_0)) \rangle.$$

We have to show that, for some $\epsilon > 0$, $f(t, t_0) < 0$ for all $(t, t_0) \in \mathbb{R}^2$ such that $0 < |t - t_0| < \epsilon$.

First note that, if such an ϵ exists, we must have $\epsilon \leqslant L$, because, if $|t - t_0| = L$, then $f(t, t_0) = f(t_0, t_0) = 0$. Also note that, since f is doubly periodic, it enough to consider only $(t, t_0) \in \mathbb{R} \times [0, L]$, which in turn implies $t \in [-L, 2L]$. Thus, it is enough to show that $f(t, t_0) < 0$ for all $(t, t_0) \in A := [-L, 2L] \times [0, L]$ such that $0 < |t - t_0| < \epsilon$.

Since $f(t_0, t_0) = 0$ and $f_1(t_0, t_0) = \langle T(\gamma(t_0)), n_\theta(\gamma(t_0)) \rangle = 0$, by Taylor's theorem we have

$$f(t, t_0) = (1/2) f_{11}(s, t_0)(t - t_0)^2,$$

where s is between t and t_0. Hence it is enough to show that $f_{11} < 0$ for all $(t, t_0) \in A$ such that $0 < |t - t_0| < \epsilon$.

Let $B := \{(t_0, t_0) \mid t_0 \in [0, L]\}$, then $f_{11}|_B < 0$, because $f_{11}(t_0, t_0) = \kappa(\gamma(t_0)) cos(\theta)$. Let $U := f_{11}^{-1}(-\infty, 0)$, then U is open, $B \subset U$ and consequently $\epsilon := \text{dist}(A - U, B) > 0$, because $A - U$ and B are disjoint and compact.

Now if $(t, t_0) \in A$ and $|t - t_0| < \epsilon$, then $\text{dist}((t, t_0), B) \leqslant |(t, t_0) - (t, t)| < \epsilon$; therefore, $(t, t_0) \in U$ and consequently $f_{11}(t, t_0) < 0$. ■

Now we can prove the sliding lemma. Let H_0 be a nonsingular support plane of Γ at γ_0, i.e., a strictly supporting plane with outer unit normal n_0 not perpendicular to the principal normal of Γ at γ_0, $N(\gamma_0)$, and making an angle of θ_0 with $N(\gamma_0)$; then, there exists an open neighborhood of Γ of radius $\delta > 0$ centered around γ_0 such that for all points γ_1 in this neighborhood the unit normal vector to Γ at γ_1 making an angle of θ_0 with the principal normal of Γ at γ_1 generates a strictly supporting plane for Γ at γ_1:

B.3.2. Lemma. *Let $\theta \in (\pi/2, 3\pi/2)$, and suppose $n_\theta(\gamma_0) \in E_{\gamma_0}(\Gamma)$, then there exists a $\delta > 0$ such that $n_\theta(\gamma_1) \in E_{\gamma_1}(\Gamma)$ for all $\gamma_1 \in \Gamma_{\delta,\gamma_0}$.*

Proof. As before, we let $(\mathbb{R}, \gamma)$ be a parameterization of Γ by arclength with period L, $t_0 \in [0, L]$, and $\gamma(t_0) = \gamma_0$. By assumption we have

$$f(t, t_0) := \langle \gamma(t) - \gamma(t_0), n_\theta(\gamma(t_0)) \rangle < 0,$$

for all $t \in \mathbb{R}$ such that $t \neq t_0 + zL$, $z \in \mathbb{Z}$. We have to show that there exists a $\delta > 0$ such that $f(t, t_1) < 0$ for all $(t, t_1) \in \mathbb{R}^2$ such that $t_1 \in (t_0 - \delta, t_0 + \delta)$ and $t \neq t_1 + zL$.

Now note that, since f is periodic, it is enough to consider only $(t, t_1) \in [0, L] \times \mathbb{R}$. Also note that $t_1 \in [-\delta, L + \delta]$, since by assumption $t_0 \in [0, L]$. Hence, if we set $\delta \leqslant L$, it is enough to show that $f(t, t_1) < 0$ for all $(t, t_1) \in A := [0, L] \times [-L, 2L]$ such that $t \neq t_1 + zL$.

Let ϵ be as in Proposition B.3.1 and define $B := \{(t, t_0) \mid |t - t_0| > \epsilon/2\}$. By assumption $f(t, t_0) < 0$ for all $t \neq t_0 + zL$, also recall that $\epsilon \leqslant L$; therefore, $f|_B < 0$. Let $U := f^{-1}(-\infty, 0)$, then U is open and $B \subset U$. Hence $\delta := \text{dist}(A - U, B) > 0$

because $A - U$ and B are disjoint and compact. Also note that since $(t_0, t_0) \notin U$, $\text{dist}(A - U, B) \leqslant \text{dist}((t_0, t_0), B) = \epsilon/2$. Thus $\delta \leqslant \epsilon/2$.

Let $|t_1 - t_0| < \delta$.

If $|t - t_0| < \epsilon/2$, then $|t - t_1| \leqslant |t - t_0| + |t_0 - t_1| < \epsilon/2 + \delta \leqslant \epsilon$. Hence by Proposition B.3.1 $f(t, t_1) < 0$ for all $t \neq t_1 + zL$.

If $|t - t_0| \geqslant \epsilon/2$, then, since $(t, t_0) \in B$, $\text{dist}((t, t_1), B) \leqslant |(t, t_1) - (t, t_0)| < \delta$. Hence $(t, t_1) \in U$, and therefore $f(t, t_1) < 0$.

So $f(t, t_1) < 0$ for all $t \neq t_1 + zL$. ■

B.4 Construction of a C^{k-2} nonsingular support

Now we are going to glue together the local one-parameter family of strictly supporting planes constructed in the previous subsection.

First we need to show that any convex combination of the outer unit normals to strictly supporting planes through a point γ_0 of Γ gives a unit normal vector to Γ at γ_0 which generates a strictly supporting plane for Γ at γ_0.

B.4.1. Lemma. *Suppose $n_{\theta_i} \in E_{\gamma_0}(\Gamma)$, $1 \leqslant i \leqslant N$, then $n_\phi \in E_{\gamma_0}(\Gamma)$, where ϕ is any convex combination of $\theta_1 \dots \theta_N$, i.e., $\phi = \sum_{i=1}^N \lambda_i \theta_i$, where $\lambda_i \geqslant 0$ and $\sum_{i=1}^N \lambda_i = 1$.*

Proof. Suppose $\theta_1 \leqslant \theta_i \leqslant \theta_N$, then $\phi \in [\theta_1, \theta_N]$. Hence, by Proposition B.2.5, $n_\phi(\gamma_0) \in E_{\gamma_0}(\Gamma)$. ■

The above lemma allows us to use a partition of unity in the proof of the following theorem.

B.4.2. Theorem. *In $\mathbb{R}^3$, every weakly strictly convex simple closed $C^{k \geqslant 2}$ curve without inflection points admits a C^{k-2} nonsingular support.*

Proof. We are going to show that there is a C^{k-2} function $\phi : \Gamma \to (\pi/2, 3\pi/2)$ such that $n(\gamma) := n_{\phi(\gamma)}(\gamma) \in E_\gamma(\Gamma)$ for all γ in Γ. Note that if such ϕ exists, then the corresponding vector field $n\colon \Gamma \to \mathbb{S}^2$ must be an immersion, because

$$\langle n(\gamma(t))', T(\gamma(t))\rangle = -\kappa(\gamma(t))\cos(\phi(\gamma(t))) \neq 0,$$

and, therefore, $n(\gamma(t))' \neq 0$.

By assumption $E_\gamma(\Gamma) \neq \emptyset$ for all $\gamma \in \Gamma$; therefore, by Lemma B.2.6, for every $\gamma \in \Gamma$ there exists a $\theta(\gamma) \in (\pi/2, 3\pi/2)$ such that

$$n_{\theta(\gamma)}(\gamma) \in E_\gamma(\Gamma).$$

By Lemma B.3.2, for every $\gamma \in \Gamma$, there exists a $\delta(\gamma) > 0$ such that

$$n_{\theta(\gamma)}(\gamma') \in E_{\gamma'}(\Gamma), \quad \forall \gamma' \in \Gamma_{\delta(\gamma),\gamma}.$$

Let $U_\gamma := \Gamma_{\delta(\gamma),\gamma}$. $\{U_\gamma\}$ is an open covering for Γ. Let $\{U_{\gamma_i}\}$, $1 \leqslant i \leqslant N$, be a finite sub-cover and $\{f_i\}$ a partition of unity subordinate to $\{U_{\gamma_i}\}$ and set

$$\phi(\gamma) := \sum_{i=1}^{N} f_i(\gamma)\theta(\gamma_i).$$

Suppose $\gamma \in \Gamma$, then $\gamma \in U_i$ if and only if $i = k_j$, for some $1 \leqslant j \leqslant m$. Consequently, $n_{\theta(\gamma_{k_j})}(\gamma) \in E_\gamma(\Gamma)$, and therefore, by Lemma B.4.1, $n_{\phi(\gamma)}(\gamma) \in E_\gamma(\Gamma)$, because $\phi(\gamma)$ is a convex combination of $\theta(\gamma_{k_1}) \ldots \theta(\gamma_{k_m})$. ■

C Strips of Positive Curvature

In step 2 of the proof of the main theorem, we showed that given a compact strictly convex submanifold $M \subset \mathbb{R}^m$, and a nonsingular support $\sigma\colon M \to \mathbb{S}^{m-1}$, it is possible to construct a strictly convex patch which contains M and is tangent to all the hyperplanes generated by σ. Here we explicitly carry out that construction for the case where $dim(M) = 1$ and $m = 3$. Furthermore, we obtain a formula for the curvature of the patch in the vicinity of M.

C.1 Basic formulas

Let $\Gamma \subset \mathbb{R}^3$ be C^2 Jordan curve. Suppose Γ is strictly convex. Then, by the main theorem, we know that there exists a C^2 nonsingular support $\sigma\colon \Gamma \to \mathbb{S}^2$. In particular, note that Γ has non-vanishing curvature; therefore, for all $p \in \Gamma$, the principal normal, $N(p)$ is well-defined. Furthermore, the angle between $\sigma(p)$ and $N(p)$ is always greater than $\pi/2$.

We are going to construct a strictly convex patch S along Γ by using σ to perturb Γ inside its convex hull by a small distance r, building a tube of radius r around the perturbed curve, and cutting from the tube a narrow strip of width ϵ neighboring Γ. This is done as follows.

C.1.1. Definition. Let $(\mathbb{R}, \gamma)$ be a periodic parameterization of Γ by arclength, and $\{T, N, B\}$ the corresponding Frenet-Serret frame. Define $s : \mathbb{R} \times [-\epsilon, \epsilon] \to \mathbb{R}^3$ by

$$s(t, \theta) := [\gamma(t) - r\sigma(t)] + r[\cos(\theta)\sigma(t) + \sin(\theta)\beta(t)],$$

where r,ϵ are *small* positive constants, $\sigma(t) := \sigma(\gamma(t))$, and $\beta(t) := T(t) \times \sigma(t)$. Let S be the trace of s.

We claim that S is the desired strip. Before we proceed further note that $s(t,0) := \gamma(t)$. Thus $\Gamma \subset S$.

In order to facilitate the computations, it is desirable to have a formula for S which involves only Γ and its standard frame $\{T, N, B\}$. To this end note that we can write:

$$\sigma(t) = \cos(\phi(t))N(t) + \sin(\phi(t))B(t),$$

where $\phi : \Gamma \to (\pi/2, 3\pi/2)$ is some C^2 function. Consequently we have

$$\beta(t) = -\sin(\phi(t))N(t) + \cos(\phi(t))B(t).$$

By substituting the above equations in the first formula and simplifying we get:

$$s(t,\theta) = \gamma(t) + r(\mathcal{C}(t,\theta)N(t) + \mathcal{S}(t,\theta)B(t)),$$

where

$$\mathcal{C}(t,\theta) := \cos(\theta + \phi(t)) - \cos(\phi(t)),$$
$$\mathcal{S}(t,\theta) := \sin(\theta + \phi(t)) - \sin(\phi(t)).$$

The following identities will further help us in simplifying the calculations:

$$\frac{\partial \mathcal{C}}{\partial t} = -\mathcal{S}\phi'(t), \qquad \frac{\partial \mathcal{S}}{\partial t} = \mathcal{C}\phi'(t),$$

$$\frac{\partial \mathcal{C}}{\partial \theta} = -\sin(\theta + \phi(t)), \qquad \frac{\partial \mathcal{S}}{\partial \theta} = \cos(\theta + \phi(t)).$$

Using these formulas we now can prove:

C.1.2. Proposition. *S is an embedded surface.*

Proof. Using the formulas given above we compute

$$\frac{\partial s}{\partial t} = (1 - r\mathcal{C}\kappa)T + r(\phi' + \tau)(-\mathcal{S}N + \mathcal{C}B),$$
$$\frac{\partial s}{\partial \theta} = r(-\sin(\theta + \phi)N + \cos(\theta + \phi)B),$$

where κ and τ are the curvature and torsion of Γ respectively. From these it follows that

$$\frac{\partial s}{\partial t} \times \frac{\partial s}{\partial \theta} = r(r\mathcal{C}\kappa - 1)[\cos(\theta + \phi)N + \sin(\theta + \phi)B]$$
$$-r^2(\phi' + \tau)Sin(\theta)T,$$

and consequently

$$\left|\frac{\partial s}{\partial t} \times \frac{\partial s}{\partial \theta}\right| = r[(r\mathcal{C}\kappa - 1)^2 + r^4(\phi' + \tau)^2 \sin^2\theta]^{\frac{1}{2}}.$$

Now note that

$$\left.\frac{\partial s}{\partial t} \times \frac{\partial s}{\partial \theta}\right|_{(t,0)} = r\sigma \qquad \text{and} \qquad \left|\frac{\partial s}{\partial t} \times \frac{\partial s}{\partial \theta}\right|_{(t,0)} = r;$$

therefore, since $r > 0$, and s is a periodic function of t, by a compactness argument it follows that for ϵ sufficiently small $\left|\frac{\partial s}{\partial t} \times \frac{\partial s}{\partial \theta}\right|_{(t,\theta)} \neq 0$, for all $|\theta| \leqslant \epsilon$. Hence we conclude that S is an immersed surface. Now since $\Gamma \subset S$ is embedded and compact, by Lemma 4.2.3, S has to be embedded as well. ■

C.1.3. Proposition. *S is tangent to the hyperplanes generated by σ.*

Proof. Let $\nu(t, \theta) := \nu(s(t, \theta))$ be the Gauss map of S. Recall that

$$\nu(t, \theta) := \frac{\frac{\partial s}{\partial t} \times \frac{\partial s}{\partial \theta}}{\left|\frac{\partial s}{\partial t} \times \frac{\partial s}{\partial \theta}\right|},$$

which is well-defined by the previous proposition. Thus

$$\nu(t, 0) = \frac{r\sigma(t)}{r} = \sigma(t).$$

■

C.2 Curvature calculations

Here we prove:

C.2.1. Proposition. *S has positive curvature.*

Proof. We begin by calculating the second derivatives:

$$\frac{\partial^2 s}{\partial t^2} = [(1 - r\mathcal{C}\kappa)' + r(\phi' + \tau)\mathcal{S}\kappa]T$$

$$+[\kappa(1 - r\mathcal{C}\kappa) + [r(\phi' + \tau)]'(-\mathcal{S}) + r(\phi' + \tau)(-\mathcal{S}' - \mathcal{C}\tau)]N$$

$$+r[(\phi' + \tau)'\mathcal{C} + (\phi' + \tau)(\mathcal{C}' - \mathcal{S}\tau)]B,$$

$$\frac{\partial^2 s}{\partial t \partial \theta} = r\{\kappa \sin(\theta + \phi)T - (\phi' + \tau)[\cos(\theta + \phi)N + \sin(\theta + \phi)B]\},$$

$$\frac{\partial^2 s}{\partial \theta^2} = -r[\cos(\theta + \phi)N + \sin(\theta + \phi)B].$$

By setting $\theta = 0$ we get :

$$\frac{\partial^2 s}{\partial t^2} = \kappa N, \qquad \frac{\partial^2 s}{\partial t \partial \theta} = r[\kappa \sin(\phi)T + (\phi' + \tau)\sigma], \qquad \frac{\partial^2 s}{\partial \theta^2} = -r\sigma.$$

Recall that $\nu(t, 0) = \sigma(t)$. Thus at $\theta = 0$:

$$\left\langle \frac{\partial^2 s}{\partial t^2}, \nu \right\rangle = -\kappa \cos(\phi), \quad \left\langle \frac{\partial^2 s}{\partial t \partial \theta}, \nu \right\rangle = -r(\phi' + \tau), \quad \left\langle \frac{\partial^2 s}{\partial \theta^2}, \nu \right\rangle = r.$$

The above are the coefficients , l_{ij}, of the second fundamental form; therefore,

$$\det(l_{ij}) = -\kappa r \cos(\phi) - r^2(\phi' + \tau)^2.$$

Also recall that

$$\det(g_{ij}) = \left| \frac{\partial s}{\partial t} \times \frac{\partial s}{\partial \theta} \right|^2 = r^2,$$

where g_{ij} are the coefficients of the first fundamental form.

Let k denote the Gaussian curvature of S and set $k(t,\theta) := k(s(t,\theta))$. Recalling that $k := \det(l_{ij})/\det(g_{ij})$, we have

$$\boxed{k(t,0) = -\frac{\kappa}{r}\cos(\phi) - (\phi' + \tau)^2}.$$

We know that s and therefore k are periodic in first variable. Suppose the period is L and set

$$M := \max_{t\in[0,L]} \left\{ \frac{(\phi'(t) + \tau(t))^2}{-\kappa(t)\cos(\phi(t))} \right\}.$$

Note that M is well-defined and positive because $\kappa(t) \neq 0$, and $\phi(t) \in (\pi/2, 3\pi/2)$, both by assumption.

Now if we let

$$0 < r < 1/M$$

then $k(t,0) > 0$ for all $t \in [0,L]$ and consequently $k(t,\theta) > 0$ for $|\theta| \leqslant \epsilon$, if ϵ is sufficiently small. ■

D Curvature Bounds

Let $M \subset \mathbb{R}^m$ be a smooth compact strictly convex submanifold, $\sigma\colon M \to S^{m-1}$ a smooth nonsingular support, and O a corresponding integral ovaloid. Let $p \in M$, and $X_p \in T_pM$. Then the normal curvature of O at p in the direction X_p is the same as the normal (Lipschitz-Killing) curvature of M at p in the direction X_p, and with respect to the normal vector $\sigma(p)$. Thus the best possible bounds for the radii of curvature of O are given by the bounds for the radii of curvature of M in the direction of σ. In general, it is not possible to achieve these bounds; however, it might be possible to do so when M is closed, i.e. compact, connected, and complete. This would follow from a suitable generalization of Blaschke's rolling theorem to higher codimensions. Towards that end, we have already taken some steps in the constructions which we used to prove the main theorem of this dissertation. Since we were primarily motivated to prove the *existence*, however, not all the additional information with regard to the curvature of the solution have been explored. The following are some preliminary investigations.

D.1 Interpolating C^1 ovaloids

Suppose $\dim(M) = 0$, i.e., M is a finite discrete subset. Denote the elements of M by x_i, $1 \leqslant i \leqslant N$. Let $\sigma_i := \sigma(x_i)$. Then

$$\langle x_i - x_j, \sigma_j \rangle < 0, \tag{D.1.1}$$

for all $1 \leqslant i, j \leqslant N$, $i \neq j$. Note also that, conversely, given a set of points $x_i \in \mathbb{R}^m$, and unit vectors $\sigma_i \in \mathbb{S}^{m-1}$, if the above inequality is satisfied, then $\{x_i\}$ is strictly convex and $\{\sigma_i\}$ generates a corresponding family of strictly supporting hyperplanes. More explicitly, let

$$H_i := \{\, x \in \mathbb{R}^m \mid \langle x - x_i, \sigma_i \rangle = 0 \,\}, \tag{D.1.2}$$

and set $H_i^- := \{\, x \in \mathbb{R}^m \mid \langle x - x_i, \sigma_i \rangle \leqslant 0 \,\}$, then $M - \{x_i\} \subset H_i^-$, if and only if (D.1.1) is satisfied.

Now suppose that $(x_i, \sigma_i) \in \mathbb{R}^m \times \mathbb{S}^{m-1}$ satisfy (D.1.1). We are going to explicitly construct a C^1 ovaloid O such that $O \cap H_i := \{x_i\}$. We shall prove this in three steps (see Figure 16):

Step 1. Let $\epsilon > O$, and set

$$\tilde{x}_i := x_i - \epsilon \sigma_i. \tag{D.1.3}$$

If ϵ is sufficiently small, then the set $\widetilde{M} := \{\tilde{x}_i\}$ will again be strictly convex, with σ_i as support vectors. To obtain an estimate for ϵ, set $\langle \tilde{x}_i - \tilde{x}_j \rangle < 0$, for $i \neq j$. After substituting (D.1.3) in this formula and performing some easy manipulations we get

$$\epsilon < \inf_{i \neq j} \left\{ \frac{-\langle x_i - x_j, \sigma_j \rangle}{1 - \langle \sigma_i, \sigma_j \rangle} \right\}. \tag{D.1.4}$$

It is easy to show that the right hand side of the above is well-defined and positive. The numerator is positive by (D.1.1). Furthermore, $\langle \sigma_i, \sigma_j \rangle \leqslant 1$, because σ_i are unit vectors. So all we need to check is that $\sigma_i \neq \sigma_j$ when $i \neq j$. This follows from (D.1.1). To see this, suppose $i \neq j$ and observe that $\langle x_i - x_j, \sigma_i - \sigma_j \rangle = \langle x_i - x_j, \sigma_j \rangle + \langle x_j - x_i, \sigma_i \rangle$. Since both quantities on the right hand side are negative, it follows that the left hand side is negative as well. In particular, $\sigma_i \neq \sigma_j$. So we conclude that there exists an $\epsilon (> 0)$ which satisfies (D.1.4).

Step 2. Now for each $(\tilde{x}_i, \sigma_i)$ we are going to construct a ball B_i such that (i) $\tilde{x}_i \in \mathrm{bd}(B_i)$, (ii) σ_i is an outer normal to $\mathrm{bd}(B_i)$, and (iii) B_i contains all points $\tilde{x}_i$, i.e., $\widetilde{M} \subset B_i$. Let $\lambda > 0$, and set

$$B_i := B(\lambda, \tilde{x}_i - \lambda\sigma_i), \tag{D.1.5}$$

i.e., a ball of radius λ centered at the point $\tilde{x}_i - \lambda\sigma_i$. It is clear that B_i satisfies properties (i) and (ii) mentioned above. Furthermore, if λ is sufficiently large, (iii) is satisfied as well. To get an estimate for λ note that we need to have $x_i \in B_j$. This happens when $\operatorname{dist}(\tilde{x}_i, \tilde{x}_j - \lambda\sigma_j) \leqslant \lambda$. Using the formula $\operatorname{dist}(p, q) = \langle p - q, p - q\rangle^{1/2}$ we obtain

$$\lambda \geqslant \sup_{i \neq j} \left\{ \frac{|\tilde{x}_i - \tilde{x}_j|^2}{-2\langle \tilde{x}_i - \tilde{x}_j, \sigma_j\rangle} \right\}. \tag{D.1.6}$$

This concludes step 2.

Step 3. Let

$$O := \operatorname{bd}(\bigcap_{i=1}^{N} B_i + \epsilon B^m), \tag{D.1.7}$$

where B^m denotes the unit ball. We claim that O is the desired ovaloid. To see this note that

$$\lambda \leqslant \quad radii\, of\, curvature\, of\, O \quad \leqslant \lambda + \epsilon, \tag{D.1.8}$$

because a ball of radius ϵ rolls freely inside O and O rolls freely inside a ball of radius $\lambda + \epsilon$. Thus O is an ovaloid. Furthermore, it is easy to see that $O \cap H_i = \{x_i\}$.

D.2 General estimates

Here we extend the results in the previous subsection to the general case, i.e., where $\dim(M)$ is arbitrary. Suppose $M \subset \mathbb{R}^m$ is a compact strictly convex smooth submanifold and $\sigma\colon M \to \mathbb{S}^{m-1}$ is a smooth nonsingular support. Set

$$0 < \epsilon < \overline{\epsilon} := \inf_{p \neq q} \left\{ \frac{-\langle p - q, \sigma(q)\rangle}{1 - \langle \sigma(p), \sigma(q)\rangle} \right\},$$

and let $\widetilde{M} := \{\, \tilde{p} \mid \tilde{p} := p - \epsilon\sigma(p), p \in M \,\}$, be an inward perturbation of M by a distance of ϵ along σ. Define $\tilde{\sigma}\colon \tilde{M} \to \mathbb{S}^{m-1}$ by $\tilde{\sigma}(\tilde{p}) = \sigma(p)$. Set

$$\infty > \lambda \geqslant \underline{\lambda} := \sup_{p \neq q} \left\{ \frac{|\tilde{p} - \tilde{q}|^2}{-2\langle \tilde{p} - \tilde{q}, \tilde{\sigma}(\tilde{q})\rangle} \right\},$$

and

$$O := \mathrm{bd}\Big(\bigcap_{p \in M} B(\lambda, \tilde{p} - \lambda\tilde{\sigma}(\tilde{p})) + \epsilon B^m\Big).$$

If ϵ and λ exist, then, much like the previous subsection, it can be shown that O is an ovaloid which contains M and is tangent to the hyperplanes generated by σ. Furthermore,

$$\epsilon \leqslant \quad \textit{radii of curvature of } O \quad \leqslant \lambda + \epsilon.$$

In the remainder of this subsection, we are going to show that ϵ and λ exist, i.e, $\overline{\epsilon} > 0$ and $\underline{\lambda} < \infty$. It turns out that $\overline{\epsilon}$ and $\underline{\lambda}$ are geometrically significant quantities. They can be expressed in terms of the Lipschitz-Killing curvature of M in the direction of σ.

First, we are going to examine $\overline{\epsilon}$. Let us see what happens to $\overline{\epsilon}$ as p and q approach each other. Let $\gamma\colon (-\delta, \delta) \to M$ be a unit speed curve with $\gamma(0) = q$ and $\gamma(t) = p$. Then we compute:

$$\begin{aligned}
\lim_{t \to 0} \frac{-\langle \gamma(t) - \gamma(0), \sigma(\gamma(0))\rangle}{1 - \langle \sigma(\gamma(t)), \sigma(\gamma(0))\rangle} &= \frac{\langle \gamma''(0), \sigma(\gamma(0))\rangle}{\langle (\sigma \circ \gamma)''(0), \sigma(\gamma(0))\rangle} \\
&= \frac{\langle \gamma'(0), (\sigma \circ \gamma)'(0)\rangle}{|(\sigma \circ \gamma)'(0)|^2} \\
&= \frac{-\sum_{i=1}^n c_i^2 k_i}{\sum_{i=1}^n c_i^2 k_i^2},
\end{aligned}$$

where $k_i = k_i(p, \sigma(p))$ are the principal curvatures of M at p with respect to $\sigma(p)$, and c_i are the components of the vector $\gamma'(0)$ with respect to the basis of principal directions of M at p with respect to $\sigma(p)$.

The first equality follows by twice applying L'Hopital's rule.

To see the second, observe that since $\langle \gamma'(t), \sigma(\gamma(t))\rangle = 0$, $\langle \gamma''(t), \sigma(\gamma(t))\rangle = -\langle \gamma'(t), (\sigma\circ\gamma)'(t)\rangle$; similarly, since $\langle \sigma(\gamma(t)), \sigma(\gamma(t))\rangle = 1$, it follows that $\langle (\sigma\circ\gamma)'(t), \sigma(\gamma(t))\rangle = 0$, which yields $\langle (\sigma\circ\gamma)''(0), \sigma(\gamma(0))\rangle = -\langle (\sigma\circ\gamma)'(0), (\sigma\circ\gamma)'(0)\rangle$.

To see the final equality, recall that $(\sigma\circ\gamma)'(0) = \sigma_*(\gamma'(0))$. Let X_q^i, $1 \leqslant i \leqslant n$, be the principal directions of M at q with respect to $\sigma(q)$. These form a basis for T_qM. Since $\gamma'(0) \in T_q(M)$, we have $\gamma'(0) = \sum_{i=1}^n c_i X_q^i$, for some constants c_i. Using these formulas we get $\langle \gamma'(0), (\sigma\circ\gamma)'(0)\rangle = -\sum_{i,j=1}^n c_i c_j \langle X_q^i, X_q^j\rangle = -\sum_{i=1}^n c_i^2 k_i$, Furthermore, $(\sigma\circ\gamma)'(0) = \sigma_*(\gamma'(0)) = \sum_{i=1}^n c_i \sigma_*(X_q^i) = \sum_{i=1}^n c_i k_i X_q^i$. Thus $|(\sigma\circ\gamma)'(0)|^2 = \sum_{i=1}^n c_i^2 k_i^2$.

Note that $\overline{\epsilon}|_{M\times M - \Delta(M)} > 0$. The above computation shows that $\overline{\epsilon}$ is positive when restricted to an open neighborhood of the diagonal, $\Delta(M)$. Hence we conclude that $\overline{\epsilon} > 0$.

Now, if $0 < \epsilon < \overline{\epsilon}$, then the inner parallel submanifold $\widetilde{M}$ will be a strictly convex submanifold, strictly supported by $\tilde{\sigma}(\tilde{p}) := \sigma(p)$.

Next we are going to show that $\underline{\lambda} < \infty$. Let $\tilde{\gamma}\colon (-\delta, \delta) \to \tilde{M}$ be a unit speed curve with $\tilde{\gamma}(0) = \tilde{q}$ and $\tilde{\gamma}(t) = \tilde{p}$. Then, much as before, we can compute the corresponding limit as $\tilde{p}$ approaches $\tilde{q}$:

$$\begin{aligned}
\lim_{t\to 0} \frac{|\tilde{\gamma}(t) - \tilde{\gamma}(0)|^2}{-2\langle \tilde{\gamma}(t) - \tilde{\gamma}(0), \tilde{\sigma}(\tilde{\gamma}(0))\rangle} &= \frac{\langle \tilde{\gamma}''(0), \tilde{\gamma}(0)\rangle}{-\langle \tilde{\gamma}''(0), \tilde{\sigma}(\tilde{\gamma}(0))\rangle} \\
&= \frac{-\langle \tilde{\gamma}'(0), \tilde{\gamma}'(0)\rangle}{\langle \tilde{\gamma}'(0), (\tilde{\sigma}\circ\tilde{\gamma})'(0)\rangle} \\
&= \frac{-1}{\sum_{i=1}^n c_i^2 \tilde{k}_i}
\end{aligned}$$

So we conclude that $\underline{\lambda}$ is well bounded as well.

E Problems

We begin with the question [Yau1, problem #26] which provided the prime stimulus for the work in this dissertation.

E.0.1. Question (S.-T. Yau). *Given a metric of positive curvature on the disk what is the condition on a space curve to form the boundary of an isometric embedding of the disk?*

We know a *necessary* condition, discovered by H. Rosenberg [Ros], involving the self-linking number, and the main result of this dissertation provides a *sufficient* criterion. Perhaps studying the following question would help us narrow the gap.

E.0.2. Question (H. Rosenberg). *Does every curve bounding a surface of positive curvature in 3-space have four vertices, i.e., points where the torsion vanishes?*

There exists an interesting literature on four vertex theorems, including the recent solution to a long standing conjecture of P. Scherk., by V. D. Sedykh [Sed2].

Next I mention a number of conjectures, starting with the one which may be the most accessible.

In Section 5.5, we proved that in 3-space the self-linking number of strictly convex Jordan curves is zero. It should be possible to weaken the hypothesis:

E.0.3. Conjecture. *Let $\Gamma \subset \mathbb{R}^3$ be a simple closed curve with non-vanishing curvature. Suppose Γ is convex, i.e., lies on the boundary of its convex hull. Then the self-linking number of Γ is zero.*

Note that if Γ is convex then it may not bound any surfaces of positive curvature (see Figure 7); therefore, we cannot apply Rosenberg's result to prove the above conjecture.

We cannot hope to learn a great deal about the boundaries, if we do not properly understand the geometry of the surfaces themselves. To this end, an important problem is deciding when a hypersurface of positive curvature with boundary is strictly convex, i.e., lies strictly one side of all of its tangent hyperplanes. Although positiveness of curvature ensures that this is always locally true, generically a hypersurface of positive curvature with nonempty boundary is far from being globally strictly convex. Still, I claim that there exists the following global property:

E.0.4. Conjecture. *Every compact connected hypersurface of positive curvature lies entirely on one side of at least one of its tangent hyperplanes.*

Furthermore, if we impose conditions on the boundary, we can obtain stronger results. Let us say a submanifold of Euclidean space is convex if it lies on the boundary of its convex hull. With this terminology in mind, I claim:

E.0.5. Conjecture. *Every compact connected hypersurface of positive curvature with connected convex boundary is embedded and its interior lies outside the convex hull of its boundary.*

With somewhat less conviction I also conjecture that the boundary does not have to be connected. Perhaps, all we need to assume is that each boundary component be convex. The above phenomenon would be the opposite to the well known convex hull property of surfaces of negative curvature, see the paper by R. Osserman [Osm1]. Also, as we mentioned in the introduction, it is a well known that the convexity of the boundary implies embeddedness of the surface when the surface is minimal. See the papers of W. Meeks & S.-T. Yau [MY], F. Almgren & L. Simon [AS], F. Tomi & A. J. Tromba [TT], and R. Gulliver & J. Spruck [GulS].

The following would extend the works of several authors, including L. Caffarelli,

L. Nirenberg, J. Spruck, N. Krylov, and B. Guan, on the Dirichlet problem for Monge-Ampére equations:

E.0.6. Conjecture. *If a closed embedded submanifold of codimension two bounds a hypersurface of positive curvature, then it bounds a hypersurface of constant positive curvature.*

As was mentioned in the introduction the above is known [GS] in the case where M projects injectively into a sphere. For completeness, we should also include the following (see Section 5.6 for references):

E.0.7. Conjecture (Carathéodory). *Every ovaloid in 3-space has at least two umbilic points, i.e., points where all the principal curvatures are equal.*

References

[Aub] T. Aubin, *Nonlinear analysis on manifolds, Monge-Ampére equations*, Springer-Verlag, New York, 1982.

[AC] S. B. Alexander, and R. J. Currier, *Hypersurfaces and Nonnegative Curvature*, Proc. of Symp. in Pure Mathematics, **54**(1993), part III.

[AS] F. Almgren, and L. Simon, *Existence of embedded solutions of Plateau's problem*, Ann. Scuola Norm. Sup. Pisa, **6**(1979), 447–495.

[Ban] T. F. Banchoff, *The two-piece property and tight n-manifolds-with-boundary in* E^n, Trans. Amer. Math. Soc., **161**(1971), 259–267.

[BF] T. Bonnesen, and W. Fenchel, *Theory of convex bodies*, BCS Associates, Moscow, Idaho, 1987.

[BG] M. Berger, and B. Gostiaux, *Differential Geometry: Manifolds, Curves, and Surfaces*, Springer-Verlag, New York, 1988.

[BS] J. N. Brooks, and J. B. Strantzen, *Blaschke's rolling theorem in* R^n, Memoirs Amer. Math. Soc., **80**(1989), 101pp.

[CFF] G. Carlos, M. Francesco, and S.-B. Federico, *On a conjecture of Carathéodory: analyticity versus smoothness*, Experiment. Math., **5** (1996), 33-37.

[CL] S.-S. Chern, and R. K. Lashof, *On total curvature of immersed manifolds*, Michigan Math. J., **5**(1958), 5–12.

[CNS1] L. A. Caffarelli, L. Nirenberg, and J. Spruck, *The Dirichlet problem for nonlinear second-order elliptic equations, I. Monge-Ampére equations*, Comm. Pure Appl. Math., **37**(1984), 369–402.

[CNS2] ——, *The Dirichlet problem for degenerate Monge-Ampére equation*, Revista Matematica Iberoamericana, **2**(1986), 19–27.

[CR] T. E. Cecil, and P. J. Ryan, *Tight and taut immersions of manifolds* (Research notes in math. vol. 107), Pitman Advanced Publishing Program, Boston-London-Melbourne, 1985.

[dCLa] M. do Carmo, and H. B. Lawson, Jr., *Spherical images of convex surfaces*, Proc. Amer. Math. Soc., **31**(1972), 635–636.

[dCLi] M. do Carmo, and E. Lima, *Immersions of manifolds with non-negative sectional curvatures*, Bol. Soc. Brasil. Mat., **2**(1971), 9–22.

[Del] J. A. Delgado, *Blaschke's theorem for convex hypersurfaces*, J. Diff. Geom., **14**(1979), 489–96.

[Ech] J.-H. Eschenburg, *Local convexity and nonnegative curvature-Gromov's proof of the sphere theorem*, Invent. math., **84**(1986), 507–522.

[Fel] E. A. Feldman, *On parabolic and umbilic points of immersed hypersurfaces*, Trans. Amer. Soc., **127**(1967), 1–28.

[Glk] H. Gluck, *Manifolds with preassigned curvature - a survey*, Bull. Amer. Math. Soc., **81**(1975), 245–276.

[GP] H. Gluck, and L. Pan, *Embedding and Knotting of Positive Curvature Surfaces in 3-Space*, Topology, to appear.

[Gry] A. Gray, *Modern differential geometry of curves and surfaces*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1993.

[GS] B. Guan, and J. Spruck, *Boundary value problems on $\mathbb{S}^n$ for surfaces of constant Gauss curvature*, Ann. of Math., **138**(1993), 601–624.

[GT] D. Gilbarg, and N. S. Trudinger, *Elliptic partial differential equations of second order, second ed.*, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983.

[GulS] R. D. Gulliver, and J. Spruck, *On embedded minimal surfaces*, Ann. of Math., **103**(1976), 331–347, with a correction in Ann. of Math., **109**(1979), 407–412.

[Gun] B. Guan, *The Dirichlet Problem for Monge-Ampére Equations in Non-Convex Domains and Spacelike Hypersurfaces of Constant Gauss Curvature*, Trans. Amer. Math. Soc., to appear.

[GW] P. M. Gruber, and J. M. Wills (editors), *Handbook of convex geometry*, Elsevier Science Publishers B.V., Amsterdam, 1993.

[Had] J. Hadamard, *Sur certaines proprietés des trajectoires en dynamique.*, J. Math. Pures Appl., **3**(1897), 331–387.

[Hjn] J. van Heijenoort, *On locally convex manifolds*, Comm. Pure Appl. Math., **5**(1952), 223–242.

[Hpf] H. Hopf, *Differential Geometry in the Large*, Lecture Notes in Mathematics No. 1000, Springer-Verlag, 1983.

[HRS] D. Hoffman, H. Rosenberg, and J. Spruck, *Boundary value problems for surfaces of constant Gauss curvature*, Comm. Pure Appl. Math., **45**(1992), 1051–1062.

[Htm] P. Hartman, *On complete hypersurfaces of nonnegative sectional curvatures and constant* m^{th} *mean curvature*, Trans. Amer. Math. Soc., **245**(1978), 363–374.

[Kft] D. Koutroufiotis, *On Blaschke's rolling theorems*, Arch. Math., **23**(1972), 655–660.

[Klz] T. Klotz, *On G. Bol's proof of Cartheodory's conjecture*, Comm. Pure Appl. Math., **12**(1959), 277–311.

[Knl] W. Kühnel, *Total curvature of manifolds with boundary in* E^n, J. London. Math. Soc. (2), **2**(1977), 173–182.

[Kpr] N. H. Kuiper, *Minimal total absolute curvature for immersions*, Invent. Math., **10**(1970), 209–238.

[Krl] N. V. Krylov, *Boundedly nonhomogeneous elliptic and parabolic equations in a domain,* Izvestia Akad. Nauk. SSSR, **47**(1983), 75–108.

[Min] H. Minkowski, *Volumen und Oberfläche.* Math. Ann., **57**(1903), 447–495.

[Mun] J. Munkres, *Elementary Differential Topology*, Princeton University Press, Princeton, New Jersey, 1966.

[MY] W. H. Meeks III and S.-T. Yau, *The classical Plateau problem and the topology of three dimensional manifolds*, Topology, **21**(1982), 409–440.

[Osm1] R. Osserman, *The convex hull property of immersed manifolds*, J. Differential Geom., **6**(1971/72), 267–270.

[Osm2] ——, *A survey of minimal surfaces*, Dover Publications, NY, 1986.

[Pog] A. V. Pogorelov, *The Minkowski multidimensional problem*, Winston & Sons, Washington DC, 1975.

[RCB] M. C. Romero Fuster, S.I. Costa, and J. J. Ballesteros, *Some global properties of closed space curves*, Differential Geometry. Proceedings, Pensicola 1988 . Lecture Notes in Math. Vol. 1410, Springer-Verlag.

[Rch] J. Rauch, *An inclusion theorem for ovaloids with comparable second fundamental form*, J. Differential Geom., **9**(1974), 501–505.

[Rod] L. L. Rodríguez, *The two-piece property and convexity for surfaces with boundary*, J. Diff. Geom., **11**(1976), 235–250.

[Ros] H. Rosenberg, *Hypersurfaces of constant positive curvature in space forms*, Bull. sc. math., **117**(1993), 211–239.

[Sac] R. Sacksteder, *On hypersurfaces with no negative sectional curvatures,* Amer. J. Math., **82**(1960), 609–630.

[Sch1] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Encyclopedia of mathematics and its applications, v. 44, Cambridge University Press, Cambridge, UK, 1993.

[Sch2] ——, *Equivariant endomorphisms of the space of convex bodies*, Trans. Amer. Math. Soc., **194**(1974), 53–78.

[Sch3] ——, *Personal communication (Letter to W. Firey, 1991).*

[Sed1] V. D. Sedykh, *Structure of the convex hull of a space curve*, J. Soviet Math., **33**(1986), 499–505.

[Sed2] ——, *Four Vertices of a Convex Space Curve*, Bull. London Math. Soc., **26**(1994), 177–180.

[Stk] J. J. Stoker, *Über die Gestalt der positiv gekrümmten offenen Flächen im dreidimensionalen Raume*, Compositio Math., **3**(1936), 55–89.

[SX] B. Smyth, and F. Xavier, *Efimov's theorem in dimension greater than two*, Invent. Math., **90**(1987), 443–450.

[Tit] C. Titus, *A proof of a conjecture of Carathéodory on umbilic points*, Acta Mathematica, **131**(1973), 43–77.

[TT] F. Tomi, and A. J. Tromba, *Extreme curves bound an embedded minimal surface of the disk type*, Math. Z., **158**(1978), 137–145.

[TU] N. S. Trudinger, and J. I. E. Urbas, *On second derivative estimates for equations of Monge-Ampére type*, Bull. Austral. Math. Soc., **30**(1984), 321–334.

[Wl] W. Weil, *Einschachtelung konvexer Körper*, Arch. Math., **26**(1975), 666–9.

[Wu] H.-H. Wu, *The spherical images of convex hypersurfaces*, J. Differential Geom., **9**(1974), 279–290.

[Yau1] S.-T. Yau, *Open Problems in Geometry*, Proc. of Symp. in Pure Mathematics, **54**(1993), part I.

[Yau2] ——, *Survey on partial differential equations in differential geometry*, Seminar on differential geometry, Princeton U. Press, Princeton NJ, 1982.

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