SHORTEST PERIODIC BILLIARD TRAJECTORIES IN CONVEX BODIES

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Abstract. We show that the length of any periodic billiard trajectory in any convex body $K \subset \mathbf{R}^n$ is always at least 4 times the inradius of K; the equality holds precisely when the width of K is twice its inradius, e.g., K is centrally symmetric, in which case we prove that the shortest periodic trajectories are all bouncing ball (2-link) orbits.

1. Introduction

Motivated by applications to inverse spectral problems ("can one hear the shape of a drum?"), S. Zelditch [5] has recently raised the question of whether every shortest periodic billiard trajectory in a bi-axisymmetric smooth convex planar body is a bouncing ball (2-link) orbit. Our main result (Theorem 1.2 below), which holds for all convex bodies in Euclidean space $\mathbf{R}^n$, implies that the answer is yes.

By a *convex body* $K \subset \mathbf{R}^n$ we mean a compact convex subset with interior points. A *periodic billiard trajectory* T in K is a sequence of distinct boundary points $p_i \in \partial K$, $i \in \mathbf{Z}/N\mathbf{Z}$, $N \geq 2$, called *links* of T, such that, for every i,

$$n_i := \frac{p_i - p_{i-1}}{\|p_i - p_{i-1}\|} + \frac{p_i - p_{i+1}}{\|p_i - p_{i+1}\|} \tag{1}$$

is an (outward) support vector of K at p_i; that is

$$\langle x - p_i, n_i \rangle \leq 0, \quad \text{for all } x \in K. \tag{2}$$

When T has exactly two links ($N = 2$) we say it is a *bouncing ball orbit*. The *length* of T is defined by

$$\text{length}(T) := \sum_{i=1}^{N} \|p_i - p_{i+1}\|.$$

The main question we are concerned with in this paper is:

Problem 1.1. In which convex bodies are the shortest periodic billiard trajectories bouncing ball orbits?

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The following result provides a sufficient criterion in terms of natural geometric measures: the *inradius* of K, which is the radius of the largest ball contained in K; and the *width* of K, which is the thickness of the narrowest slab which contains K.

Theorem 1.2. *Let $K \subset \mathbf{R}^n$ be a convex body, and T be a periodic billiard trajectory in K. Then*

$$\text{length}(T) \geq 4 \,\text{inradius}(K).$$

Further, the equality holds for some T, if and only if $\text{width}(K) = 2\,\text{inradius}(K)$. *In this case, every shortest periodic trajectory of K is a bouncing ball orbit.*

We say K is *centrally symmetric* provided that (after a translation) K coincides with its reflection through the origin of $\mathbf{R}^n$ (this notion is weaker than "bi-axisymmetric"). The above theorem yields:

Corollary 1.3. *If $K \subset \mathbf{R}^n$ is a centrally symmetric convex body, then each of the shortest periodic billiard trajectories in K is a bouncing ball orbit.*

Proof. By theorem 1.2 it is enough to check that $\text{width}(K) = 2\,\text{inradius}(K)$. Set $r := \text{inradius}(K)$. Then K contains a ball B of radius r. Let B' be the "central symmetrization" of B, that is,

$$B' := \frac{B + (-B)}{2} = \left\{ \frac{x - y}{2} \,\middle|\, x, y \in B \right\}.$$

Then B' is a ball of radius r centered at the origin of $\mathbf{R}^n$. Further, since K is centrally symmetric ($K = -K$), it follows that $B' \subset K$. Since the inradius of K is r, the boundary of K, ∂K, must intersect B' at a point p. Then $-p$ also has to lie in $\partial K \cap B'$ by symmetry. Let $H \subset \mathbf{R}^n$ be a support hyperplane of K at p. Then H is also a support hyperplane of B' which yields that H is orthogonal to the segment $(-p)p$. By symmetry, the reflection of H, which we denote by $-H$, is also a supporting hyperplane, and is orthogonal to $(-p)p$ at $-p$. So the distance between H and $-H$ is $2r$, which yields that $\text{width}(K) \leq 2r$. On the other hand, we always have $\text{width}(K) \geq 2\,\text{inradius}(K) = 2r$, which completes the proof. ☐

Note 1.4. Without the central symmetry assumption, Corollary 1.3 does not in general remain valid. An equilateral triangle, whose corners may be rounded off by a small amount, would be a counterexample [1]. Here the shortest orbit is the (orthic) triangle which is obtained by connecting the midpoints of the sides (Figure 1(a)).

Note 1.5. As is known, and easy to show, any smooth convex planar body has at least two bouncing ball orbits (one is determined by the "diameter" and the other by the "width"). If one forgoes convexity, however, one may construct smooth planar regions without any bouncing ball orbits that are contained in

[1] This observation has been known to Yves Colin de Verdiere [6].

the region (we say that a billiard trajectory is contained in a region if the line segment connecting any pairs of consecutive links is contained in that region). An example of a smooth region which does not contain any bouncing ball orbits is illustrated in Figure 1(b). This region has been known to Bos [1].

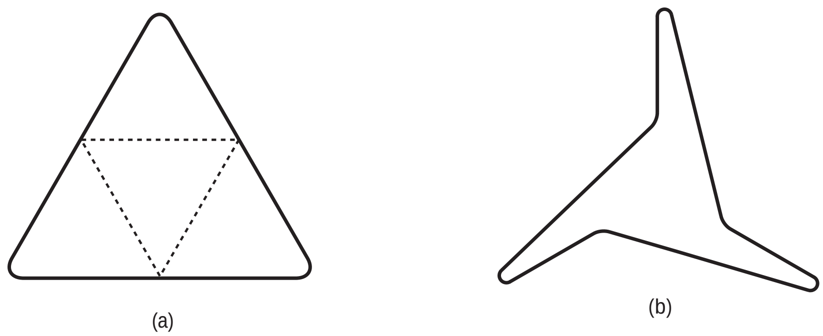

FIGURE 1

Note 1.6. Theorem 1.2 gives a lower bound for the length of shortest periodic trajectories of a convex body; however, this bound is sharp only when $2\,\mathrm{inradius}(K)$ is equal to $\mathrm{width}(K)$. Can one find a lower bound which is sharp in all cases? (Of course this lower bound should coincide with $4\,\mathrm{inradius}(K)$ whenever $2\,\mathrm{inradius}(K)$ is equal to $\mathrm{width}(K)$.) Can one find a sharp *upper* bound for the length of shortest periodic billiard trajectories of a convex body? One estimate, in terms of volume, is given by Viterbo [4].

2. Proof of Theorem 1.2

The proof of Theorem 1.2 follows from the two propositions below. By a *polygon* P we mean a sequence of distinct points $p_i \in \mathbf{R}^n$, $i \in \mathbf{Z}/N\mathbf{Z}$ (in particular, every periodic billiard trajectory is a polygon). Each p_i is called a vertex of P, and the length of P is defined as $\sum_{i=1}^{N} \|p_i - p_{i+1}\|$. The *convex hull* of a set $X \subset \mathbf{R}^n$, which we denote by $\mathrm{conv}\, X$, is the intersection of all convex sets which contain X.

Proposition 2.1. *Let $B \subset \mathbf{R}^n$ be a ball of radius r, and $P \subset \mathbf{R}^n$ be a polygon such that the center of B lies in* $\mathrm{conv}\, P$, *and the vertices of P lie outside the interior of B. Then* $\mathrm{length}(P) \geq 4r$, *with equality only if P has exactly two vertices.*

The proof of the above proposition follows from a little trigonometry (Lemma 3.1) together with some integral geometry (Crofton's formula). The proof of the following proposition, uses the above proposition and some basic convexity theory.

Proposition 2.2. *Let $K \subset \mathbf{R}^n$ be a convex body, and T be a periodic billiard trajectory in K. Suppose that there exists a ball B of radius r in K whose center* *does not lie in* $\operatorname{conv} T$*. Then* $\operatorname{length}(T) \geq 4r$*, with equality only if* T *is a bouncing ball orbit.*

The proof of the above propositions appear in the next two sections. Using these, we prove Theorem 1.2 as follows:

Let $r := \operatorname{inradius}(K)$ and $B \subset K$ be a ball of radius r. Then either $\operatorname{conv} T$ contains the center of B or not. Thus Proposition 2.1 or Proposition 2.2 respectively establish that $\operatorname{length}(T) \geq 4r$.

Now suppose that $\operatorname{length}(T) = 4r$, then, either by Proposition 2.1 or 2.2, T is a bouncing ball orbit. In particular, the support hyperplanes of K which pass through each link of T, and are orthogonal to the support vectors n_i given by (1), are parallel. So $\operatorname{width}(K) \leq 2r$. But we always have that $\operatorname{width}(K) \geq 2r$. So $\operatorname{width}(K) = 2r$.

Conversely, suppose $\operatorname{width}(K) = 2r$. Then there exists a ball B of radius r in K, and K is contained in between a pair of parallel support hyperplanes of separation $2r$. So B must intersect both support hyperplanes, and it is clear that the intersection points generate a bouncing ball orbit T with $\operatorname{length}(T) = 4r$, since the line connecting the intersection points is orthogonal to both support hyperplanes.

So we conclude that $\operatorname{width}(K) = 2r$ if and only if K contains some bouncing ball orbit of length $4r$. In particular, whenever $\operatorname{width}(K) = 2r$, every shortest periodic trajectory $T \subset K$ has length $4r$. Thus, since $\operatorname{inradius}(K) = r$, it follows from Propositions 2.1 and 2.2, that every shortest periodic trajectory of K is a bouncing ball orbit.

3. Proof of Proposition 2.1

Lemma 3.1. *Let* $0 \leq \alpha_i \leq \pi/2$*,* $i = 1, \dots N$*, and* $\sum_{i=1}^N \alpha_i \geq \pi$*. Then*

$$\sum_{i=1}^{N} \sin \alpha_i \geq 2, \tag{3}$$

with equality if and only if all $\alpha_i = 0$ *save two.*

Proof. Since $2/\pi < 1$, comparing the graph of $y = \sin x$ with that of $y = (2/\pi)x$ quickly yields that

$$\sin x \geq \frac{2}{\pi} x$$

for all $0 \leq x \leq \pi/2$. Thus, since by assumption $\sum_{i=1}^N \alpha_i \geq \pi$, we have

$$\sum_{i=1}^{N} \sin \alpha_i \geq \frac{2}{\pi} \sum_{i=1}^{N} \alpha_i \geq 2.$$

If $\sum_{i=1}^N \sin \alpha_i = 2$, then, the above expression yields

$$\sum_{i=1}^{N} \sin \alpha_i = \frac{2}{\pi} \sum_{i=1}^{N} \alpha_i.$$

But $\sin\alpha_i \geq (2/\pi)\alpha_i$, thus the above equality yields

$$\sin\alpha_i = \frac{2}{\pi}\alpha_i.$$

The last equality holds only when $\alpha_i = 0$, or $\alpha_i = \pi/2$. Thus we conclude that the equality holds in (3) precisely when all $\alpha_i = 0$ save two. □

Let p_i, $i = 1, \dots, N$, be the vertices of P, o be the center of the ball B, which we may assume to coincide with the origin of $\mathbf{R}^n$, and $2\alpha_i := \angle p_i o p_{i+1}$ be the corresponding angles. We claim that

$$\sum_{i=1}^{N} \alpha_i \geq \pi. \tag{4}$$

To see this note that $\sum_{i=1}^{N} 2\alpha_i$ is equal to the length of the projection of P, say P', into a sphere S of radius 1 centered at o. Since o is in the convex hull of P, o lies in the convex hull of P' as well. In particular, every great circle in S intersects P'; thus, it follows from Crofton's formula [2] that length$(P') \geq 2\pi$. So (4) holds which in turn allows us to apply Lemma 3.1 to conclude that (3) holds as well.

Next note that, since by assumption $\|p_i\| \geq r$,

$$\begin{aligned} \|p_i - p_{i+1}\|^2 &= \|p_i\|^2 + \|p_{i+1}\|^2 - 2\|p_i\|\|p_{i+1}\|\cos 2\alpha_i \\ &\geq 2r^2(1 - \cos 2\alpha_i) \\ &= 4r^2 \sin^2\alpha_i. \end{aligned}$$

Thus, the above inequality together with (3) yield that

$$\begin{aligned} \text{length}(P) &\geq \sum_{i=1}^{N} \|p_i - p_{i+1}\| \\ &\geq 2\,r \sum_{i=1}^{N} \sin\alpha_i \\ &\geq 4r. \end{aligned}$$

Now suppose that the equality in the above inequality holds. Then, by Lemma 3.1, all $\alpha_i = 0$, save two. In this case the remaining two angles must each be equal to $\pi/2$ by (4). Thus all vertices of P must lie on a line which passes through the center o of B. Since the vertices of P lie outside of the interior of B, and are distinct, it then follows that P has exactly two vertices.

4. Proof of Proposition 2.2

Since, by assumption, the center o of B does not lie in $\operatorname{conv} T$, which is a convex set, there exists a unique point p of $\operatorname{conv} T$ which is closest to o [3]. After a rigid motion we may assume that o is the origin of $\mathbf{R}^n$, and p lies on the negative half of the x_n-axis. So

$$\frac{p}{\|p\|} = (0, 0, \dots, -1), \tag{5}$$

and p is the "highest" point of $\operatorname{conv} T$. That is,

$$\pi(p) \geq \pi(p_i),$$

where $\pi\colon \mathbf{R}^n \to \mathbf{R}$ is projection into the n^{th} coordinate and p_i, $i = 1, \dots, N$ are the links of T. Suppose, towards a contradiction, that $p \in T$. That is $p = p_k$, a link of T. Let $n := n_k$ be as in (1). That is,

$$n := \frac{p - p_{k-1}}{\|p - p_{k-1}\|} + \frac{p - p_{k+1}}{\|p - p_{k+1}\|}.$$

Combining the previous two displayed expressions yields

$$\pi(n) \geq 0.$$

On the other hand, since o is an interior point of K and n is a support vector, (5) and (2) yields that

$$\pi(n) = \langle -\frac{p}{\|p\|}, n\rangle = \frac{1}{\|p\|}\langle o - p, n\rangle < 0,$$

which is a contradiction. So we conclude that $p \notin T$.

Since $p \in \partial \operatorname{conv} T$, but $p \notin T$, it follows that p lies in the (relative) interior of a *face* F of T. Let p_k be a vertex of F. Note that since no link of T is higher than p, F has to be "horizontal" (parallel to $x_n = 0$ hyperplane), which yields that

$$\pi(p_k) = \pi(p) \geq \pi(p_i),$$

for all i. So, as we had argued earlier, since

$$n_k := \frac{p_k - p_{k-1}}{\|p_k - p_{k-1}\|} + \frac{p_k - p_{k+1}}{\|p_k - p_{k+1}\|},$$

it follows that

$$\pi(n_k) \geq 0,$$

which in turn yields

$$\langle p, n_k\rangle = \|p\|\langle \frac{p}{\|p\|}, n_k\rangle = -\|p\|\pi(n_k) \leq 0.$$

Further, note that, since $o \in K$, (2) yields

$$\langle p_k, n_k\rangle = -\langle o - p_k, n_k\rangle \geq 0.$$

The previous two inequalities yield:

$$\langle p_k - p, n_k\rangle = \langle p_k, n_k\rangle - \langle p, n_k\rangle \geq \langle p_k, n_k\rangle. \tag{6}$$

Next, recall that, since $p \notin T$, $p \neq p_k$. Thus

$$r\frac{p_k - p}{\|p_k - p\|} \in B \subset K.$$

This via (2) yields

$$\left\langle r\frac{p_k - p}{\|p_k - p\|} - p_k, n_k \right\rangle \leq 0,$$

which we may rewrite as

$$r\langle p_k - p, n_k\rangle \leq \|p_k - p\|\langle p_k, n_k\rangle.$$

Comparing the above inequality with (6), we obtain

$$r \leq \|p_k - p\|.$$

Thus all the vertices of the face F are at least a distance r away from p. Let T_F be the subpolygon of T composed of the vertices of F, and with the ordering inherited from T, then the triangle inequality together with Proposition 2.1 yield

$$\text{length}(T) \geq \text{length}(T_F) \geq 2r.$$

If $\text{length}(T) = 2r$, then the above expression yields that $\text{length}(T_F) = \text{length}(T)$, which implies $T_F = T$. Further, we also get $\text{length}(T_F) = 2r$, which, by Proposition 2.1, implies that T_F has only two vertices, which completes the proof.

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