Topology of Riemannian Submanifolds with Prescribed Boundary and Curvature Bounded Below

Mohammad Ghomi

Georgia Institute of Technology
Atlanta, USA

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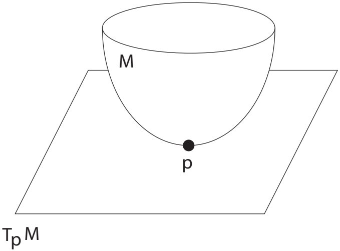
The most basic example of submanifolds with curvature bounded below are probably hypersurfaces with positive (sectional) curvature in Euclidean space

These surfaces are *locally convex*, i.e., each point has a neighborhood which lies locally on one side of its tanget plane.



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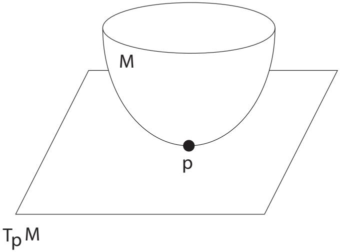

The basic structure theorem for complete hypersurfaces of positive curvature:

Theorem (Hadamard/Stoker/van Heijenoort/Sacksteder)

Let M be a (geodesically) complete hypersurface immersed in $\mathbf{R}^n$. Suppose that M is locally convex and is locally strictly convex at one point; or that M has nonnegative sectional curvature and has positive sectional curvature at one point. Then M is convex (i.e., forms the boundary of a convex set).

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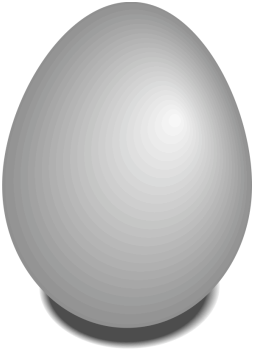
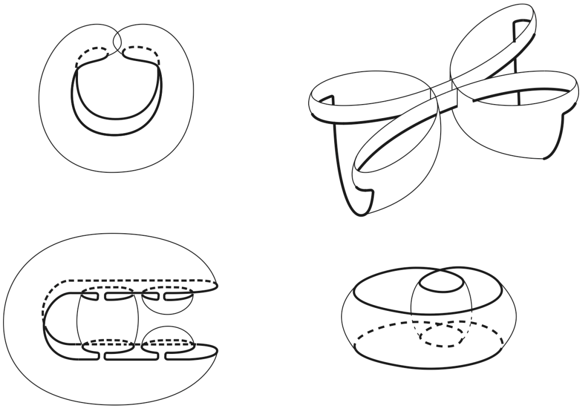
So complete surfaces of positive curvature (without boundary) have a very simple structure:



Positively curved surfaces with boundary, on the other hand, may be quite complicated both geometrically and topologically.



In particular, these surfaces may not be convex or embedded, not even when their boundary lies on a convex body.

Furthermore, any type of topology may occure:

Theorem (H. Gluck and L. Pan)

Every compact 2-dimensional manifold with boundary admits an embedding in $\mathbf{R}^3$ *with positive curvature.*

Proof.

By explicit constructions. □

Theorem (M. Ghomi and M. Kossowki)

The same results holds for hyprsurfaces in $\mathbf{R}^n$

Proof.

An application of the "h-principle" in the sense of Gromov, via the "holonomic of approximation theorem " of Eliashberg and Mishachev. □

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Can one control the geometry/topology of a locally convex hypersurface by imposing conditions on its boundary?

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The most fundamental open problem which is concerned with locally convex hypersurfaces with boundary is perhaps the following:

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When does a simple closed curve Γ in $\mathbf{R}^3$ bound a surface of positive curvature?

Yau's Question and Related Problems

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Connection with surfaces of constant curvature

Theorem (Guan and Spruck)

Let $\Gamma \subset \mathbf{R}^n$ be a closed submanifold of codimension 2. If Γ bounds a surface of positive curvature, then it bounds a locally convex hypersurface of constant positive Gaussian curvature.

(So characterizing boundaries of surfaces with positive curvature is equivalent to characterizing boundaries of surfaces with constant positive curvature.)

The proof uses the Peron's method and a convergence theorem of S. Alexander and M.G.

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Definition (Linking Number)

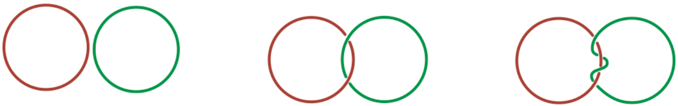
Let M, N be a pair of closed disjoint submanifolds of $\mathbf{R}^n$. Suppose that

$$\dim(M) + \dim(N) = n - 1.$$

Then the *linking number* of M and N is defined as the degree of the Gauss map $g\colon M \times N \to \mathbf{S}^{n-1}$ given by

$$g(p, q) := \frac{p-q}{\|p-q\|}.$$

Examples



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Definition (Self-Linking Number)

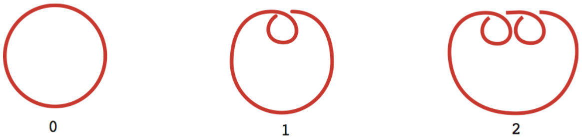
Let $\gamma \colon I \to \mathbf{R}^3$ be a C^2 closed curve without inflection points. Then the principal normal vectorfield $N \colon I \to \mathbf{R}^3$ of γ is well defined. Let

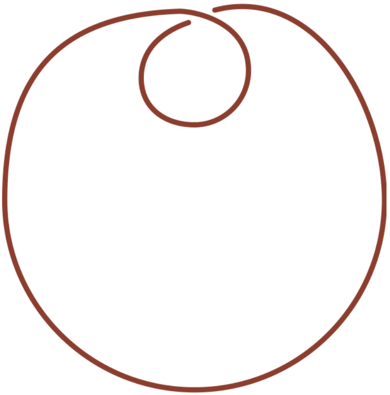
$$\gamma_\epsilon(t) := \gamma(t) + \epsilon N.$$

For ϵ sufficiently small, γ and γ_ϵ are disjoint. The *self linking number* of γ is defined as the linking number of γ and γ_ϵ:

$$SL(\gamma) = L(\gamma, \gamma_\epsilon).$$

Examples





This curve will never bound a positively curved surface (not even an immersed one).

Theorem (H. Rosenbrg)

If $\Gamma \subset \mathbf{R}^3$ bounds an embedded surface M of positive curvature, then its self-linking number must be zero.

Proof.

Let n be the inward normal of M and N be the principle normal vector field of Γ. Since M is locally convex,

$$\Big\langle \gamma(t) - \gamma(t_0), n\big(\gamma(t_0)\big) \Big\rangle \geq 0.$$

for t near t_0. Differentiating twice yields

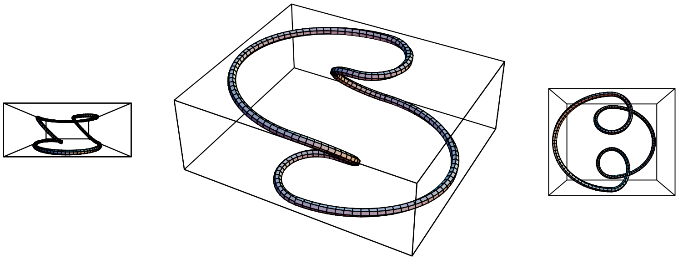
$$\langle N, n \rangle \geq 0.$$

Let $\Gamma_\epsilon := \Gamma + \epsilon N$ and $\tilde{\Gamma}_\epsilon := \Gamma + \epsilon n$. Then

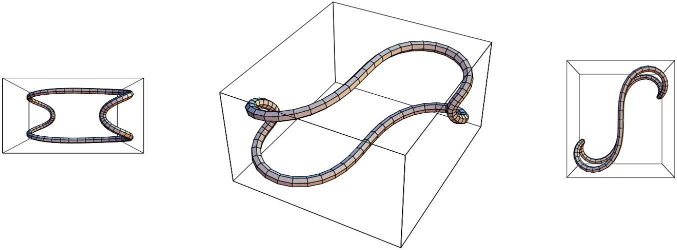
$$\tilde{\Gamma}_\epsilon^\theta := \Gamma + \epsilon(N \cos\theta + n \sin\theta)$$

is a regular isotopy between Γ_ϵ and $\tilde{\Gamma}_\epsilon$ *in the complement of* Γ. So $L(\Gamma, \Gamma_\epsilon) = L(\Gamma, \tilde{\Gamma}_\epsilon)$. But the latter is zero, because $\tilde{\Gamma}_\epsilon \cap M = \emptyset$.

Herman Gluck was the first to show that the condition $SL(\Gamma) = 0$ is *not sufficient* for the existence of a positively curved surface bounded by Γ:



There is even a curve with $SL(\Gamma) = 0$ which lies on the boundary of its convex hull, but still does not bound a positively curved surface:



To prove this, one uses the condition $\langle n, N \rangle \neq 0$ along the boundary curve Γ to show that if Γ bounded a positively curved surface M, then M would have to enter the convex hull of Γ. This would violate a convex hull property for locally convex hypersurfaces by S. Alexander and M.G.

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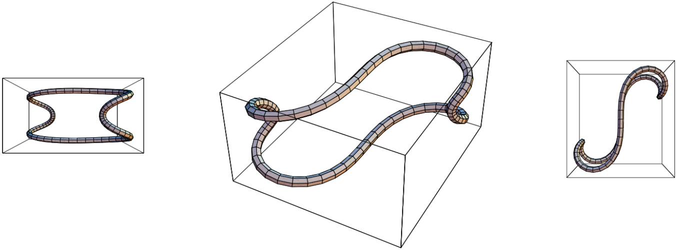

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What other obstructions can there be?

Question (H. Rosenberg)

Does the boundary of every positively curved surface has at least 4 *vertices*, i.e., points where the torsion vanishes?

Note

It is plausible that the answer is yes, since by a Theorem of V. D. Sedykh, any closed curve which lies on the boundary of a convex body must have at least 4 vertices.

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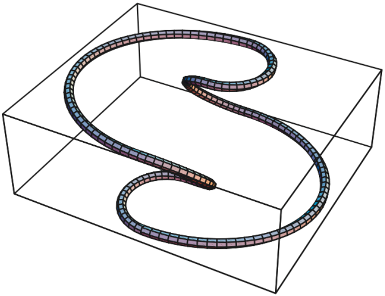
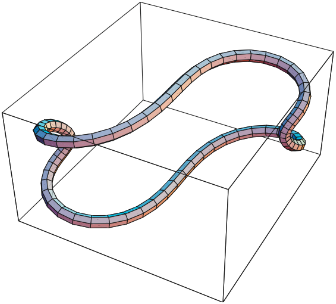
Perhaps one could also discover another obstruction if one could figure out what property the following curves have in common

besides having vanishing self-linking number and bounding no positively curved surfaces.

Topological finiteness

Question (Bo Guan and Joel Spruck)

Does a compact submanifold of codimension 2 in $\mathbf{R}^n$ always bound at most finitely many topological types of compact hypersurfaces with positive curvature?

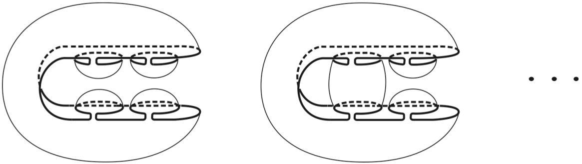
The Convex Hull Property

Theorem (S. Alexander and M.G.)

Let $M \subset \mathbf{R}^n$ be a compact one-sidedly locally convex hypersurface. Suppose that ∂M is embedded and lies on the boundary of its convex hull C. Then the interior of M lies completely outside C. Further, M is homeorphic to the closure of a component of $\partial C - \partial M$ (So topology of M is determined up to at most 2 choices).

Note

The fact that the convexity of the boundary determines the topology of the surface up to at most two choices is significant; because for every integer n there exists a simple closed curve in $\mathbf{R}^3$ which bounds n topologically distinct surfaces of positive curvature.

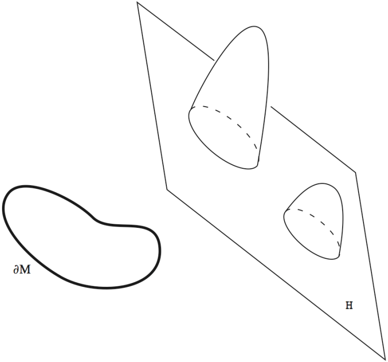


Lemma

Let M be a compact one sided locally convex hypersurface in $\mathbf{R}^{n+1}$ with convex boundary $\Gamma = \partial M$, i.e., suppose that Γ lies on the boundary of its own convex hull C. If M lies inside C, then M has to be homeomorphic to a component of $\partial C - \Gamma$.

Proposition (The clipping lemma)

Let $M \subset \mathbf{R}^n$ be a compact locally convex surface. Suppose that ∂M lies on one side of a hyperplane H. Then the portion of M lying on the other side of H consists of a finite number of convex caps.



Corollary

Let $M \subset \mathbf{R}^n$ be a compact locally convex surface. Suppose that ∂M lies inside a convex polytope P. Then there exists a locally convex surface $\tilde{M}$, homeomorphic to M, and with $\partial M = \partial \tilde{M}$, such that $\tilde{M} \subset P$.

Next we need a convergence result:

Recall:

Theorem (W. Blaschke)

Suppose we have a family of convex bodies $K_i \subset \mathbf{R}^n$ whose inradii are bounded below and outradii bounded above, then K_i contains a convergent subsequence.

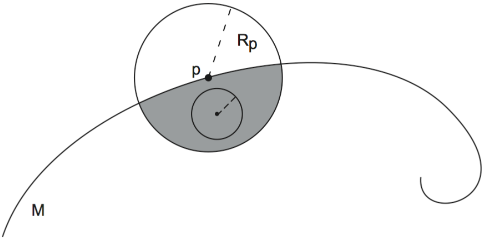
We need an analogues result for locally convex surfaces with boundary:

Definition (Radii of convexity of a locally convex surface)

For every $p \in M$ let R_p be the largest radius such that $M \cap B_{R_p}(p)$ is convex. The *radius of convexity* of M is

$$R(M) := \inf_p R_p.$$

Further, we define the *inradius of convexity* $r(M)$ as the supremum of the inradius of the convex bodies in $B_R(p)$ on which $M \cap B_R(p)$ lies.



Theorem (S. Alexander and M.G.)

Let $M_i \subset \mathbf{R}^n$ be a family of homeomorphic one-sided locally convex hypersurfaces with indentical boundary $\partial M_i = \Gamma$. Suppose that the radius of convexity and inradius of M_i are uniformly bounded. Further, for any $\lambda > 0$, the number of elements in some λ-net of M_i has a uniform upper bound. Then M_i has a convergent subsequence whose limit is a locally convex hypersurface homeomorphic to M_i.

More on topological finiteness

Let us again recall:

Question (Bo Guan and Joel Spruck)

Does a compact submanifold of codimension 2 in $\mathbf{R}^n$ always bound at most finitely many topological types of compact hypersurfaces with positive curvature?

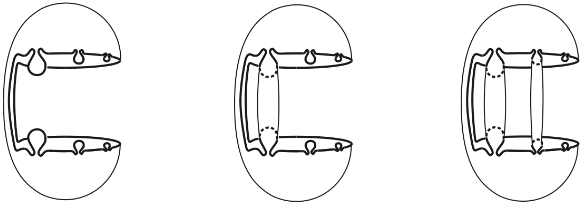
The Topological Finiteness Theorem

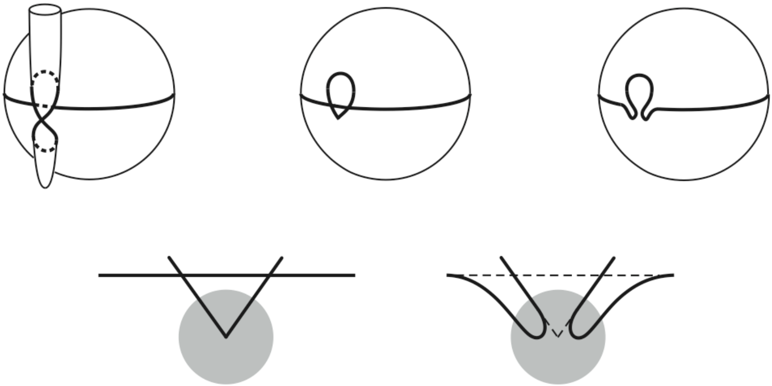
The answer is yes provided the boundary is sufficiently smooth:

Theorem (S. Alexander, M.G., J. Wong)

A smooth, compact, immersed (not necessarily connected) submanifold of codimension 2 *in* $\mathbf{R}^{n+1}$, $n \geq 2$, *bounds at most finitely many topologically distinct, compact, nonnegatively curved immersed hypersurfaces.*

On the other hand, there exists a rectifiable closed curve in $\mathbf{R}^3$ that is differentiable in its arclength parameter, is C^∞ in the complement of two points, and bounds infinitely many topologically distinct, compact, embedded, positively curved C^∞ surfaces.





Theorem

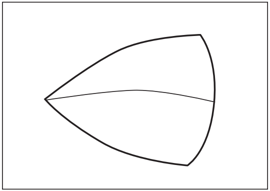
Let Σ_1 and Σ_2 be smooth strictly convex compact disks of positive curvature in $\mathbf{R}^3$ which are tangent to each other at a common interior point p. Suppose that Σ_1 and Σ_2 lie locally on the same side of their common tangent plane at p, and that their intersection near p consists of a finite number of smooth curve segments each of which emanates from p in a distinct direction. Then for every open neighborhood U_1 of p in Σ_1 there exists an open neighborhood U_2 of p in Σ_2 and a smooth positively curved compact disk Σ such that $(\Sigma_1 - U_1) \cup U_2 \subset \Sigma$, and Σ contains $\Sigma_1 \cap \Sigma_2$.

The finiteness theorem is proved by placing it in the context of Alexandrov spaces of curvature bounded below.

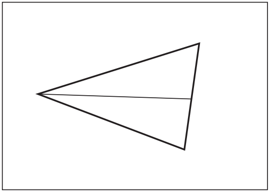
Definition

An *Alexandrov space of curvature* $\geq K$ is a length metric space in which geodesic triangles are *wider* than comparison triangles in the simply connected space form S_K of constant curvature K.

Specifically, for any geodesic triangle the distance from a vertex to a point on the opposite side is at least the distance between corresponding points on a geodesic triangle with the same sidelengths in S_K.



K > c



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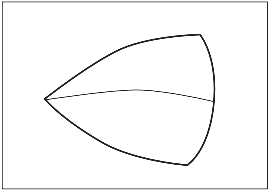

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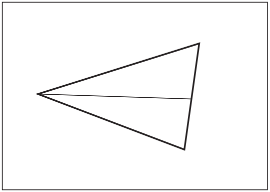
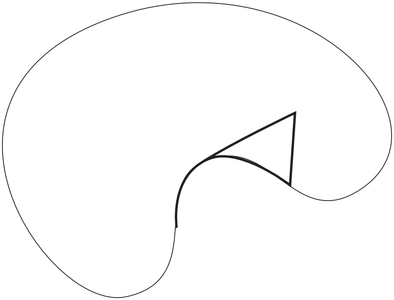


K = c

Note

A complete Riemannian manifold M with boundary is an Alexandrov space of curvature bounded below if and only if its interior sectional curvatures are bounded below and its boundary is locally convex, with respect to the inward normal.

If the boundary is not locally convex, we may have branching of geodesics which will cause infinite negative curvature:



There is a deep relation between Alexandrov spaces of curvature bounded below and topological finitenes:

Theorem (Gromov Compactness)

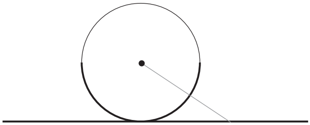
Let $\mathcal{M} = \mathcal{M}(K, n, V, D) =$ Alexandrov spaces of curvature $\geq K$, dim $= n$, vol $\geq V$, and diam $\leq D$, where $\mathcal{M}$ carries the Gromov-Hausdorff metric. Then $\mathcal{M}$ is compact.

Theorem (Perelman Stability)

If $X \in \mathcal{M}$, then any $Y \in \mathcal{M}$ sufficiently close to X is homeomorphic to X.

To get the upper diameter bound we use a projective transformation:

Let Γ be a compact, immersed submanifold of codimension 2 in $\mathbf{R}^{n+1}$. Let $\mathcal{F}_\Gamma$ denote the family of all compact, nonnegatively curved, immersed hypersurfaces having Γ as boundary.



Since $\mathbf{R}^{n+1}$ is projectively equivalent to a hemisphere in $\mathbf{S}^{n+1}$, we may regard the elements $M \in \mathcal{F}_\Gamma$ as hypersurfaces of sectional curvature ≥ 1 in $\mathbf{S}^{n+1}$ (as was done by do Carmo and Warner).

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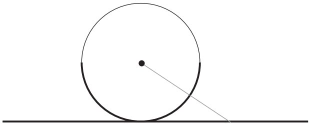

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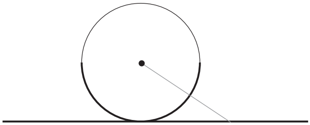

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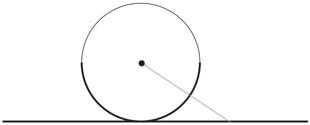

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Denote the components of Γ by Γ_i, $1 \leq i \leq l$. The following lemma bounds the intrinsic diameter of any $M \in \mathcal{F}_\Gamma$ in terms of the intrinsic diameters $\operatorname{diam}(\Gamma_i)$ of the Γ_i.

Lemma

The diameter of $M \in \mathcal{F}_\Gamma$ is uniformly bounded above by

$$(l+1)\pi + \Sigma_{i=1}^{l} \operatorname{diam}(\Gamma_i).$$

Proof.

By Rauch's comparison theorem, $d_M(p, \Gamma) \leq \pi$ for all $p \in M$. $\square$

Next we have to turn our manifolds with boundary into Alexandrov spaces:

Lemma (Kossovskii's)

Let M_1 and M_2 be two Riemannian manifolds-with-boundary, each having sectional curvatures bounded below by K, and whose boundaries are isometric and have respective second fundamental forms the sum of which is positive semidefinite. Then the space obtained by isometrically gluing M_1 to M_2 along their common boundary is an Alexandrov space of curvature bounded below by K.

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Lemma (Wong's Thesis)

Suppose Γ *is any compact Riemannian manifold. Then for any* $\lambda > 0$*, there exists a Riemannian manifold* $M = \Gamma \times [0, 1]$ *such that*

- $\partial M_0 = \Gamma \times \{0\}$ *is isometric to* Γ*;*
- $II_{\partial M_0} - \lambda I$ *is positive definite;*
- $\partial M_1 = \Gamma \times \{1\}$ *is locally convex.*

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- $\partial M_0 = \Gamma \times \{0\}$ *is isometric to* Γ;
- $II_{\partial M_0} - \lambda I$ *is positive definite;*
- $\partial M_1 = \Gamma \times \{1\}$ *is locally convex.*

Lemma (Wong's Thesis)

Suppose Γ *is any compact Riemannian manifold. Then for any* $\lambda > 0$, *there exists a Riemannian manifold* $M = \Gamma \times [0, 1]$ *such that*

- $\partial M_0 = \Gamma \times \{0\}$ *is isometric to* Γ;
- $II_{\partial M_0} - \lambda I$ *is positive definite;*
- $\partial M_1 = \Gamma \times \{1\}$ *is locally convex.*

Note

The finiteness theorem also holds for noncompact hypersurfaces.

This follows from a theorem of Cai which states that the number of ends is finite, for any complete Riemannian manfiold which has nonnegative Ricci curvature off of a ball and its Ricci curvature is bounded below everywhere.

Further, there is a structure theorem of Alexander and Currier which states that each end has a convex representative.
So the ends may be clipped off.

Also note that:

Theorem (S. Alexander, M.G., J. Wong)

A compact submanifold Γ of an arbitrary Riemannian manifold bounds at most finitely many compact submanifolds with curvature bounded below, and diameter bounded above.

The diameter bound is not needed for surfaces:

Theorem (S. Alexander, M. G., J. Wong)

A finite collection of closed C^3 curves Γ immersed in a given Riemannian manifold $\overline{M}$ bounds at most finitely many topologically distinct, complete, immersed C^3 surfaces M whose total curvature is uniformly bounded below.

The proof is an easy corollary of Gauss-Bonnet:

Let $\gamma \colon [0, L] \to \overline{M}$ be a unit speed parametrization for Γ, and $v \colon [0, L] \to \mathbf{R}^3$ be the unit normal vector field along γ which is tangent to M and points inside M .

$$\int_\Gamma \kappa_g := \int_0^L \langle \overline{\nabla}_{\gamma'(t)}\gamma', v(t)\rangle \, dt \geq -\int_0^L \|\overline{\nabla}_{\gamma'(t)}\gamma'\| \, dt =: -\int_\Gamma \overline{\kappa}_g \, dt.$$

where $\overline{\kappa}_g$ is the curvature of Γ in the ambient space $\overline{M}$. So,

$$\chi(M) = \int_M K + \int_\Gamma \kappa_g$$

is bounded below.

Question

Does the topological finiteness theorem hold for all Riemannian submanifolds of any dimension and codimension (without the diameter bound)?

No!

Theorem

The standard embedding of $\mathbf{S}^2$ in $\mathbf{R}^3 \subset \mathbf{R}^{18}$ bounds infinitely many topologically distinct C^∞ compact 3-dimensional submanifolds with positive curvature.

The proof follows from a relative version of Nash's isometric embedding theorem. Let $N(n,k)$ be the smallest integer such that every C^k Riemannian n-manifold (M,g) admits a C^k isometric embedding into the Euclidean space $\mathbf{R}^{N(n,k)}$.

Theorem (M.G., and R. Greene)

Let (M,g) be a $C^{k\geq 1}$ Riemannian n-manifold, $p \in M$, and U be a neighborhood of p. Suppose there exists a C^k isometric embedding $f\colon U \to \mathbf{R}^m$, $m > n$. Then there exists an isometric embedding $\overline{f}\colon M \to \mathbf{R}^{N(n,k)+m}$, and a neighborhood $V \subset U$ of p, with closure $\overline{V}$ diffeomorphic to a ball, such that

$$\overline{f}|_{\overline{V}} = f,$$

$\overline{f}$ is C^1 on M, and C^k on $M - \partial V$.

Theorem (M.G., and R. Greene)

Let (M, g), p, U, and $f\colon U \to \mathbf{R}^m$, be as in the last theorem. Then there exists a neighborhood $V \subset U$ of p and a C^k embedding $\overline{f}\colon M \to \mathbf{R}^{N(n,k)+m}$, such that

$$\overline{f}|_{\overline{V'}} = f,$$

for any open neighborhood V' of p with $\overline{V}' \subset V$; the pull-back metric $\overline{f}^*\langle , \rangle$ is as C^k-close to g as desired; and, $\overline{f}^*\langle , \rangle = g$ outside any given open neighborhood of ∂V.

Apply the last result to lens spaces $M_i := \mathbf{S}^3/Z_i$, so that a nighborhood of each M_i coincides with a piece of the standard $\mathbf{S}^3 \subset \mathbf{R}^4 \subset \mathbf{R}^{18}$. (Here "18" is due to a result of Gunther). Then after a clipping and rescaling, we obtain submanifolds bounded by $\mathbf{S}^2$.

Question

What is the highest codimension where we have topological finiteness without the diameter bound (it has to be between 2 and 15)?

Perhaps it has to be 2. Can one embed all lens spaces S^3/Z_i in $\mathbf{R}^6$ with positive curvature?

(In fact there is another question of Yau which asks if all compact orientable n-manifolds with almost constant curvature can be isometrically embedded in $\mathbf{R}^{2n}$.)

The End