Isoperimetric problems arising in the physics of thin structures and in geometry

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Nanoelectronics

- Quantum wires
- Quantum waveguides
- Designer potentials - STM places individual atoms on a surface; quantum dots
- Semi- and non-conducting “threads”

    Simplified mathematical models

An electron near a charged thread

LMP 2006, with Exner and Loss

$$H_{\alpha,\Gamma} = -\Delta - \alpha\delta(x - \Gamma)$$

Fix the length of the thread. What shape binds the electron the least tightly? Conjectured for about 3 years that answer is circle.

Reduction to an isoperimetric problem of classical type.

Is it true that:

$$\int_0^L |\Gamma(s+u) - \Gamma(s)|\,\mathrm{d}s \le \frac{L^2}{\pi} \sin \frac{\pi u}{L}$$

Reduction to an isoperimetric problem of classical type.

Birman-Schwinger reduction. A negative eigenvalue of the Hamiltonian corresponds to a fixed point of the Birman-Schwinger operator:

$$\mathcal{R}^{\kappa}_{\alpha,\Gamma}\phi = \phi\,, \quad \mathcal{R}^{\kappa}_{\alpha,\Gamma}(s,s') := \frac{\alpha}{2\pi} K_0\big(\kappa|\Gamma(s)-\Gamma(s')|\big)$$

K_0 is the Macdonald function (Bessel function that is the kernel of the resolvent in 2 D).

About Birman-Schwinger

With a factorization due to Birman and Schwinger, an operator H will have eigenvalue λ iff the family of operators $B(\lambda)$ has eigenvalue 1.

Birman - Schwinger

Consider the negative eigenvalues of $-\Delta + V(x)$ on $\mathbb{R}^n$, $V(x)$ minimally regular (Kato class) and $\to 0$ at ∞.

Suppose u is an eigenfunction,

$$(-\Delta + V(x))\, u = \lambda u, \quad \lambda < 0. \qquad ☆$$

$$(-\Delta + |\lambda|)\, u = - V(x)\, u$$

$$u = -(-\Delta + |\lambda|)^{-1} V(x)\, u.$$

Let $\phi = \sqrt{|V(x)|}\, \operatorname{sgn} V(x)\, u$. Then

$$\phi = \left[-\sqrt{|V(x)|}\, \operatorname{sgn} V(x)\, (-\Delta + |\lambda|)^{-1} \sqrt{|V(x)|} \right] \phi$$

$$=: B_\lambda\, \phi.$$

Simplify: $V(x) < 0$. Then $B_\lambda = \sqrt{|V(x)|}\, (-\Delta + |\lambda|)^{-1} \sqrt{|V(x)|}$

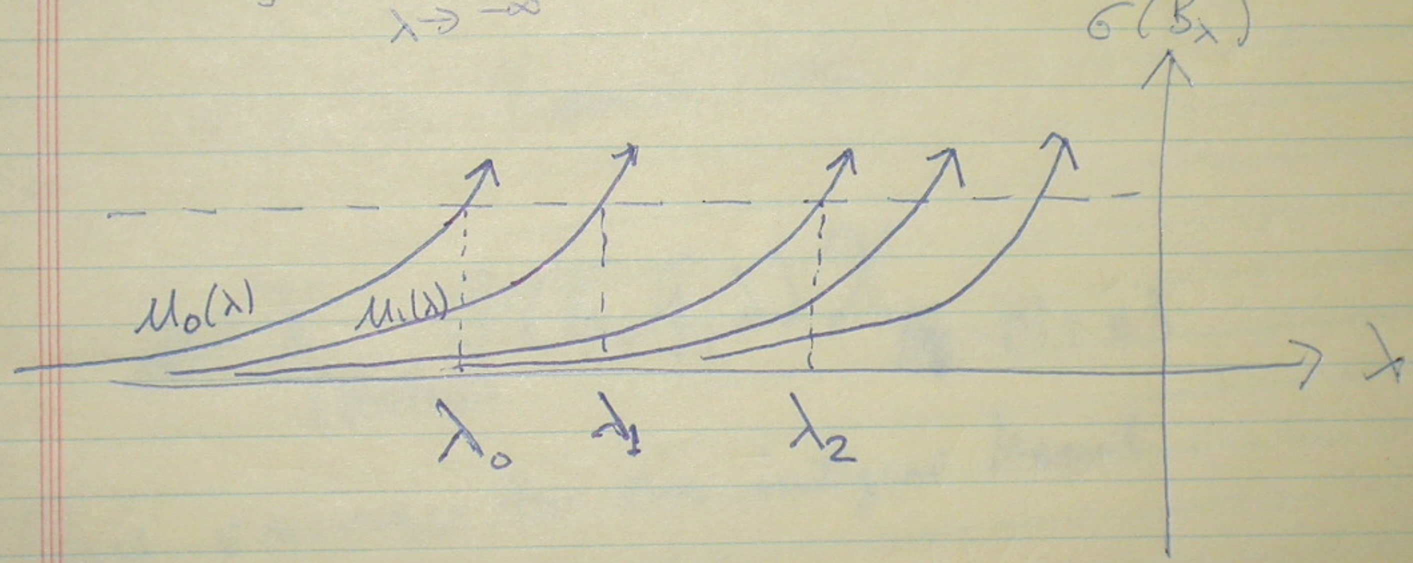
and $\boxed{\phi = B_\lambda \phi}$ (eigenvalue 1)

$\iff$ ☆.

Now note that in sense of operators,

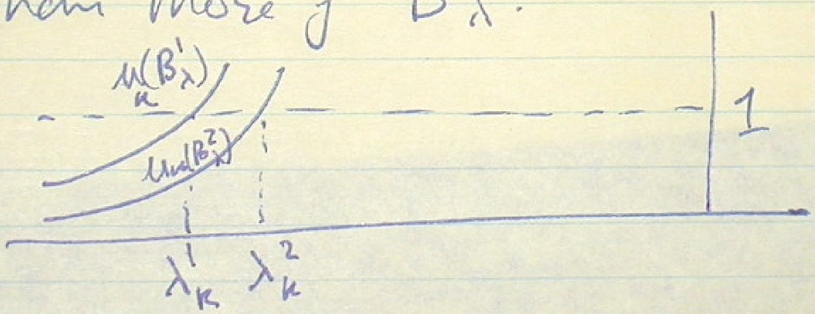
$0 > \lambda > \lambda' \Rightarrow B_\lambda > B_{\lambda'}$

Also, $\lim_{\lambda \to -\infty} B_\lambda = 0$



Consider now two potentials V_1, V_2, and suppose $B_\lambda^1 \geq B_\lambda^2$ for all $\lambda < 0$.

Then $\lambda_k^1(-\Delta + V_1) \leq \lambda_k^2(-\Delta + V_2)$, because all the curves ~~of~~ B_λ^1 are higher than those of B_λ^2:



A positively charged thread can be modeled as

$V(x) = -\alpha\, \delta_\Gamma(\vec{x})$

Approximate by honest functions supported in $\{x : \text{dist}(x, \Gamma) \leq \varepsilon\}$.

B_λ has a kernel like

$$-N(\varepsilon)\, \chi_{\{|x-\Gamma| \leq \varepsilon\}}\, G(\vec{x}, \vec{y}, \lambda)\, \chi_{\{|y-\Gamma| \leq \varepsilon\}}$$

This converges to an integral kernel

$$G(\vec{x}, \vec{y}, \lambda) \Big|_{\vec{x}, \vec{y} \in \Gamma}$$

Specifically, with $\kappa = \sqrt{-\lambda}$,

$$\boxed{\frac{\alpha}{2\pi} K_0\left(\kappa |\Gamma(s) - \Gamma(s')|\right)}$$

It suffices to show that the largest eigenvalue of $\mathcal{R}^{\kappa}_{\alpha,\Gamma}$ is uniquely minimized by the circle, i.e.,

$$\int_0^L \int_0^L K_0\big(\kappa|\Gamma(s)-\Gamma(s')|\big)\,\mathrm{d}s\mathrm{d}s' \geq \int_0^L \int_0^L K_0\big(\kappa|\mathcal{C}(s)-\mathcal{C}(s')|\big)\,\mathrm{d}s\mathrm{d}s'$$

with equality only for the circle.

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with equality only for the circle. Equivalently, show that

$$F_\kappa(\Gamma) := \int_0^{L/2} \mathrm{d}u \int_0^L \mathrm{d}s \left[K_0\big(\kappa|\Gamma(s+u)-\Gamma(s)|\big) - K_0\left(\frac{\kappa L}{\pi}\sin\frac{\pi u}{L}\right)\right]$$

is positive (0 for the circle).

Since K_0 is decreasing and strictly convex, with Jensen's inequality,

$$\frac{1}{L} F_\kappa(\Gamma) \geq \int_0^{L/2} \left[K_0\left(\frac{\kappa}{L}\int_0^L |\Gamma(s+u) - \Gamma(s)| \mathrm{d}s\right) - K_0\left(\frac{\kappa L}{\pi} \sin\frac{\pi u}{L}\right)\right] \mathrm{d}u \,,$$

where the inequality is strict unless $\int_0^L |\Gamma(s+u) - \Gamma(s)|\mathrm{d}s$ is independent of s,

i.e. for the circle. The conjecture has been reduced to:

$$\int_0^L |\Gamma(s+u) - \Gamma(s)| \, \mathrm{d}s \leq \frac{L^2}{\pi} \sin\frac{\pi u}{L}$$

A family of isoperimetric conjectures for p > 0:

$$C_L^p(u): \qquad \int_0^L |\Gamma(s+u) - \Gamma(s)|^p \, \mathrm{d}s \;\le\; \frac{L^{1+p}}{\pi^p} \sin^p \frac{\pi u}{L}\,,$$

$$C_L^{-p}(u): \qquad \int_0^L |\Gamma(s+u) - \Gamma(s)|^{-p} \, \mathrm{d}s \;\ge\; \frac{\pi^p L^{1-p}}{\sin^p \frac{\pi u}{L}}\,,$$

Right side corresponds to circle.

Proposition. 2.1.

$$C_L^p(u) \;\; implies \;\; C_L^{p'}(u) \;\; if \; p > p' > 0.$$

$$C_L^p(u) \;\; implies \;\; C_L^{-p}(u)$$

First part follows from convexity of x → x^a for a > 1:

$$\begin{aligned} \frac{L^{1+p}}{\pi^p} \sin^p \frac{\pi u}{L} &\geq \int_0^L \left(|\Gamma(s+u) - \Gamma(s)|^{p'} \right)^{p/p'} \mathrm{d}s \\ &\geq L \left(\frac{1}{L} \int_0^L |\Gamma(s+u) - \Gamma(s)|^{p'} \mathrm{d}s \right)^{p/p'} . \end{aligned}$$

Proof when p = 2

$$\Gamma(s) = \sum_{0 \neq n \in \mathbb{Z}} c_n \, \mathrm{e}^{ins}$$

$$c_{-n} = \bar{c}_n \, .$$

$$\dot{\Gamma}(s) = i \sum_{0 \neq n \in \mathbb{Z}} n c_n \, \mathrm{e}^{ins} \, .$$

By assumption, $|\dot{\Gamma}(s)| = 1$, and hence from

$$2\pi = \int_0^{2\pi} |\dot{\Gamma}(s)|^2 \, \mathrm{d}s = \int_0^{2\pi} \sum_{0 \neq m \in \mathbb{Z}} \sum_{0 \neq n \in \mathbb{Z}} nm \, c_m^* \cdot c_n \, \mathrm{e}^{i(n-m)s} \, \mathrm{d}s \,,$$

$$\sum_{0 \neq n \in \mathbb{Z}} n^2 |c_n|^2 = 1 \,. \tag{2.5}$$

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$$\sum_{0\neq n\in\mathbb{Z}} n^2 |c_n|^2 = 1 \,. \tag{2.5}$$

$$\int_0^{2\pi} \left| \sum_{0\neq n\in\mathbb{Z}} c_n \left(\mathrm{e}^{inu} - 1\right) \mathrm{e}^{ins} \right|^2 \,\mathrm{d}s = 8\pi \sum_{0\neq n\in\mathbb{Z}} |c_n|^2 \left(\sin\frac{nu}{2}\right)^2 \,,$$

Inequality equivalent to

$$\sum_{0\neq n\in\mathbb{Z}} n^2|c_n|^2 \left(\frac{\sin\frac{nu}{2}}{n\sin\frac{u}{2}}\right)^2 \leq 1.$$

It is therefore sufficient to prove that

$$|\sin nx| \leq n \sin x$$

Inductive argument based on

$$(n+1)\sin x \mp \sin(n+1)x = n\sin x \mp \sin nx \cos x + \sin x(1 \mp \cos nx)$$

What about $p > 2$?

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The conjecture is false for $p = \infty$. The family of maximizing curves for $\|\Gamma(s+u) - \Gamma(s)\|_\infty$ consists of all curves that contain a line segment of length > s.

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What about $p > 2$?

At what critical value of p does the circle stop being the maximizer?

This problem is open. We calculated $\|\Gamma(s+u) - \Gamma(s)\|_p$ for some examples:

Two straight line segments of length π:

$\|\Gamma(s+u) - \Gamma(s)\|_p^p = 2^{p+2}(\pi/2)^{p+1}/(p+1)$.

Better than the circle for $p > 3.15296\ldots$

What about $p > 2$?

Examples that are more like the circle are not better than the circle until higher p:

Stadium, small straight segments $p > 4.27898\ldots$

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Polygon with many sides, $p > 6$

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Stadium, small straight segments $p > 4.27898\ldots$

Polygon with many sides, $p > 6$

Polygon with rounded edges, similar.

Circle is local maximizer for all p < ∞

Let $\Gamma(\gamma, s)$ be a closed curve in the complex plane parametrized by arc length s, of the form $(1-\gamma)e^{is} + \Theta(\gamma, s)$, where $\gamma \geq 0$. Suppose that Θ is smooth (say, C^2 in γ and s), and that for each γ, $\Theta(\gamma, s)$ is orthogonal to e^{is}. Then $\Gamma(0, s)$ is a circle of radius 1, and for any u, $0 < u < 2\pi$,

$$\left.\frac{\partial I(\Gamma(\gamma), p, u)}{\partial \gamma}\right|_{\gamma=0} < 0.$$

On a (hyper) surface, what object is most like the Laplacian?

(Δ = the good old flat scalar Laplacian of Laplace)

Answer #1 (Beltrami's answer):

Consider only tangential variations.

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Difficulty:

- **The Laplace-Beltrami operator is an intrinsic object, and as such is unaware that the surface is immersed!**

Answer #2
The nanoelectronics answer

- **E.g., Da Costa, Phys. Rev. A 1981**

$$-\Delta_{LB} + q,$$

$$q(\mathbf{x}) = \frac{1}{4}\left(\sum_{j=1}^{d} \kappa_j\right)^2 - \frac{1}{2}\sum_{j=1}^{d} \kappa_j^2$$

d=1, $q = -\kappa^2/4 \leq 0$ d=2, $q = -(\kappa_1-\kappa_2)^2/4 \leq 0$

Some other answers

- **In other physical situations, such as reaction-diffusion, q(x) may be other quadratic expressions in the curvature, usually $q(x) \leq 0$.**

- **The conformal answer: q(x) is a multiple of the scalar curvature.**

Heisenberg's Answer

(if he had thought about it)

$$q(\mathbf{x}) = \frac{1}{4}\left(\sum_{j=1}^{d} \kappa_j\right)^2 .$$

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$$q(\mathbf{x}) = \frac{1}{4}\left(\sum_{j=1}^{d} \kappa_j\right)^2 .$$

Note: $q(\mathbf{x}) \geq 0$!

Some more loopy problems

$$H(g) := -\frac{d^2}{ds^2} + g\,\kappa^2.$$

The isoperimetric theorems for $-\nabla^2 + q(\kappa)$

I. One dimension

$$-\frac{d^2}{ds^2} + g\kappa^2$$

Ω - curve.

A. Ω infinitely long, asymptotically straight

$g<0$

$\lambda_1 < 0$ unless Ω is a line

Duclos - Exner

B. Ω closed, say $|\Omega| = 1$

(i) $g \leq 0$,

λ_1 uniquely maximized by ○

Duclos – Exner

(ii)

$g = -1$

λ_2 uniquely maximized by ○

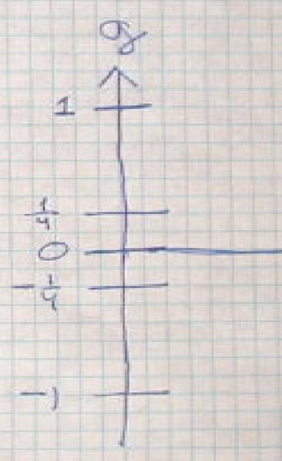
Harrell – Loss

(iii)

$0 \leq g \leq \frac{1}{4}$

λ_1 uniquely minimized by ○

Exner – Harrell – Loss



Minimality when g ≤ 1/4.

Proof. a) Assume first that $0 < g < 1/4$. The minimal value of λ_1, which we
λ_*, is

$$\inf_{\kappa} \inf_{\zeta} \int \left(\left(\frac{d\zeta}{ds} \right)^2 + g\, \kappa^2 \zeta^2 \right) ds,$$

Because the quantity in question is an iterated infimum, it may be calculated in th other order. By Cauchy-Schwarz's inequality

$$2\pi = \int \frac{\kappa}{\zeta}\zeta \mathrm{d}s \le \left(\int \frac{1}{\zeta^2}\mathrm{d}s\right)^{1/2} \left(\int \kappa^2\zeta^2\mathrm{d}s\right)^{1/2},$$

with equality only if

$$\kappa = \left(2\pi / \int \frac{1}{\zeta^2}\mathrm{d}s\right)\frac{1}{\zeta^2}.$$

A non linear functional

$$E(\zeta) := \int \left(\frac{\mathrm{d}\zeta}{\mathrm{d}s}\right)^2 \mathrm{d}s + \frac{4\pi^2 g}{\int \left(\frac{1}{\zeta^2}\right) \mathrm{d}s}.$$

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$$\lambda_* \leq E(\zeta) = 4g\pi^2 < \pi^2 \text{ for } g < 1/4.$$

Lemma 5: *If $E(\zeta) \leq \pi^2$ for a positive test function ζ normalized in L^2, then*

$$\inf_s (\zeta(\mathrm{s})) \; > \; 1 \; - \; \frac{\sqrt{E(\zeta)}}{\pi}.$$

Proof of Lemma 5.

$$E(\zeta) \; > \; \int_0^1 (\zeta')^2 \, \mathrm{d}s \; = \; \int_0^1 (\zeta \; - \; \zeta_{\mathrm{min}})'^2 \mathrm{d}s \; \geq \; \pi^2 \int_0^1 (\zeta \; - \; \zeta_{\mathrm{min}})^2 \mathrm{d}s,$$

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Minimizer therefore exists.
Its Euler equation is

$$-\zeta_*'' + M \frac{1}{\zeta_*^3} = C \zeta_*,$$

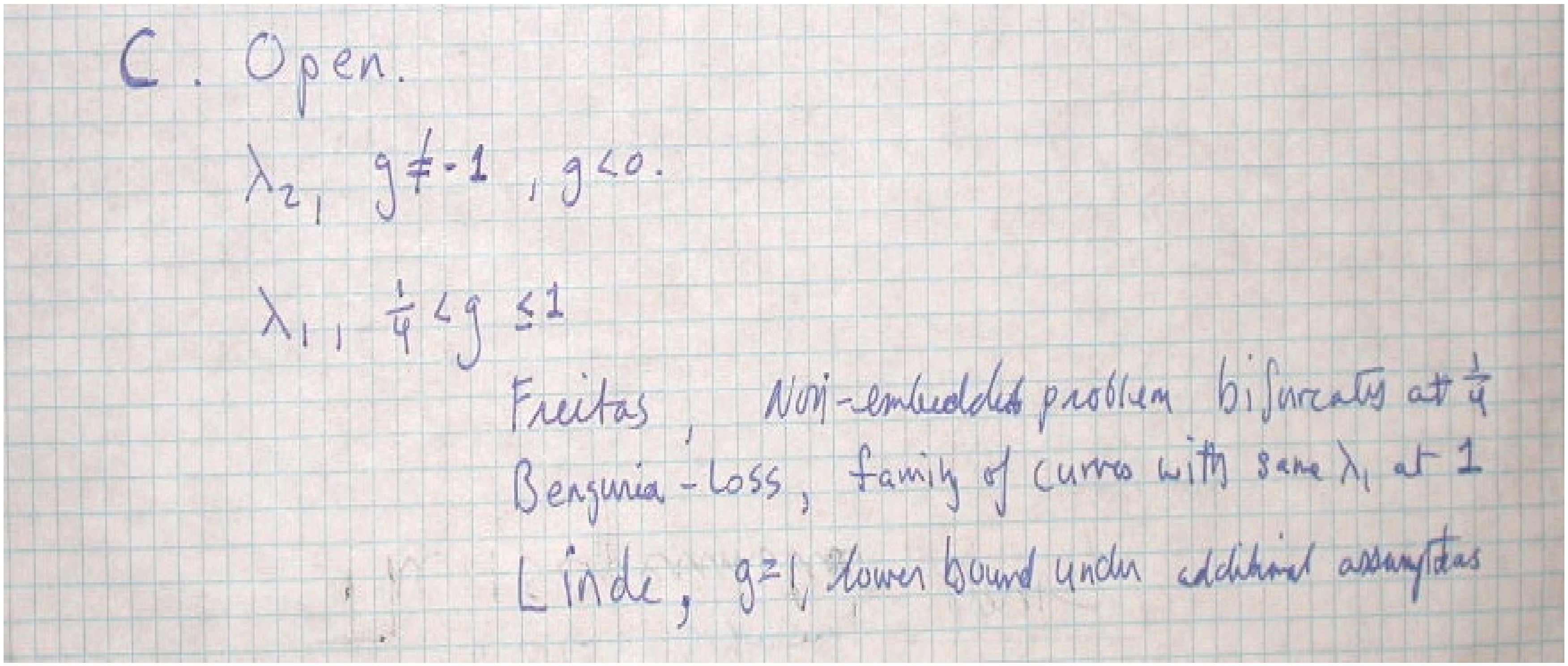
$$M = \frac{4\pi^2 g}{\left(\int_0^1 \frac{1}{\zeta_*^2} \, ds \right)^2}$$

Solution of Euler equation of the form:

$$\zeta_*^2 \;=\; 1 + \sqrt{1 - M/\lambda_*}\;\cos\left(2\sqrt{\lambda_*}\,(s - s_0)\right)$$

Nonconstant solution of this form excluded because $\lambda_* \;<\; \pi^2.$

The isoperimetric theorems for $-\nabla^2 + q(\kappa)$



C. Open.

λ_2, $g \neq -1$, $g < 0$.

λ_1, $\frac{1}{4} < g \leq 1$

Freitas, Non-embedded problem bifurcates at $\frac{1}{4}$

Benguria - Loss, family of curves with same λ_1 at 1

Linde, g=1, lower bound under additional assumptions

Progress on $-d^2/ds^2 + g\,\kappa^2$

- Benguria-Loss, *Contemp*. *Math*. 2004
  - Connnection to Lieb-Thirring in one-D

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  - Family of curves with same λ_0 as circle when g=1
  - $\lambda_0 > 1/2$.

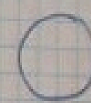
Progress on $-d^2/ds^2 + g\,\kappa^2$

- Linde Proc AMS 2005
- $\lambda_0 > 0.6085$ (convex, etc.)

II. Two dimensions

A. $q(K) = g\, K_1 K_2$ (Gauß curvature), genus$(\Omega)=0$, $|\Omega|=1$.

(i) Hersch 1970, $g=0$, λ_1 trivially $= 0$

$d=2$: λ_2 uniquely maximized by $S^2 \subset \mathbb{R}^3$ 

(ii) Harrell 1996 any g, $\lambda_{1,2}$ both uniquely maximized by 

* certain other potentials, $g(K_1^2+K_2^2)$ $g<0$.

<u>Open</u> - other genera

* Special facts in 2-D about conformal equivalence.

II. Two dimensions

A. $q(K) = g\, K_1 K_2$ (Gauß curvature), genus$(\Omega) = 0$, $|\Omega| = 1$.

False in high dim.

(i) Hersch 1970, $g = 0$, λ_1 trivially $= 0$

$d = 2$: λ_2 uniquely maximized by $S^2 \subset \mathbb{R}^3$ ○

(ii) Harrell 1996 any g, $\lambda_{1,2}$ both uniquely maximized by ○

* certain other potentials, $g(K_1^2 + K_2^2)$ $g < 0$.

<u>Open</u> - other genera

* Special facts in 2-D about conformal equivalence.

III Two or more dimensions.

A Ω - hypersurface of codimension 1

$$-\nabla^2 - \frac{1}{\dim}\left(\sum \kappa_\ell\right)^2$$

λ_2 uniquely maximized by sphere
(Harrell - Loss '98).

$\Rightarrow$ Same for $g(\kappa) = -\sum(\kappa_\ell^2)$

B) Ω - embedded in $\mathbb{R}^{n+\ell}$, $\mathbb{H}^{n+\ell}$, $\mathbb{S}^{n+\ell}$

El Soufi - Ilias.

Actually show $\lambda_2(-\nabla^2 + V(x)) \le \frac{1}{|S| \dim} \int \sum \kappa_\ell^2 + V_{ave}$

Universal Bounds using Commutators

- A “sum rule” identity (Harrell-Stubbe, 1997):

$$1 = \frac{4}{d} \sum_{k:\lambda_k \neq \lambda_j} \frac{|\langle u_k, \mathbf{p} u_j \rangle|^2}{\lambda_k - \lambda_j}$$

Here, H is *any* Schrödinger operator, **p** is the gradient
(times -i if you are a physicist and you use atomic units)

Commutators: [A,B] := AB-BA

3a. The equations of space curves are commutators:

$$\left[\frac{d}{ds}, \mathbf{x}\right] = \mathbf{t}$$

$$\left[\frac{d}{ds}, \mathbf{t}\right] = \kappa\mathbf{n}$$

Note: curvature is defined by a second commutator

The Serret-Frenet equations as commutator relations:

$$[H, X_m] = -\frac{d^2 X_m}{ds^2} - 2\,\frac{dX_m}{ds}\,\frac{d}{ds} = -\kappa n_m - 2t_m \frac{d}{ds}, \qquad (2.2)$$

$$[X_m\,[H, X_m]] = 2t_m^2. \qquad (2.3)$$

Lemma. *Let M be a smooth curve in* R^d, d = *2 or 3. Then for*

$$H = -\frac{d^2}{ds^2} + V(s) \quad and\ \varphi \in W_0^1(M),$$

$$\sum_{m=0}^{d} \|[H, X_m]\,\varphi\|^2 = 4\int_M \left(\left|\frac{d\varphi}{ds}\right|^2 + \frac{\kappa^2}{4}\,|\varphi|^2 \right) ds.$$

Proposition 2.1 *Let M be a smooth curve in $\mathbb{R}^\nu$, $\nu = 2$ or 3. Then for*

$H = -\dfrac{d^2}{ds^2}$ +V(s) *and* $\varphi \in W_0^1(M)$,

$$\sum_{m=0}^{\mathrm{d}} \|[H, X_m]\,\varphi\|^2 = 4\int_M \left(\left|\frac{d\varphi}{ds}\right|^2 + \frac{\kappa^2}{4}|\varphi|^2 \right) ds.$$

Proof. By closure it may be assumed that $\varphi \in C_c^\infty(M)$. Apply (2.2) to φ and square the result, to obtain

$$4\left(t_m^2 \left(\frac{d\varphi}{ds}\right)^2 + \frac{1}{4}\kappa^2 n_m^2 \varphi^2 + \frac{1}{2}\kappa n_m t_m \varphi \frac{d\varphi}{ds} \right).$$

Sum on m and integrate. QED

Interpretation:

Algebraically, for quantum mechanics on a wire, the natural H_0 is not

$$\mathbf{p}^2,$$

but rather

$$H_{1/4} := \quad \mathbf{p}^2 + \kappa^2/4.$$

Corollary 2.2 *Let M be as in* Proposition 2.1 *and suppose that H is a Schrödinger Hamiltonian with a bounded measurable potential $V(s)$. Then*

$$\Gamma \leq 4 \int_M \left(\left(\frac{du_1}{ds} \right)^2 + \frac{\kappa^2}{4} u_1^2 \right) ds. \tag{2.5}$$

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That is, the gap for *any* H is controlled by an expectation value of $H_{1/4}$.

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$$\Gamma \leq 4 \int_M \left(\left(\frac{du_1}{ds} \right)^2 + \frac{\kappa^2}{4} u_1^2 \right) ds. \tag{2.5}$$

Furthermore, if H is of the form

$$H_g := -\frac{d^2}{ds^2} + g\kappa^2,$$

then

$$\Gamma \leq \max \left(4, \frac{1}{g} \right) \lambda_1. \tag{2.6}$$

Equivalently, the universal ratio bound

$$\frac{\lambda_2}{\lambda_1} \leq \max \left(5, 1 + \frac{1}{g} \right)$$

holds.

Bound is sharp for the circle:

$$\frac{\lambda_2}{\lambda_1} = \frac{4\pi^2 (1 + g)}{4\pi^2 g} = 1 + \frac{1}{g}.$$

Gap bounds for (hyper) surfaces

Let M be a d-dimensional manifold immersed in $\mathbb{R}^{d+1}$.

Theorem 3.1 *Let H be a Schrödinger operator on M with a bounded potential, i.e.,*

$$H = -\Delta + V, \tag{3.1}$$

$$\Gamma(H) \leq \frac{1}{d} \int_M \left(4 \left| \nabla_{||} u_1 \right|^2 + h^2 u_1^2 \right) dVol$$
$$= \frac{4}{d} \left\langle u_1, \left(-\Delta + \frac{h^2}{4} \right) u_1 \right\rangle .$$

Here h is the sum of the principal curvatures.

Corollary 3.2 *Let H be as in (3.1) and define* $\delta := \sup_M \left(\frac{h^2}{4} - V\right)$. *Then*

$$\Gamma(H) \leq \frac{4}{d}\left(\lambda_1 + \delta\right).$$

Bound is sharp for the sphere:

$$\lambda_1 = gd^2, \qquad \lambda_2 = gd^2 + d$$

$$d = \lambda_2 - \lambda_1 \leq \left(\frac{gd^2}{gd} \right) = d.$$

Spinorial Canonical Commutation

$$\mathbf{P} = \sum_{j=1}^{d} \left(\mathbf{t}_j \frac{\partial}{\partial s_j} \pm \frac{1}{2} \kappa_j \mathbf{n} \right) \tag{4.1}$$

and for a dense set of functions φ,

$$\|\mathbf{P}\varphi\|^2 = \langle \varphi, H_{1/4}\varphi \rangle . \tag{4.2}$$

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and for a dense set of functions φ,

$$\|\mathbf{P}\varphi\|^2 = \langle \varphi, H_{1/4}\varphi \rangle . \tag{4.2}$$

Thus $\mathbf{P}$ plays the rôle of a momentum operator, with which there is a version of canonical commutation (cf. (1.9)) as follows. Defining a variant commutator bracket for operators $L^2(M) \to \mathbb{R}^{d+1} \otimes L^2(M)$ by $[A; B] := A \cdot B - B \cdot A$, a calculation shows that $[\mathbf{P}; X_k \mathbf{e}_k] = \sum_{j=1}^{d} \mathbf{t}_j \cdot \frac{\partial X_k \mathbf{e}_k}{\partial s_j} = \mathbf{1}$ (identity operator), and by averaging on k,

$$\mathbf{1} = \frac{1}{d} [\mathbf{P}; \mathbf{X}] \tag{4.3}$$

which is a coordinate-independent formula.

Sum Rules

Proposition 4.1 *Let H be as in (3.1), with eigenvalues $\{\lambda_k\}$ and normalized eigenfunctions $\{u_k\}$. Then*

$$1 = \frac{4}{d} \sum_{\substack{k \\ \lambda_k \neq \lambda_j}} \frac{|\langle u_k, \mathbf{P} u_j \rangle|^2}{\lambda_k - \lambda_j}. \tag{4.4}$$

Corollaries of sum rules

- Sharp universal bounds for all gaps

- Some estimates of partition function

  $Z(t) = \sum \exp(-t\, \lambda_k)$

Speculations and open problems

- Can one obtain/improve Lieb-Thirring bounds as a consequence of sum rules?
- Full understanding of spectrum of H_g.
  - What spectral data needed to determine the curve?
  - What is the bifurcation value for the minimizer of λ_1?
- Physical understanding of H_g and of the spinorial operators it is related to.

Sharp universal bound for all gaps

Corollary 4.4 b) *For H_g be of the form (1.10) on a smooth, compact submanifold. Then*

$$[\lambda_n, \lambda_{n+1}] \subseteq \left[\left(1+\frac{2\sigma}{d}\right)\overline{\lambda_n} - \sqrt{D_n}), \left(1+\frac{2\sigma}{d}\right)\overline{\lambda_n} + \sqrt{D_n}\right],$$

with

$$D_n := \left(\left(1+\frac{2\sigma}{d}\right)\overline{\lambda_n}\right)^2 - \left(1+\frac{4\sigma}{d}\right)\overline{\lambda_n^2}.$$

This bound is sharp for every non–zero eigenvalue gap of $H_{\frac{1}{4}}$ on the sphere.

Partition function

$$Z(t) := tr(\exp(-tH)).$$

Partition function

$$Z(t) \leq \left(\frac{2t}{d}\right) \sum_j \left(\exp\left(-t\lambda_j\right)\right) \|\mathbf{P} u_j\|^2,$$

which implies

Corollary 4.5 a) *Let H be as (3.1), with M a compact, smooth submanifold. Then $t^{\frac{d}{2}} \exp(-\delta t) Z(t)$ is a nondecreasing function;*

b) *For H_g be of the form (1.10) on a smooth, compact submanifold M, $t^{\frac{d}{2\sigma}} Z(t)$ is a nondecreasing function.*