An Inverse Random Source Problem for the Helmholtz Equation

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Abstract

We study the one-dimensional Helmholtz equation with a spatially random source function. Our main goal is to reconstruct the statistical distribution of the source function from boundary measurements of the radiation field. First, we present the model problem and convert it into a two-point spatially stochastic boundary value problem, for which we prove there exists a unique pathwise solution. Furthermore, we deduce an explicit formula for the solution by using the integrated solution method. Based on the analysis and solution formula, we propose a novel and efficient strategy, which only uses fast fourier transforms (FFT), to reconstruct the statistical properties, such as the mean and the standard deviation or the variance, of the random source function from measurements at one boundary point. Numerical examples are presented to demonstrate the validity and effectiveness of the proposed method.

Key words. inverse source problem, Helmholtz equation, stochastic differential equation

AMS subject classifications. 65N21, 78A46

1 Introduction

The inverse source problem for wave propagation has been considered as a basic tool for the solution of reflection tomography, diffusion-based optical tomography, and more recently fluorescence microscopy [25]. This problem is largely motivated by medical applications in which it is desirable to use electric or magnetic measurements on the surface of the human body, such as head, to infer the source currents inside of the body, such as the brain, that produced these measured data. A major advantage of such imaging modalities over the traditional ones is that it allows systematic imaging studies of protein localization in living cells and of the structure and function of living tissues.

The problem has been extensively investigated in the literature both from the point of view of applied biomedical engineering and also as a mathematical problem. There are a number of work on the scalar and the full vector electromagnetic inverse source problem in the free space as well as

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in nonhomogeneous background media, see e.g. Albanese and Monk [1], Ammari *et al.* [2], Devaney *et al.* [12], Eller and Valdivia [14], and references cited therein. Most of the work make use of the fact that the radiation pattern determines the field everywhere outside the source volume. In other words, the inverse source problem is to determine a source function that generates a prescribed radiation pattern.

It is also known that there exist an infinity of sources that radiate fields which vanish identically outside their support volumes so that the inverse source problem does not have a unique solution, i.e., an infinity of solutions can be obtained by adding any one of these nonradiating sources to any given solution, see e.g. Bleistein and Cohen [6], Devaney and Sherman [13], and Hauer *et al.* [16]. Therefore, it is clear that the inverse source problem is ill-posed. In order to obtain a unique solution, it is necessary to give additional constraints that the source must satisfy. A typical choice of the constraint is to pick up the minimum energy solution, which represents the pseudo-inverse of the inverse source problem, see e.g. Marengo and Devaney [19]. Recently Bao *et al.* [5] investigates the multi-frequency inverse source problem in which the uniqueness is shown and some stability estimates are established from the radiated fields outside the source volume for a set of frequencies. We refer to Chen and Rokhlin [9] for an inverse medium scattering problem for the one-dimensional Helmholtz equation. See also Gelfand and Levitan [15] for the related Sturm-Liouville problem.

In many applications the source and hence the radiated field may not be deterministic but rather are modeled by random processes, such as Gaussian random field. Therefore their governing equations are some forms of stochastic differential equations instead of their deterministic counterparts. In fact, stochastic partial differential equations are known to be effective tools in modeling complex physical and engineering phenomena including the wave propagation, see e.g. Papanicolaou [22]. In this paper, we are concerned with the wave propagation in the one-dimensional stochastic Helmholtz equation with source generated by a spatial Wiener process.

Unlike deterministic differential equations, solutions of stochastic differential equations are random functions. Hence it is more important to study their statistical characteristics such as mean value, variance, and even higher order moments in many practical problems. In the context of the inverse random sources problem, the goal is thus to deduce the statistical structure such as the mean value and standard deviation or variance of the source from physically realizable measurements of the radiated fields, such as the measurements taken on the boundaries. We refer to Devaney [11] for an inverse random source problem where it was shown that the auto-correlation function of the random source is uniquely determined everywhere outside the source region by the auto-correlation function of the radiated field. Recently, a novel and efficient Wiener chaos expansion based technique has been developed for modeling and simulation of spatially incoherent sources in photonic crystals by Badieirostami *et al.* [3]. See Bao et al. [4] for a related inverse medium scattering problem with a stochastic source which is to reconstruct the refractive index of an inhomogeneous medium from the boundary measurements of the scattered random field. We refer to Cao *et al.* [7] for the finite element and discontinuous Galerkin method for solving the stochastic Helmholtz equation, and Kloeden and Platen [18] for an account of various numerical methods and approximation schemes for general stochastic partial differential equations.

This work is devoted to the one-dimensional stochastic Helmholtz equation in a homogeneous background medium. The random source function, representing the electric current density, is assumed to have a compact support contained in a finite interval. The problem is also modeled with an outgoing wave condition imposed on the lateral end points of the finite interval, which reduces the model to a second order stochastic two-point boundary value problem. We first convert this model problem into an equivalent first order stochastic two-point boundary value problem and show the pathwise existence and uniqueness of the solution. Then we explicitly deduce the solution by using the integrated solution method, which transforms the first order stochastic two-point boundary value problem into another equivalent problem: stochastic initial value problems plus an algebraic linear system. The solution is given by a combination of a regular integral and an Itô integral. And it also connects the random wave field with the Fourier transform of the mean and variance of the random source function in an explicit manner. By studying the expectation and variance of the integral equation, we are able to develop an efficient algorithm to reconstruct the mean and variance, which is based on the fast Fourier transform (FFT). Our numerical examples, including the reconstructions of both smooth and non-smooth functions, demonstrate the validity and effectiveness of the proposed method.

The paper is organized as follows. In Section 2, we present the model problem and formulate it as a first order two-point stochastic boundary value problem. The existence and uniqueness of the direct problem are established, and the solution formula is explicitly derived from the integrated solution method. Base on the solution, we propose an inversion method for the reconstruction of the mean and variance of the random source. In Section 3, we discuss numerical implementation of the method and present three numerical examples to demonstrate the validity and effectiveness of the proposed approach. The paper is concluded with general remarks and directions for future research in Section 4.

2 Inverse Random Source Problem

In this section, we introduce a mathematical model for the inverse random source problem in wave propagation. To study this model, we first convert it into a stochastic boundary value problem. We establish a theoretical framework for the model by an integrated solution method, which allows us to derive an explicit formula for the solution of the inverse random source problem, and design the computational methods.

2.1 The Model Problem

Consider the one-dimensional Helmholtz equation in homogeneous background medium

$$u''(x,\omega)+\omega^2 u(x,\omega)=f(x), \tag{2.1}$$

where the magnetic permeability and the electric permittivity of the vacuum are assumed to be the unity for simplicity, $\omega>0$ is the angular frequency, and f, representing the electric current density, is a stochastic source function assumed to have the form

$$f(x)=g(x)+h(x)W'_x.$$

Here g and h are deterministic functions with compact supports contained in $[0,1]$, and W_x is a one-dimensional spatial Wiener process, and W'_x is its stochastic differential in the Itô sense which is commonly used as a model for the white noise, i.e, a spatial Gaussian random field. Following from the standard stochastic theory on the white noise, we have

$$\mathbb{E}[f(x)]=g(x) \quad \text{and} \quad \mathbb{V}[f(x)]=h^2(x),$$

where $\mathbb{E}$ and $\mathbb{V}$ are the expectation and variance operators, respectively. Obviously, because of the random source, the solution u, the radiate field, is also a random function. Typical boundary conditions imposed on u are the so called outgoing radiation boundary conditions, which are equivalent to the boundary conditions at two lateral end points of the interval $[0,1]$:

$$u'(0,\omega)+\mathrm{i}\omega u(0,\omega)=0 \quad \text{and} \quad u'(1,\omega)-\mathrm{i}\omega u(1,\omega)=0. \tag{2.2}$$

Given the mean g and the standard deviation h of the random source function f, the direct (forward) problem is to determine the random wave field u. On the contrary, the inverse source problem is to determine the mean value g and the standard deviation h or the variance h^2 of the random source from the boundary measurements of the random wave field $u(0,\omega)$. Our main goal is to investigate both direct and inverse problems and propose a novel and efficient numerical algorithm to solve the inverse source problem. We remark that we use the left boundary point $x = 0$ in our discussion, all of the results are still true if the measurements are taken at the right boundary point $x = 1$.

First, we show that the direct problem has a unique pathwise solution for each realization of the random field dW_x, and the solution is given by an explicit formula, which serves as the foundation of our numerical algorithm for the inverse problem. To begin with, we convert the second order wave equation in the direct problem into a first order two-point stochastic boundary value problem.

Let $u_1 = u$ and $u_2 = u'$, the second order stochastic boundary value problem (2.1)–(2.2) can be equivalently written as

$$d\mathbf{u} = (M\mathbf{u} + \mathbf{g})dx + \mathbf{h}dW_x, \tag{2.3}$$

$$A_0\mathbf{u}(0) = 0, \tag{2.4}$$

$$B_1\mathbf{u}(1) = 0, \tag{2.5}$$

where

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} 0 \\ g \end{bmatrix}, \quad \mathbf{h} = \begin{bmatrix} 0 \\ h \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix},$$

and

$$A_0 = [\mathrm{i}\omega\ 1], \quad B_1 = [-\mathrm{i}\omega\ 1].$$

We use this equivalent problem to establish our analysis in the rest of this section.

2.2 Two-point stochastic boundary value problem

To solve the two-point boundary value problem, we apply the integrated solution method to a more general initial value problems together with an algebraic linear system. The reader is referred to Nualart and Pardoux [20] for discussions on general boundary value problems for stochastic differential equations, and to Zhang [26] for the details of the integrated solution method for solving the deterministic two-point boundary value problems.

Consider the stochastic differential equation in the interval $[0, 1]$

$$d\mathbf{u} = M\mathbf{u} + \mathbf{g} + \mathbf{h}dW_x \tag{2.6}$$

together with a partial boundary condition for $\mathbf{u}(0) = \mathbf{u}_0$ at $x = 0$, which is assumed to be given in the following form of the linear equations

$$A_0\mathbf{u}_0 = \mathbf{v}_0, \tag{2.7}$$

where $\mathbf{u}(x) \in \mathbb{C}^n, \mathbf{g}(x) \in \mathbb{C}^n$, and $\mathbf{h}(x) \in \mathbb{C}^n$ are n-dimensional vector fields, $\mathbf{v}_0 \in \mathbb{C}^{n_1}$ $(n_1 < n)$ is a given n_1-dimensional vector field, $M \in \mathbb{C}^{n\times n}$ is a constant matrix, $A_0 \in \mathbb{C}^{n_1\times n}$ is a full rank matrix, i.e., $\mathrm{rank}(A_0) = n_1$ or equivalently $\det(A_0A_0^{\mathrm{H}}) \neq 0$.

The problem (2.6)–(2.7) has infinitely many solutions since only n_1 linearly independently boundary conditions are given for the n-dimensional vector field $\mathbf{u}(x)$. Denote the set of solutions by

$$U = \{\mathbf{u}(x) : \mathbf{u}(x) \text{ satisfies (2.6) and (2.7)}\}.$$

For any given $x \in [0,1]$, denote U_x the set of points in the n-dimensional Euclidean space

$$U_x = \{\mathbf{p} = \mathbf{u}(x) : \mathbf{u}(x) \in U\}.$$

Due to the linearity of the problem (2.6)–(2.7), the point set U_x spans an $n_2 = n - n_1$ dimensional hyperplane, which can be represented by

$$U_x = \{\mathbf{p} : A(x)\mathbf{p} = \mathbf{v}_0(x)\},$$

where $\mathbf{v}_0(x) \in \mathbb{C}^{n_1}$ is an n_1-dimensional vector and $A(x) \in \mathbb{C}^{n_1 \times n}$ is a full rank $n_1 \times n$ matrix.

Definition 2.1. *The pair of functions $\{A(x), \mathbf{v}_0(x)\}$ is called the integrated solution for the problem (2.6)–(2.7) if they satisfy*

$$A(x)\mathbf{u}(x) = \mathbf{v}_0(x) \quad \textit{for any } x \in [0,1]. \tag{2.8}$$

The following representation theorem characterizes how the integrated solution can be expressed in terms of $M, A_0, \mathbf{g}, \mathbf{h}$, and $\mathbf{v}_0$ from the problem (2.6)–(2.7).

Theorem 2.1. *The pair of functions $\{A(x), \mathbf{v}_0(x)\}$ is the integrated solutions to the problem (2.6)–(2.7) if and only if there exists $n_1 \times n_1$ matrix $D_0(x)$ such that*

$$dA(x) = (-A(x)M + D_0(x)A(x))dx, \quad A(0) = A_0, \tag{2.9}$$

$$d\mathbf{v}_0(x) =(A(x)\mathbf{g}(x) + D_0(x)\mathbf{v}_0(x))dx + A(x)\mathbf{h}(x)dW_x, \quad \mathbf{v}_0(0) = \mathbf{v}_0. \tag{2.10}$$

Proof. Assume that $\{A(x), \mathbf{v}_0(x)\}$ is the integrated solution to the problem (2.6)–(2.7), thus they satisfy Eq. (2.8). Taking the differentiation of Eq. (2.8) and substituting (2.6), we have

$$[dA(x) + A(x)Mdx]\,\mathbf{u}(x) = d\mathbf{v}_0(x) - A(x)\mathbf{g}(x)dx - A(x)\mathbf{h}(x)dW_x \tag{2.11}$$

for any $\mathbf{u}(x) \in U$. Therefore for any fixed point $x \in [0,1]$, the hyperplane defined from (2.11) contains the n_2-dimensional hyperplane defined from (2.8). It follows from the linear algebra that every row of the $n_1 \times (n+1)$ matrix

$$[dA(x) + A(x)Mdx, d\mathbf{v}_0(x) - A(x)\mathbf{g}(x)dx - A(x)\mathbf{h}(x)dW_x]$$

is a linear combination of the row vectors for the $n_1 \times (n+1)$ matrix $[A(x), \mathbf{v}_0(x)]$, i.e., there exists an $n_1 \times n_1$ matrix $D_0(x)$ such that

$$[dA(x) + A(x)Mdx, d\mathbf{v}_0(x) - A(x)\mathbf{g}(x)dx - A(x)\mathbf{h}(x)dW_x] = D_0(x)[A(x), \mathbf{v}_0(x)],$$

which gives Eqs. (2.9) and (2.10).

On the other hand, we assume that $\{A(x), \mathbf{v}_0(x)\}$ satisfies Eqs. (2.9) and (2.10). It follows from Eq. (2.6) that for any $\mathbf{u}(x) \in U$ we have

$$d[A(x)\mathbf{u}(x) - \mathbf{v}_0(x)] = D_0(x)[A(x)\mathbf{u}(x) - \mathbf{v}_0(x)]dx$$

and the homogeneous initial condition

$$A(0)\mathbf{u}(0) - \mathbf{v}_0(0) = 0$$

Solving the above initial value problem leads to

$$A(x)\mathbf{u}(x) = \mathbf{v}_0(x) \quad \text{for any } \mathbf{u}(x) \in U.$$

Finally we prove that the rank of the matrix $A(x)$ is n_1.

Consider an initial value problem for the $n_1 \times n_1$ matrix $E_R(x)$

$$
\begin{aligned}
dE_R(x) - D_0(x)E_R(x)dx &= 0, \\
E_R(0) &= I_{n_1 \times n_1},
\end{aligned}
$$

where $I_{n_1 \times n_1}$ is the $n_1 \times n_1$ identity matrix. Let the pair of functions $\{A_R(x), \mathbf{v}_R(x)\}$ be the solution of the following initial value problem

$$
\begin{aligned}
A_R(x) - A_R(x)M = 0, &\quad A_R(0) = A_0, \\
d\mathbf{v}_R(x) - A_R(x)\mathbf{g}(x)dx - A_R(x)\mathbf{h}(x)dW_x = 0, &\quad \mathbf{v}_R(0) = \mathbf{v}_0.
\end{aligned}
$$

It can be verified that

$$
A(x) = E_R(x)A_R(x) \quad \text{and} \quad \mathbf{v}_0(x) = E_R(x)\mathbf{v}_R(x).
$$

The rank of $A_R(x)$ is the same as that of A_0, i.e., $\mathrm{rank}(A_R(x)) = \mathrm{rank}(A_0) = n_1$, since $A_R(x)$ is the solution of the initial value problem of the homogeneous ordinary differential equation. A simple calculation yields

$$
[\mathrm{det}E_R(x)]' = t(x)\mathrm{det}E_R(x),
$$

where $t(x) = \mathrm{trace}D_0(x)$. It follows from

$$
\mathrm{det}E_R(x) = \mathrm{det}E_R(0)\exp\int_0^x t(s)ds = \exp\int_0^x t(s)ds \neq 0
$$

that $E_R(x)$ is nonsingular. Thus we have $\mathrm{rank}A(x) = \mathrm{rank}A_R(x) = n_1$, which completes the proof. $\square$

Based on the integrated solution method, we may consider the first order linear stochastic differential equation

$$
d\mathbf{u} = (M\mathbf{u} + \mathbf{g})dx + \mathbf{h}dW_x, \tag{2.12}
$$

together with the boundary condition which involves both $\mathbf{u}_0$ and $\mathbf{u}_1 = \mathbf{u}(1)$ in the form of linear equations

$$
A_0\mathbf{u}_0 = \mathbf{v}_0, \tag{2.13}
$$

$$
B_1\mathbf{u}_1 = \mathbf{v}_1, \tag{2.14}
$$

where $B_1 \in \mathbb{C}^{n_2 \times n}$ and $\mathbf{v}_1 \in \mathbb{C}^{n_2}$ with $n_1 + n_2 = n$. We give a necessary and sufficient condition for the pathwise existence and uniqueness of solution for any fixed realization of Wiener process W_x.

We note that a solution to Eq. (2.12), if any, takes the form

$$
\mathbf{u}(x) = e^{Mx}\left[\mathbf{u}_0 + \int_0^x e^{-My}\mathbf{g}(y)dy + \int_0^x e^{-My}\mathbf{h}(y)dW_y\right],
$$

where the last expression is given in the sense of the Itô integral. The solution is required to satisfy the boundary condition at $x = 1$

$$
B_1e^M\left[\mathbf{u}_0 + \int_0^1 e^{-My}\mathbf{g}(y)dy + \int_0^1 e^{-My}\mathbf{h}(y)dW_y\right] = \mathbf{v}_1.
$$

We denote the random vector

$$B_1 e^M \int_0^1 e^{-My} \mathbf{h}(y) dW_y = \mathbf{w} \in \mathbb{C}^n.$$

The well-posedness of the two-point stochastic boundary value problem (2.12)–(2.14) can be equivalently formulated as follows: Given $\mathbf{v}_0$ and $\mathbf{v}_1$, for any random process $\mathbf{w}$, there exists a unique solution $\mathbf{u}_0$ to the linear equations

$$\begin{aligned} A_0 \mathbf{u}_0 &= \mathbf{v}_0, \\ B_1 e^M \mathbf{u}_0 &= \mathbf{v}_1 - \mathbf{w} - B_1 e^M \int_0^1 e^{-My} \mathbf{g}(y) dy. \end{aligned}$$

It follows from the linear algebra that the unique solvability of the above linear system can be obtained if the coefficient matrix is nonsingular. Therefore we obtain the necessary and sufficient condition for the well-posedness of the two-point stochastic boundary value problem.

Theorem 2.2. *The two-point stochastic boundary value problem (2.12)–(2.14) has a unique solution if and only if*

$$\det \begin{bmatrix} A_0 \\ B_1 e^M \end{bmatrix} \neq 0. \tag{2.15}$$

We next construct the solution $\mathbf{u}(t)$ as a function of the input W_x so that the fact that W_x is a Wiener process is actually irrelevant in the derivation.

Let $\{A(x), \mathbf{v}_0(x)\}$ and $\{B(x), \mathbf{v}_1(x)\}$ be the integrated solutions to Eqs. (2.12)–(2.13) and (2.12)–(2.14), respectively. It follows from the representation of the integrated solution in Theorem 2.1 that there exist $n_1 \times n_1$ matrix $D_0(x)$ and $n_2 \times n_2$ matrix $D_1(x)$ such that

$$dA(x) = [-A(x)M + D_0(x)A(x)]dx, \quad A(0) = A_0, \tag{2.16}$$
$$d\mathbf{v}_0(x) = [A(x)\mathbf{g}(x) + D_0(x)\mathbf{v}_0(x)]dx + A(x)\mathbf{h}(x)dW_x, \quad \mathbf{v}_0(0) = \mathbf{v}_0. \tag{2.17}$$

and

$$dB(x) = [-B(x)M + D_1(x)B(x)]dx, \quad B(1) = B_1, \tag{2.18}$$
$$d\mathbf{v}_1(x) = [B(x)\mathbf{g}(x) + D_1(x)\mathbf{v}_1(x)]dx + B(x)\mathbf{h}(x)dW_x, \quad \mathbf{v}_1(1) = \mathbf{v}_1. \tag{2.19}$$

Define the $n \times n$ matrix $Y(x)$ as

$$Y(x) = \begin{bmatrix} A(x) \\ B(x) \end{bmatrix}.$$

Lemma 2.1. *Assume that the two-point stochastic boundary value problem (2.12)–(2.14) has a unique solution, then the matrix $Y(x)$ is nonsingular.*

Proof. We have in Theorem 2.1 that $A(x) = E_R(x)A_R(x)$. It is easily verified that $A_R(x) = A_0 e^{-Mx}$. Thus we have $A(x) = E_R(x)A_0 e^{-Mx}$. Similarly we may obtain

$$B(x) = E_L(x)B_L(x) = E_L(x)B_1 e^M e^{-Mx},$$

where $E_L(x)$ is an $n_2 \times n_2$ nonsingular matrix. We have

$$Y(x) = \begin{bmatrix} E_R & \\ & E_L \end{bmatrix} \begin{bmatrix} A_0 \\ B_1 e^M \end{bmatrix} e^{-Mx},$$

which completes the proof from a direct application of Theorem 2.2. $\square$

Theorem 2.3. *The two-point boundary value problem (2.12)–(2.14) is equivalent to the linear system*

$$\begin{bmatrix} A(x) \\ B(x) \end{bmatrix} \mathbf{u}(x) = \begin{bmatrix} \mathbf{v}_0(x) \\ \mathbf{v}_1(x) \end{bmatrix}, \tag{2.20}$$

where $\{A(x), \mathbf{v}_0(x)\}$ *and* $\{B(x), \mathbf{v}_1(x)\}$ *are the integrated solutions to Eqs. (2.12), (2.13) and (2.12), (2.14), respectively.*

Proof. Taking the differentiation to Eq (2.20), and substituting (2.16)–(2.19), we obtain

$$\begin{aligned} 0 &= d\begin{bmatrix} A(x) \\ B(x) \end{bmatrix} \mathbf{u}(x) + \begin{bmatrix} A(x) \\ B(x) \end{bmatrix} d\mathbf{u}(x) - d\begin{bmatrix} \mathbf{v}_0(x) \\ \mathbf{v}_1(x) \end{bmatrix} \\ &= \begin{bmatrix} A(x) \\ B(x) \end{bmatrix} [d\mathbf{u}(x) - (M\mathbf{u}(x) + \mathbf{g}(x))dx - \mathbf{h}(x)dW_x]. \end{aligned}$$

Since the matrix $Y(x)$ is nonsingular due to Lemma 2.1, we get

$$d\mathbf{u}(x) - (M\mathbf{u}(x) + \mathbf{g}(x))dx - \mathbf{h}(x)dW_x = 0.$$

The boundary conditions (2.13) and (2.14) follows from the initial conditions in (2.16) and (2.18), which complete the proof.

□

Therefore the two-point stochastic boundary value problem (2.12)–(2.14) is reduce to the initial stochastic ordinary differential equations (2.16)–(2.17) and (2.18)–(2.19), and a linear algebraic equation (2.20).

2.3 Solutions for the inverse random source problem

Using the theory developed in the previous section, we easily obtain the existence and uniqueness for the inverse random source problem, which is stated as following,

Corollary 2.1. *The two-point boundary value problem (2.3)–(2.5) attains a unique solution.*

Proof. Since M is a non-singular matrix, there exists a non-singular matrix Q such that

$$Q^{-1}MQ = \Lambda,$$

where

$$\Lambda = \begin{bmatrix} \mathrm{i}\omega & \\ & -\mathrm{i}\omega \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 \\ \mathrm{i}\omega & -\mathrm{i}\omega \end{bmatrix}, \quad \text{and} \quad Q^{-1} = \frac{1}{2\mathrm{i}\omega}\begin{bmatrix} \mathrm{i}\omega & 1 \\ \mathrm{i}\omega & -1 \end{bmatrix}.$$

A simple calculation yields

$$\det \begin{bmatrix} A_0 \\ B_1 e^M \end{bmatrix} = \begin{vmatrix} \mathrm{i}\omega & 1 \\ -\mathrm{i}\omega e^{-\mathrm{i}\omega} & e^{-\mathrm{i}\omega} \end{vmatrix} = 2\mathrm{i}\omega e^{-\mathrm{i}\omega} \neq 0.$$

It follows from Theorem 2.2 that the two-point boundary value problem (2.3)–(2.5) has a unique solution.

□

Furthermore, if we let $\{A(x), v_0(x)\}$ and $\{B(x), v_1(x)\}$ be the integrated solutions for Eqs. (2.3), (2.4) and (2.3), (2.5), respectively, and take

$$D_0(W, x) = \mathrm{i}\omega \quad \text{and} \quad D_1(V, x) = -\mathrm{i}\omega,$$

we obtain the equations for the integrated solutions

$$dA = (-AM + \mathrm{i}\omega A)dx, \quad A(0) = A_0, \tag{2.21}$$
$$dv_0 = (A\mathbf{g} + \mathrm{i}\omega v_0)dx + A\mathbf{h}dW_x, \quad v_0(0) = 0, \tag{2.22}$$

and

$$dB = (-BM - \mathrm{i}\omega B)dx, \quad B(1) = B_1, \tag{2.23}$$
$$dv_1 = (B\mathbf{g} - \mathrm{i}\omega v_1)dx + B\mathbf{h}dW_x, \quad v_1(1) = 0. \tag{2.24}$$

It is easy to solve the above ordinary differential equation and the integrated solutions can be obtained

$$A(x) = [\mathrm{i}\omega\ 1], \quad B(x) = [-\mathrm{i}\omega\ 1] \tag{2.25}$$

and

$$v_0(x) = \int_0^x e^{\mathrm{i}\omega(x-y)} g(y)dy + \int_0^x e^{\mathrm{i}\omega(x-y)} h(y)dW_y \tag{2.26}$$
$$v_1(x) = -\int_x^1 e^{\mathrm{i}\omega(y-x)} g(y)dy - \int_x^1 e^{\mathrm{i}\omega(x-y)} h(y)dW_y. \tag{2.27}$$

Therefore, we obtain an explicit formula for the solution for the direct problem.

Corollary 2.2. *The unique solution of the two-point stochastic boundary value problem (2.1)–(2.2) is*

$$u(x, \omega) = \frac{1}{2\mathrm{i}\omega} \int_0^1 e^{\mathrm{i}\omega|x-y|} g(y)dy + \frac{1}{2\mathrm{i}\omega} \int_0^1 e^{\mathrm{i}\omega|x-y|} h(y)dW_y. \tag{2.28}$$

Proof. By Theorem 2.3, the two-point boundary value problem (2.3)–(2.5) is equivalent to the linear system

$$\begin{bmatrix} \mathrm{i}\omega & 1 \\ -\mathrm{i}\omega & 1 \end{bmatrix} \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} = \begin{bmatrix} v_0(x) \\ v_1(x) \end{bmatrix}.$$

Using Cram's rule yields

$$u(x) = u_1(x) = \frac{1}{2\mathrm{i}\omega}[v_0(x) - v_1(x)], \tag{2.29}$$

which completes the proof after substituting Eqs. (2.26) and (2.27).

□

We are ready to derive the formulas to reconstruct the mean and variance of the random source function. Let us evaluate both sides of Eq. (2.28) at $x = 0$, which yields

$$2\mathrm{i}\omega u(0, \omega) = \int_0^1 e^{\mathrm{i}\omega y} g(y)dy + \int_0^1 e^{\mathrm{i}\omega y} h(y)dW_y. \tag{2.30}$$

We easily obtain the relation between the data $u(0, \omega)$ and the mean value g after taking the expectation on both sides of (2.30). However, it is not convenient to derive the connection between the boundary measurements $u(0,\omega)$ and the standard deviation h, so we split all the complex functions into the sum of real part and imaginary part as following.

Denote

$$u(0,\omega) = \mathrm{Re}u(0,\omega) + \mathrm{i}\,\mathrm{Im}u(0,\omega).$$

Then Eq. (2.30) can be decomposed into two equations corresponding to the real part and the imaginary part:

$$2\omega\mathrm{Re}u(0,\omega) = \int_0^1 \sin(\omega y)g(y)dy + \int_0^1 \sin(\omega y)h(y)dW_y, \tag{2.31}$$

$$2\omega\mathrm{Im}u(0,\omega) = -\int_0^1 \cos(\omega y)g(y)dy - \int_0^1 \cos(\omega y)h(y)dW_y. \tag{2.32}$$

Recalling the basic property for the Itô integrals

$$\mathbb{E}\left[\int_0^1 \sin(\omega y)h(y)dW_y\right] = \mathbb{E}\left[\int_0^1 \cos(\omega y)h(y)dW_y\right] = 0,$$

we take the expectation on both sides of Eqs. (2.31) and (2.32) and obtain

$$2\omega\mathbb{E}\left[\mathrm{Re}u(0,\omega)\right] = \int_0^1 \sin(\omega y)g(y)dy, \tag{2.33}$$

$$2\omega\mathbb{E}\left[\mathrm{Im}u(0,\omega)\right] = -\int_0^1 \cos(\omega y)g(y)dy. \tag{2.34}$$

Therefore the mean value g can be recovered from either the inverse sine transform from Eq. (2.33) or the inverse cosine transform from Eq. (2.34).

Both Eqs. (2.33) and (2.34) are only valid for positive angular frequency $\omega > 0$. The zero Fourier mode is missing which leads to the non-uniqueness of the reconstruction, i.e., any vertical shift of the reconstructed function will give the same nonzero Fourier modes corresponding to the positive angular frequencies. In practice, the zero Fourier mode is set to be zero. After the inverse sine or cosine transform, the reconstructed function can be artificially shifted in vertical direction to make the value vanish at the lateral point $x = 0$ or $x = 1$ since the function g is assumed to have a compact support contained in the interval $[0, 1]$.

Using the Itô isometry, we have

$$\mathbb{E}\left[\left(\int_0^1 \sin(\omega y)h(y)dW_y\right)^2\right] = \int_0^1 \sin^2(\omega y)h^2(y)dy = \frac{1}{2}\int_0^1 [1-\cos(2\omega y)]h^2(y)dy,$$

$$\mathbb{E}\left[\left(\int_0^1 \cos(\omega y)h(y)dW_y\right)^2\right] = \int_0^1 \cos^2(\omega y)h^2(y)dy = \frac{1}{2}\int_0^1 [1+\cos(2\omega y)]h^2(y)dy.$$

Taking the variance on both sides of Eqs. (2.31) and (2.32) and using the Itô isometry, we get

$$4\omega^2\mathbb{V}\left[\mathrm{Re}u(0,\omega)\right] = \frac{1}{2}\int_0^1 [1-\cos(2\omega y)]h^2(y)dy,$$

$$4\omega^2\mathbb{V}\left[\mathrm{Im}u(0,\omega)\right] = \frac{1}{2}\int_0^1 [1+\cos(2\omega y)]h^2(y)dy.$$

Subtracting the above two equations we deduce

$$4\omega^2\{\mathbb{V}\left[\mathrm{Im}u(0,\omega)\right] - \mathbb{V}\left[\mathrm{Re}u(0,\omega)\right]\} = \int_0^1 \cos(2\omega y)h^2(y)dy. \tag{2.35}$$

The variance h^2 or the standard deviation h of the random source function can thus be retrieved from taking the inverse cosine transform on both sides of Eq. (2.35). Furthermore the zero Fourier mode can also be recovered by adding the above two equations

$$4\omega^2\{\mathbb{V}\left[\mathrm{Im}u(0,\omega)\right] + \mathbb{V}\left[\mathrm{Re}u(0,\omega)\right]\} = \int_0^1 h^2(y)dy. \tag{2.36}$$

Following Eqs. (2.31), (2.32), and (2.35), we may conclude that the inverse problem has a unique solution, i.e., the mean value g and the standard deviation h can be uniquely determined, if the data $\mathbb{E}\left[\mathrm{Re}u(0,\omega)\right]$, $\mathbb{E}\left[\mathrm{Im}u(0,\omega)\right]$, $\mathbb{V}\left[\mathrm{Re}u(0,\omega)\right]$, and $\mathbb{V}\left[\mathrm{Im}u(0,\omega)\right]$ m given for all frequencies $\omega > 0$. This is certainly an ideal situation since the data may only be available at a finite number of discrete set of frequencies in practice. The uniqueness is still valid as long as the data covers all the Fourier modes of the mean value g and the standard deviation h. Otherwise the uniqueness will not hold if some Fourier coefficients of the functions g and h are missing. In the reconstruction, we adopt so-called filtered backprojection algorithm which assumes that the Fourier coefficients at all of the unobserved frequencies are zero. This algorithm produces the reconstruction with minimal energy under the observation constraints. An alternative approach is the ℓ_1-minimization based method proposed by Candès *et al.* [8]. We refer to Yin *et al.* [24] for a discussion on the efficient Bregman iterative algorithms for the ℓ_1-minimization problems.

3 Numerical experiments

In this section, we discuss the algorithmic implementation and present three numerical examples to demonstrate the validity and effectiveness of the proposed method.

The scattering data $u(0,\omega)$ is obtained from two different approaches to avoid the so-called inverse crime. One is based on an integral equation and another is based on differential equation. Both approaches are numerically implemented and give the same performance of the reconstructions. We briefly introduce how we obtain the scattering data in the following.

In the integral equation approach, we use the solution representation in Eq. (2.28). We evaluate both sides at $x = 0$

$$u(0,\omega) = \frac{1}{2\mathrm{i}\omega}\int_0^1 e^{\mathrm{i}\omega y} g(y)dy + \frac{1}{2\mathrm{i}\omega}\int_0^1 e^{\mathrm{i}\omega y} h(y)dW_y.$$

Numerically the integrals are approximated by the trapezoidal rule

$$u(0,\omega) \approx \frac{1}{2\mathrm{i}\omega}\left[\Delta y \sum_{m=0}^{M-1} e^{\mathrm{i}\omega y_m} g(y_m) + \sum_{n=0}^{N-1} e^{\mathrm{i}\omega y_n} h(y_n)dW_n\right],$$

where $\Delta y = 1/M, y_m = m\Delta y = m/M, y_n = n/N$, and the spatial Brownian motion $dW_n = \xi_n/\sqrt{N}$, in which $\xi_n \in N(0,1)$ is a random variable in the standard Gaussian distribution with zero mean and unit variance. We generate ξ_n by a random number generator in FORTRAN90. In the following examples, M and N are taken as $M = N = 256$.

In another approach based on solving the stochastic initial value problem, we use equations (2.29) and (2.26)–(2.27) to get

$$u(0,\omega) = \frac{1}{2\mathrm{i}\omega}\left[v_0(0) - v_1(0)\right] = -\frac{v_1(0)}{2\mathrm{i}\omega}.$$

To obtain the data $u(0,\omega)$, it suffices to solve the stochastic ordinary differential equation

$$dv_1 = (g - \mathrm{i}\omega v_1)dx + hdW_x,$$
$$v_1(1) = 0,$$

Then we apply a numerical method over $[0,1]$ to compute the solution. We first discretize the interval. Let $\Delta x = 1/N$ for some positive integer N, and $x_i = i\Delta x = i/N$. Denote the numerical approximation to $v_1(x_n)$ by v_1^n. The Euler–Maruyama method takes the form

$$v_1^{n+1} = v_1^n + \left[g(x_{n+1}) - \mathrm{i}\omega v_1^{n+1}\right]\Delta x + h(x_{n+1})\left[W(x_{n+1}) - W(x_n)\right]$$

for $n = N-1, N-2, \ldots, 0$. We refer to [17] for an introduction to numerical simulation of stochastic differential equations.

Example 1. Let

$$g(x) = 0.3\left[(1-\cos(2x)) - \frac{16}{21}(1-\cos(3x)) + \frac{5}{28}(1-\cos(4x))\right],$$
$$h(x) = 0.6 - 0.3\cos(x) - 0.3\cos(2x),$$

reconstruct the mean value and the standard deviation given by

$$g_1(x) = g(2\pi x) \quad \text{and} \quad h_1(x) = h(2\pi x)$$

inside the interval $[0,1]$. This is a relatively simple example as both functions g_1 and h_1 contain few low frequency Fourier modes. For the reconstruction of the mean value g_1, the scattering data $u(0,\omega_k)$ is computed at discrete frequencies $\omega_k = k\pi, k = 1,2,\ldots,8$; while the scattering data $u(0,\omega_k)$ is computed at frequencies $\omega_k = k\pi/2, k = 1,2,\ldots,8$, for the reconstruction of the standard deviation h_1. The data covers all the frequency coefficients of this example. To test the stability of the method, we reconstruct the mean value and the standard deviation or the variance using different numbers of realization. This is equivalent to using data with different level of error. Figure 1 shows the reconstructed mean value and variance and the exact ones with different numbers of realizations. As expected, the relative $L^2([0,1])$ error "err" decreases from $\mathrm{err} = 3.05\times10^{-1}$ to $\mathrm{err} = 6.44\times10^{-3}$ and from $\mathrm{err} = 1.16\times10^{-1}$ to $\mathrm{err} = 4.00\times10^{-3}$ for the mean value and the variance, respectively, as the number of realization "nr" increases from $\mathrm{nr} = 10^3$ to $\mathrm{nr} = 10^6$. It is obvious that the better reconstruction may be obtained when the more accurate data is used. In fact, the reconstruction corresponding to the number of realizations $\mathrm{nr} = 10^6$ is actually indistinguishable from the exact functions from the graphs.

Example 2. Let

$$g(x) = 0.4\left[(1-\cos(3x)) - \frac{1215}{2783}(1-\cos(11x)) + \frac{7}{23}(1-\cos(12x))\right],$$
$$h(x) = 0.5e^1 - 0.3e^{\cos(2x)} - 0.3e^{\cos(3x)},$$

reconstruct the mean value and the standard deviation given by

$$g_2(x) = g(2\pi x) \quad \text{and} \quad h_2(x) = h(2\pi x)$$

inside the interval $[0,1]$. This example is more complicated than Example 1 since both functions contain more higher frequency modes. Correspondingly, the data at high frequencies should be computed to recover the mean value g_2 and the standard deviation h_2. For the reconstruction of the mean value g_2, the scattering data $u(0,\omega_k)$ is computed at discrete frequencies $\omega_k =$

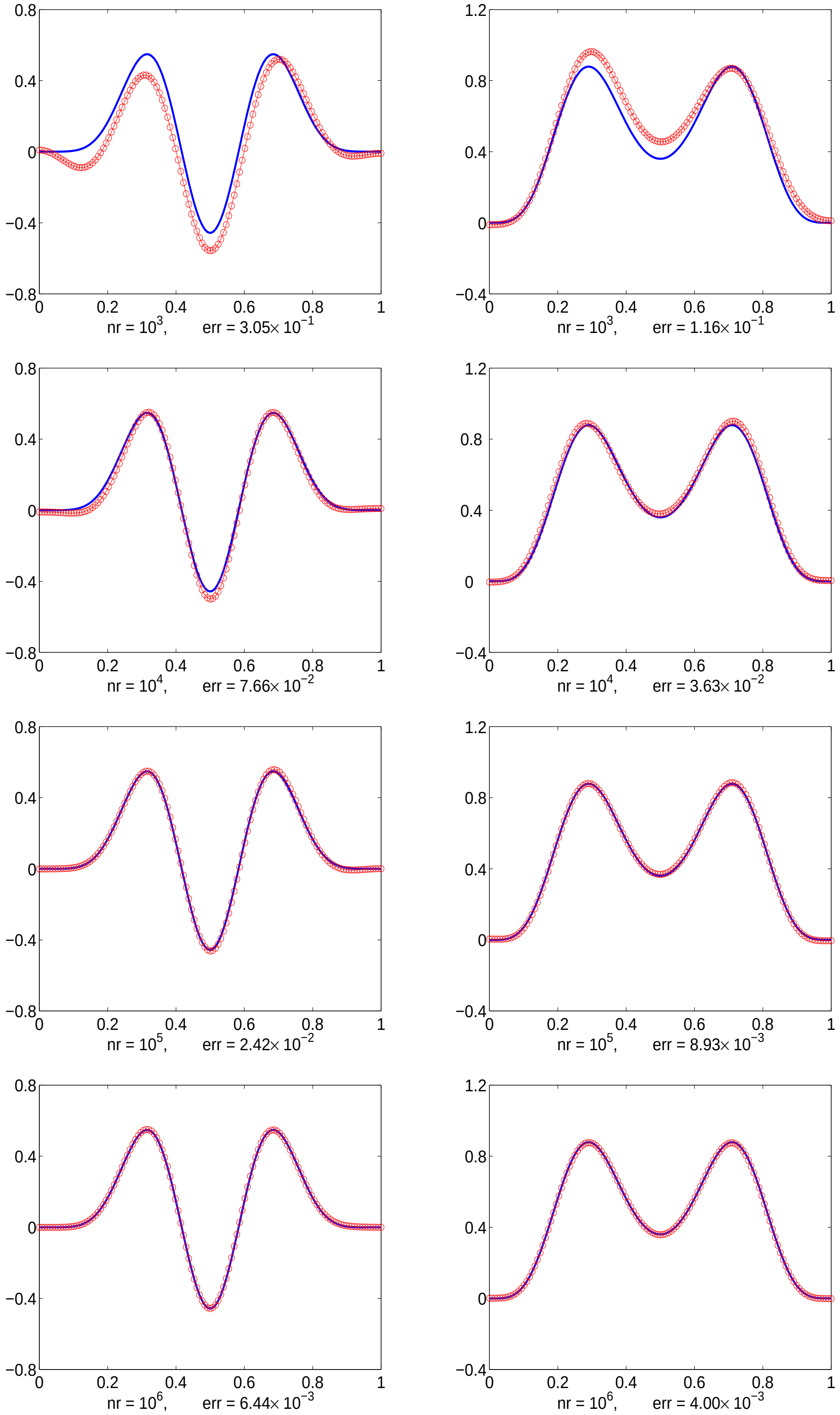

Figure 1: Example 1. (left) reconstruction of the mean value; (right) reconstruction of the variance.

$k\pi, k = 1, 2, \ldots, 16$; while the scattering data $u(0, \omega_k)$ is computed at frequencies $\omega_k = k\pi/2, k = 1, 2, \ldots, 16$, for the reconstruction of the standard deviation h_2. Figure 2 shows the reconstructed mean value and variance and the exact ones with different numbers of realizations. Not surprisingly, the relative error decreases from $\mathrm{err} = 3.84 \times 10^{-1}$ to $\mathrm{err} = 9.15 \times 10^{-3}$ and from $\mathrm{err} = 1.67 \times 10^{-1}$ to $\mathrm{err} = 5.03 \times 10^{-3}$ for the mean value and the variance, respectively, as the number of realization increases from $\mathrm{nr} = 10^3$ to $\mathrm{nr} = 10^6$.

Example 3. Reconstruct the mean value

$$g_3(x) = \begin{cases} 0.5 & \text{for } 0.15 < x < 0.35 \\ 0.5 & \text{for } 0.65 < x < 0.85 \\ 0 & \text{otherwise} \end{cases}$$

and the standard deviation

$$h_3(x) = \begin{cases} 0.5 & \text{for } 0.3 < x < 0.7 \\ 0 & \text{otherwise} \end{cases}$$

inside the interval $[0, 1]$. In this example, the functions are discontinuous. It is well known that the piecewise constant function contains infinitely many Fourier coefficients that decay slowly. To show the effect of the maximum frequency on the reconstruction, we use the number of realization $\mathrm{nr} = 10^6$ to generate the data which is intended to reduce the effect of the data error. For the reconstruction of the mean value g_3, the scattering data $u(0, \omega_k)$ is computed at frequencies $\omega_k = k\pi, k = 1, 2, \ldots, \mathrm{nw}$; while the scattering data $u(0, \omega_k)$ is computed at $\omega_k = k\pi/2, k = 1, 2, \ldots, \mathrm{nw}$, for the reconstruction of the standard deviation h_2, where "nw" is the maximum number of frequency. Figure 3 shows the reconstructed mean value and variance and the exact ones with different numbers of frequencies. As one can see, the relative error decreases from $\mathrm{err} = 3.13 \times 10^{-1}$ to $\mathrm{err} = 1.90 \times 10^{-1}$ and from $\mathrm{err} = 2.32 \times 10^{-1}$ to $\mathrm{err} = 1.23 \times 10^{-1}$ for the mean value and the variance, respectively, as the number of frequency increases from $\mathrm{nw} = 8$ to $\mathrm{nw} = 32$.

In summary, the following observations can be made from Figure 1 to Figure 3. When the functions contain few low Fourier modes or fast decaying Fourier coefficients, accurate and stable reconstructions can be obtained easily. When the functions are discontinuous, the oscillatory behavior near the discontinuities displays the well-known Gibbs phenomenon. To encounter this challenge, we have also implemented an alternative approach of the ℓ_1-minimization based method together with the Bregman iteration for all presented three examples. Generally speaking, the alternative approach produces similar results for smooth functions and may reduce the oscillations for discontinuous functions if appropriate parameters are chosen in the iteration. However, we feel the alternative approach is beyond the scope of the current paper, since our main intention is to report the novel numerical methods to reconstruct random source functions from boundary measurements. And thus we decide to only show the numerical results based on filtered backprojection algorithm which involves just one FFT in the computation.

4 Concluding remarks

We studied an inverse problem for the stochastic Helmholtz equation in one dimension with a random source function. Based on integrated solution method, we formulated the model problem as a two-point stochastic boundary value problem and derived an integral equation for the solution which sets up the relation between the data and the target functions through FFT. The method is extremely efficient and accurate for smooth functions. Although we only consider Gaussian random field in this paper, the strategy can be extended to other types of randomness in the

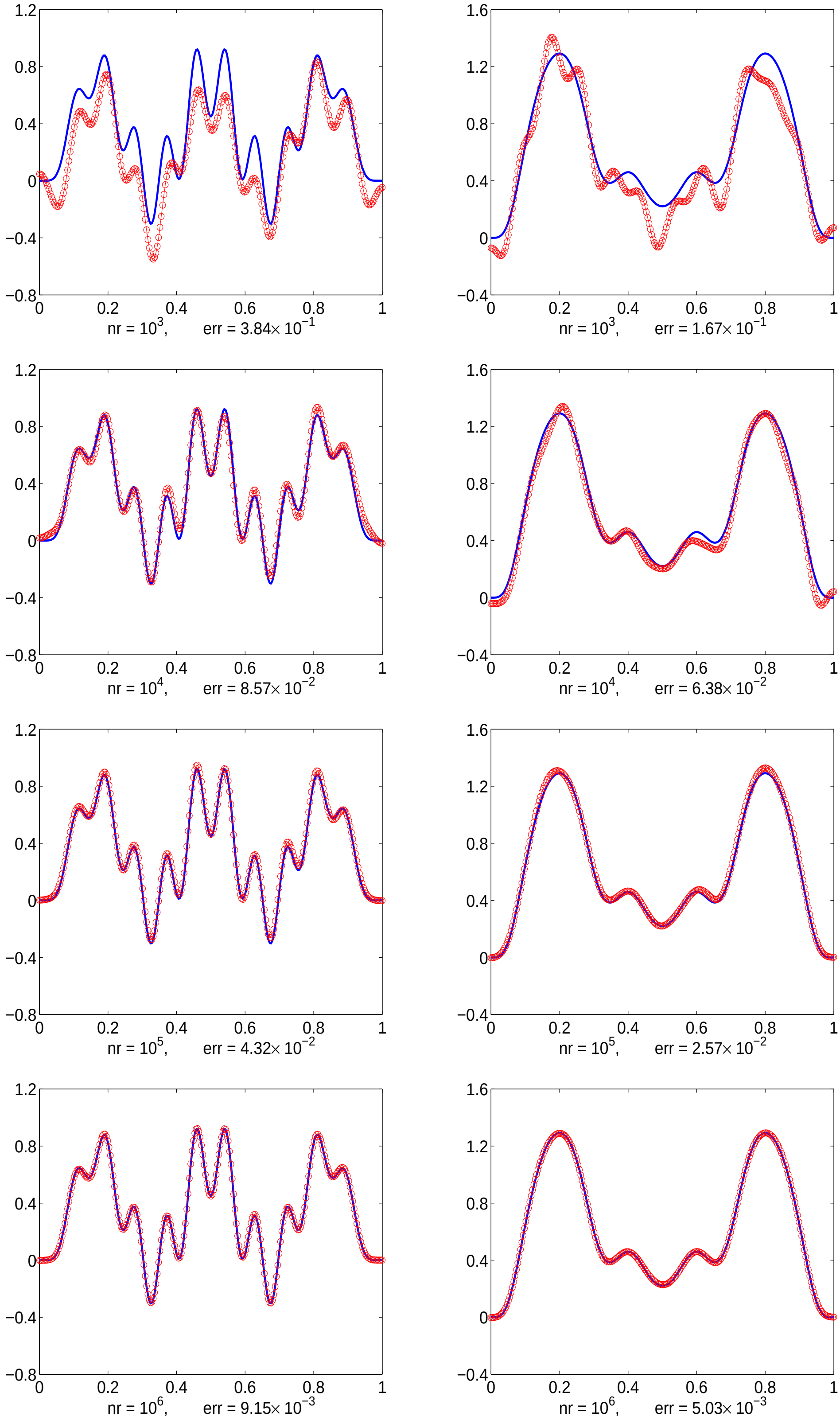

Figure 2: Example 2. (left) reconstruction of the mean value; (right) reconstruction of the variance.

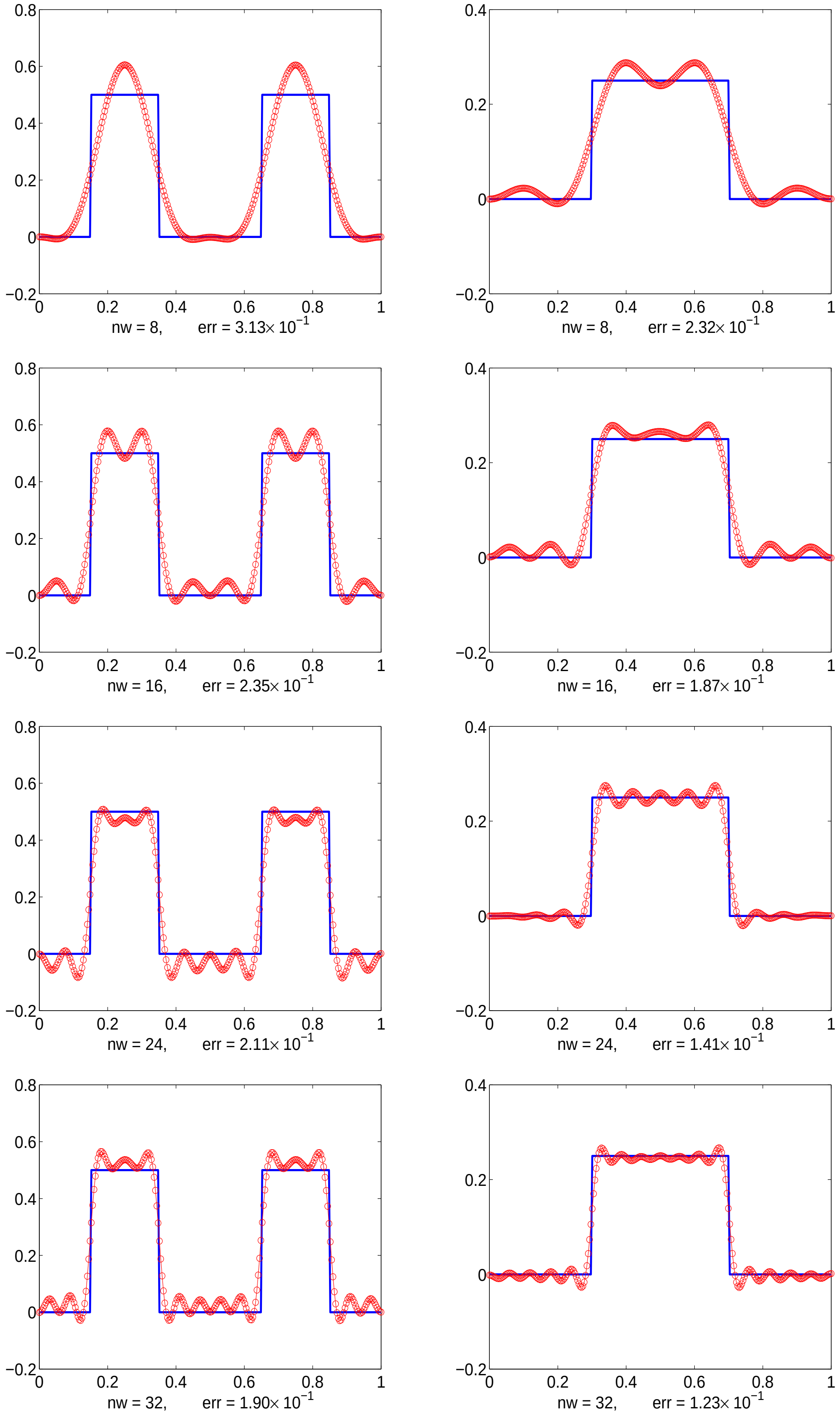

Figure 3: Example 3. (left) reconstruction of the mean value; (right) reconstruction of the variance.

source function with minor modification. Furthermore, the techniques used in this paper can be naturally generalized to two and three dimensional problems. We are currently investigating the inverse random source problem in inhomogeneous background medium and will report the progress elsewhere in the future.

References

[1] R. Albanese and P. Monk, The inverse source problem for Maxwell's equations, Inverse Problems, 22 (2006), 1023–1035.

[2] H. Ammari, G. Bao, and J. Fleming, An inverse source problem for Maxwell's equations in magnetoencephalography, SIAM J. Appl. Math., 62 (2002), 1369–1382.

[3] M. Badieirostami, A. Adibi, H. Zhou, and S. Chow, Model for efficient simulation of spatially incoherent light using the Wiener chaos expansion method, Opt. Lett., 32 (2007), 3188–3190.

[4] G. Bao, S.-N. Chow, P. Li, and H. Zhou, Numerical solution of an inverse medium scattering problem with a stochastic source, preprint.

[5] G. Bao, J. Lin, and F. Triki, A multi-frequency inverse source problem, preprint.

[6] N. Bleistein and J. Cohen, Nonuniqueness in the inverse source problem in acoustic and electromagnetics, J. Math. Phys., 18 (1977), 194–201.

[7] Y.-Z. Cao, R. Zhang, and K. Zhang, Finite element and discontinuous Galerkin method for stochastic Helmholtz equation in two- and three-dimensions, J. Comput. Math., 25 (2007), 368–373.

[8] E. Candès, J. Romberg, and T. Tao, Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information, IEEE Trans. Inform. Theory, 52 (2006), 489–509.

[9] Y. Chen and V. Rokhlin, On the inverse scattering problem for the Helmholtz equation in one dimension, Inverse Problems, 8 (1992), 365–391.

[10] D. Colton and R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, 2nd ed., Appl. Math. Sci. 93, Springer-Verlag, Berlin, 1998.

[11] A. Devaney, The inverse problem for random sources, J. Math. Phys., 20 (1979), 1687–1691.

[12] A. Devaney, E. Marengo, and M. Li, The inverse source problem in nonhomogeneous background media, SIAM J. Appl. Math., 67 (2007), 1353–1378.

[13] A. Devaney and G. Sherman, Nonuniqueness in inverse source and scattering problems, IEEE Trans. Antennas Propagat., 30 (1982), 1034–1037.

[14] M. Eller and N. Valdivia, Acoustic source identification using multiple frequency information, Inverse Problems, 25 (2009), 115005.

[15] I. M. Gelfand and B. M. Levitan, On the determination of a differential equation from its spetral functions, Amer. Math. Soc. Transl. Ser. 2, 1 (1955), 253–304.

[16] K.-H. Hauer, L. Kühn, and R. Potthast, On uniqueness and non-uniqueness for current reconstruction from magnetic fields, Inverse Problems, 21 (2005), 955–967.

[17] D. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, SIAM Review, 43 (2001), 525–546.

[18] P. Kloeden and E. Platen, Numerical Solution of Stochastic Differential Equations, Springer-Verlag, New York, 1992.

[19] E. Marengo and A. Devaney, The inverse source problem of electromagnetics: linear inversion formulation and minimum energy solution, IEEE Trans. Anntennas Propagat., 47 (1999), 410–412.

[20] D. Nualart and E. Pardoux, Boundary value problems for stochastic differential equations, Ann. Probab., 19 (1991), 1118–1144.

[21] B. Øksendal, Stochastic Differential Equations, 6th ed., Springer-Verlag, Berlin, 2005.

[22] G. Papanicolaou, Wave propagation in a one-dimensional random medium, SIAM J. Appl. Math., 21 (1971), 13–18.

[23] W. Press, S. Teukolsky, W. Vetterling, and B. Flannery, Numerical Recipes in Fortran 90: The Art of Parallel Scientific Computing, 2nd ed., Cambridge University Press, 1996.

[24] W. Yin, S. Osher, D. Goldfarb, and J. Darbon, Bregman iterative algorithms for ℓ_1-minimization with applications to compressed sensing, SIAM J. Imaging Sci., 1 (2008), 143-168.

[25] R. Yuste, Fluorescence microscopy today, Nat. Methods, 2 (2005), 902–904.

[26] G.-Q. Zhang, Integrated solutions of ordinary differential equation system and two-point boundary value problems, I. Integrated solution method, J. Comp. Math., 3 (1981), 245–254.