

# TOTAL VARIATION IMPROVED WAVELET THRESHOLDING IN IMAGE COMPRESSION

*Tony F. Chan and H.M. Zhou*

Department of Mathematics, the University of California, Los Angeles, CA90095-1555.  
email: {chan,hmzhou}@math.ucla.edu

## ABSTRACT

In this paper, we propose using Partial Differential Equation (PDE) techniques in wavelet based image processing to reduce edge artifacts generated by wavelet thresholding. We employ minimization techniques, in particular the minimization of total variation (TV), to modify the retained standard wavelet coefficients so that the reconstructed images have less oscillations near edges. Numerical experiments show that this approach improves the reconstructed image quality in wavelet compression and in denoising.

## 1. INTRODUCTION

In this paper, we are concerned with the suppression of edge artifacts caused by wavelet thresholding in digital image denoising and compression. It is well known that wavelet thresholding, including linear (i.e. truncating the high frequencies) and nonlinear thresholding (i.e. retaining large coefficients) may generate oscillations near discontinuities, especially when the images contain high level noise. This Gibbs' phenomenon is the primary reason for edge artifacts in digital image processing.

Many methods have been proposed to overcome this problem. Donoho's soft thresholding truncates wavelet coefficients on different scale levels subject to different thresholds [8]. A different approach also due to Donoho is to construct special basis such as curvelets [1] for discontinuities. A different approach is to modify the wavelet transforms so that fewer large high frequency coefficients are generated near discontinuities, resulting in fewer large coefficients are truncated in the thresholding process. Along this direction, Claypoole, Davis, Sweldens and Baraniuk [3] proposed an adaptive lifting scheme which lowers the order of approximation near jumps, thus minimizing the Gibbs' effects. We have proposed ENO-wavelet transforms which apply the one-side approximation idea of constructing Essentially Non-Oscillatory (ENO) schemes in numerical shock capturing to design an adaptive wavelet transform

such that no large high frequency coefficients are generated through differencing across discontinuities [6], essentially eliminating oscillations in the restored images.

In this paper, we propose an alternative method, which uses other PDE techniques, especially PDE's from variational principles, to reduce the oscillations in wavelet thresholding approximations. In fact, variational-PDE models have been commonly used in image processing since the end of the 1980's, for example, Mumford-Shah's functional [10] in segmentation, and Rudin-Osher-Fatemi's Total Variation in restoration [11]. The crucial observation which makes these methods successful is in viewing images as piecewise smooth functions connected by large jumps (edges) and realizing the similarity between images and piecewise smooth solutions of certain kinds of PDE's, and employing well-developed PDE's techniques to handle the edges. Based on this observation, one can also apply PDE techniques to wavelet image processing to reduce the edge oscillations. Our goal is to use minimization techniques, in particular, the techniques for minimizing total variation (TV), to modify the retained standard wavelet coefficients so that the restored images have fewer oscillations near the edges.

It has been shown through many simulations that the TV model can effectively suppress the noise while retaining sharp edges in images because it allows the existence of discontinuities, see [11] and [5]. Chambolle, DeVore, Lee and Lucier [2] attempted to use wavelet based variational forms to accomplish compression and denoising. Using wavelet coefficients, they compute the best fitting of the observed images subject to minimizing certain norms in Besov spaces, which are close to the Bounded Variation (BV) space with the TV norm. An essential difference between the Besov spaces and the BV space is that Besov spaces do not allow the existence of discontinuities. Therefore, sharp edges are unavoidably smoothed out in the restored images. In [7], we demonstrate that compressing TV denoised images may produce higher ratio compression and better quality than denoising and compressing the images by directly using wavelets. On the other hand, edge oscillations caused by standard wavelet thresholding significantly increase the TV norm of the restored images. All this mo-

---

Research supported in part by grants ONR-N00017-96-1-0277 and NSF DMS-96-26755. Report website: <http://www.math.ucla.edu/applied/cam/index.html>

tivates us to select and modify the nonzero wavelet coefficients in the thresholding procedure subject to minimizing the TV norm of the restored images so that they can produce fewer edge artifact approximations while retain sharp edges.

In general, minimizers of such variational problems can be found by solving their associated Euler-Lagrangian equations, which are PDE's. In particular, the PDE produced by the TV minimization problem is highly nonlinear and usually degenerate at flat regions. Many works have been advocated to speed up the solvers in physical space, for instance, see [13] and [4] for some different methods proposed. In the present work, we deduce the corresponding PDE's in wavelet space and solve them in analogous ways. We will discuss some aspects of the numerics in this paper as well.

The above described method can be easily embedded into image compression by simply replacing the standard wavelet thresholding step by TV regularized wavelet thresholding. The produced non-zero wavelet coefficients then can be forwarded for quantizing and coding in the standard ways. In this situation, at the reconstruction end, the standard wavelet procedure will automatically restore the images with fewer edge artifacts. We will concentrate on selecting and modifying the non-zero wavelet coefficients subject to minimizing the TV norm of the restored images, and we will not consider the quantization and coding steps. In addition, the introduced idea can also be used as a post-processing technique for the restored images so that it can suppress the edge oscillations generated in the compression process.

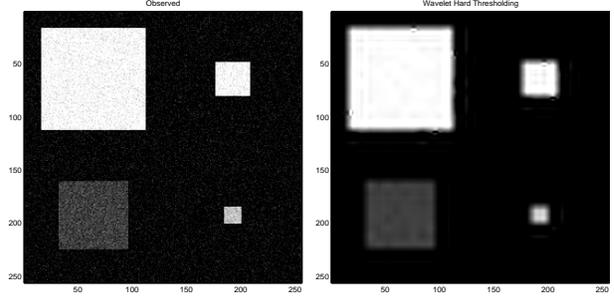
This paper is arranged in the following way: In section 2, we give the TV regularized wavelet compression model for wavelet thresholding. In section 3, we discuss some numerical aspects of solving its associated PDE. And in section 4, we show some examples to illustrate the results of the model.

## 2. TV REGULARIZED WAVELET COMPRESSION MODEL FOR THRESHOLDING

In this section, we give our TV regularized Wavelet Compression model for suppressing the oscillations generated by wavelet thresholding.

Suppose we are given an observed image  $z(x) = u_0(x) + n(x)$ , where  $u_0(x)$  is the original noise free image and  $n(x)$  the Gaussian white noise with  $\|n(x)\|_2 = \sigma$ . Let us denote the standard orthonormal wavelet transform of  $z(x)$  by:  $z(\vec{\alpha}, x) = \sum_{j,k} \alpha_{j,k} \phi_{j,k}(x)$ , where  $\phi_{j,k}(x)$  are wavelet basis functions and  $\vec{\alpha} = \{\alpha_{j,k}\}$  the corresponding coefficients.

One way to describe the wavelet thresholding technique is to prescribe a wavelet coefficient index set  $I$ , then retain all coefficients if their indices belong to  $I$  and truncate the



**Fig. 1.** Left: The observed image has features with sharp edges despite of the present of noise. Right: The 4-level DB6 wavelet hard thresholding reconstruction which retains the largest  $16 \times 16$  coefficients. Edge artifacts are clearly seen along the boundaries.

other coefficients to zero:

$$\alpha_{j,k}^* = \begin{cases} \alpha_{j,k} & (j,k) \in I \\ 0 & otherwise \end{cases}$$

For example, in linear thresholding,  $I$  is taken as the set of low frequencies; and in hard thresholding,  $I$  is defined as the set of all coefficients whose magnitude is larger than a given tolerance, otherwise, it is smaller than the tolerance. Since orthonormal wavelets form an orthonormal basis of the  $L^2$  space, it is obvious that the hard thresholding selection of  $I$  minimizes the  $L^2$  error between the compressed image  $u(x)$  and the observed image  $z(x)$ .

The hard thresholding approximations introduce oscillations at the edges, although they are optimal in the  $L^2$  space. This is due to the fact that the  $L^2$  norm minimization does not penalize oscillations. Fig 1 (left) is a 2-D image containing four noisy squares with different sizes and intensities. We show its 4-level Daubechies 6 wavelet hard thresholding approximation in Fig 1 (right). The approximation contains edge artifacts along the boundaries of the objects, while in the observed image, these objects have sharp edges.

Wavelet thresholding can cause oscillations near edges, this consequently increases the TV norm of the restored image. To suppress these oscillations, we propose the following model to modify the values of the retained wavelet coefficients  $\beta_{j,k}$  such that the restored image  $u(\vec{\beta}, x)$  forms a less severe oscillatory approximation:

$$\min_{\beta_{j,k}, (j,k) \in I_H} F(u, z) = \lambda \int |\nabla_x u(\vec{\beta}, x)| dx + \frac{1}{2} \|u - z\|_2^2 \quad (1)$$

where  $I_H$  is the retained index set in the standard hard thresholding. Here we have  $\beta_{j,k} = 0$  if  $(j,k) \notin I_H$ , and  $\lambda$  the regularization parameter.

The first term in the objective functional reduces the oscillations of  $u(x)$  by diminishing its TV norm. The second

term is the standard  $L^2$  fitting term which controls the difference between  $u(x)$  and the observed image  $z(x)$ . The regularization parameter  $\lambda$  is used to balance the trade-off between the suppression of oscillations and the fitting term. When  $\lambda$  tends to zero,  $u(x)$  goes to the standard hard thresholding approximation. On the other hand, when  $\lambda$  tends to infinity, the suppression term dominates the objective functional, and therefore  $u(x)$  tends to a constant. As a TV regularization parameter,  $\lambda$  also controls the smallest scale of features to preserve [12], i.e. for a given value of  $\lambda$ , there exists a size of feature such that the model treats all features smaller than this size as oscillations and diminishes them. On the other hand, it preserves features which are bigger than this critical scale. In practice,  $\lambda$  can be determined in many ways, for instance, using  $L$ -curve techniques [9] to select the best  $\lambda$ , or determining it by studying the best  $\lambda$  for a set of training images in certain classes of images. In this paper, we do not discuss them in detail, though we use the latter choice to select  $\lambda$  in our numerical experiments.

Compared to the approach proposed in [7], which uses the TV denoising followed by standard wavelet thresholding to obtain high ratio compression for noisy data, the advantage of the proposed TV regularized wavelet compression model is that the TV regularized model can reduce the oscillations generated by wavelet thresholding as well as the noise, while TV denoising followed by standard thresholding may generate new oscillations after denoising. Also, the TV regularized wavelet compression model can directly work on wavelet coefficients. Therefore, it is easier to be fit into practical compression schemes, especially for images given in wavelet coefficient format (e.g. the upcoming wavelet based JPEG 2000 compression standard). In addition, the TV regularized model may work on a fewer number of coefficients (in the hard thresholding case). Potentially, it could be faster than TV denoising followed by standard thresholding.

**Remark:** (1) The TV regularization term in the model can be replaced by the H-1 regularization term  $\|\nabla u\|_2^2$ , or other regularization terms. Compared to the TV term, the other norms usually smooth out sharp edges in the reconstructed images. We will show a comparison in our numerical experiments in section 4. (2) The TV regularization idea can also be applied to selecting the retained index set  $I$  (in contrast to setting  $I = I_H$  as is done in this work) and modifying the retained coefficients to suppress the edge oscillations. We will not explore this more general model in this paper due to the lack of space.

### 3. NUMERICS

The above minimization problem is convex and unconstrained, and has a unique solution  $u(x)$  in the subspace of  $I_H$ . The solution  $u(x)$  satisfies the E-L equation in wavelet space:

$$-\lambda \int \nabla_x \left( \frac{\nabla_x u}{|\nabla_x u|} \right) \phi_{j,k}(x) dx + 2(\beta_{j,k} - \alpha_{j,k}) = 0, \quad (2)$$

$$(j, k) \in I_H.$$

To find the solutions for the TV regularized wavelet compression model, we want to solve its E-L equations (2). In fact, many numerical methods for similar equations in physical space have been proposed in literature. All those methods can be adapted to the wavelet space. Here, we use the fixed-point method [13] as an example to show some numerical aspects involved in the computation.

The fixed-point method discretizes the E-L equation by linearizing the nonlinear terms with previous approximations. We denote  $D_{x,+}$  ( $D_{x,-}$ ) as the forward (backward) finite differences in physical space, Then the fixed-point scheme is:

$$-\lambda \int D_{x,-} \left( \frac{D_{x,+} u^{n+1}}{\sqrt{|D_{x,+} u^n|^2 + \epsilon_1}} \right) \phi_{j,k}(x) dx \quad (3)$$

$$+ 2(\beta_{j,k}^{n+1} - \alpha_{j,k}) = 0, \quad (j, k) \in I_H,$$

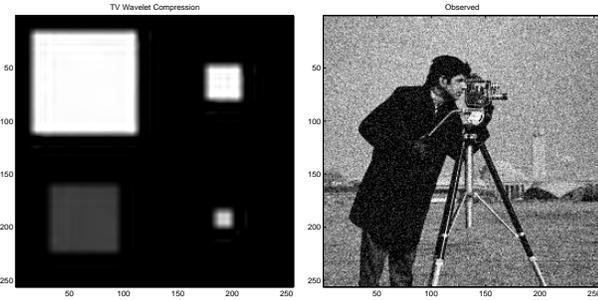
where  $u^n = u(x, \vec{\beta}^n)$ ,  $\epsilon_1$  and is a small positive number which are used to prevent blow-up in regions where  $\nabla u = 0$ . (3) is a linear equation of unknowns  $\beta_{j,k}$ , which we solve by Conjugate Gradient (CG) without preconditioning.

### 4. EXAMPLES

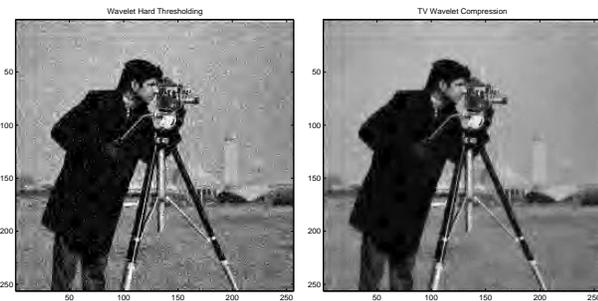
In this section, we show some 2-D examples to demonstrate the improvement in images of the TV regularized model for wavelet thresholding.

We apply the TV regularized model to the example in Fig 1. and show the result in Fig 2 (left). It is obvious that in this picture, the edge artifacts are less severe than in the standard case (Fig 1 (right)). Meanwhile, since the regularization parameter  $\lambda$  also controls the smallest size of features to preserve, in the TV regularized restored image, smaller features (such as the smallest square) are altered more than the large features, i.e. the intensities are lower than the standard approximation. In Fig 2 (right), we show the cameraman image with Gaussian white noise. We display the  $64 \times 64$  non-zero coefficient reconstruction calculated by standard hard thresholding in Fig 3 (left), and the TV regularized wavelet compression model in Fig 3 (right). Compared to the standard hard thresholding image, the edge artifacts in the TV model approximations are much less severe.

In conclusion, we have used the TV regularized model to select and modify the non-zero wavelet coefficients in the thresholding procedure. The resulting compressed images contain less severe edge artifacts than those in the standard



**Fig. 2.** Left: The TV regularized Compression with hard thresholding. It keeps the largest  $16 \times 16$  coefficients. Less severe edge artifacts present in the left compressed image compared to the image in Fig 1 (right). Right: The noisy cameraman image.



**Fig. 3.** The standard hard thresholding approximation (left) and the TV regularized wavelet compression (right), both keep the largest  $64 \times 64$  coefficients. Severe edge artifacts present in the left compressed image while in the right one, there are much fewer edge artifacts.

thresholding images, especially when large noise is present in the image. The model can directly operate on the wavelet coefficients, therefore, it can easily be embedded into practical compression schemes, and it also does not affect the reconstruction schemes at all. More work needs to be done to improve the speed of convergence and to make the methods more practical.

## 5. REFERENCES

- [1] E. Candes and D. Donoho, *Curvelets: A Surprisingly Effective Nonadaptive Representation of Objects with Edges*, Tech. Report, Dept. of Stat., Stanford Univ., 1999.
- [2] A. Chambolle, R. DeVore, N. Lee and B. Lucier, *Nonlinear Wavelet Image Processing: Variational Problems, Compression, and Noise Removal Through Wavelet Shrinkage*, IEEE Tran. Image Proc., Vol. 7, No. 3, Mar. 1998, pp319-333.
- [3] P. Claypoole, G. Davis, W. Sweldens and R. Baraniuk, *Nonlinear Wavelet Transforms for Image Coding*, Correspond. Author: Baraniuk, Dept. of Elec. and Comp. Sci., also Submit to IEEE Tran. on Image Proc., Preprint, 1999.
- [4] T. F. Chan, G. H. Golub, and P. Mulet, *A Nonlinear Primal-Dual Method for Total Variation-Based Image Restoration*, in ICAOS'96, 12th International Conference on Analysis and Optimization of Systems: Images, Wavelets, and PDEs, Paris, June 26-28, 1996, number 219 in Lecture Notes in Control and Information Sciences, pp. 241-252.
- [5] T. F. Chan and C. K. Wong, *Total Variation Blind Deconvolution*, CAM Report, No. 96-44, Dept. of Math., UCLA, 1996.
- [6] T. F. Chan and H. M. Zhou, *Adaptive ENO-Wavelet Transforms for Discontinuous functions*, CAM Report, No. 99-21, Dept. of Math., UCLA, Submit to SIAM Numer. Anal., 1999.
- [7] T. F. Chan and H. M. Zhou, *Feature Preserving Lossy Image Compression Using Nonlinear PDE's*, SPIE Proceedings on Advanced Signal Processing Algorithms, Architectures, and Implementations VIII, Vol. 3461, F. T. Luk, ed., San Diego, California, July 1998, pp 316-327.
- [8] D. Donoho, *De-noising by Soft Thresholding*, IEEE Trans. Inf. Th. 41(1995), pp613-627.
- [9] P. C. Hensen, *The L-curve and Its Use in the Numerical Treatment of Inverse Problems*, Tech. Report, IMM-REP 99-15, Dept. of Math. Model., Tech. Univ. of Denmark, 1999.
- [10] D. Mumford and J. Shah, *Optimal Approximation by Piecewise Smooth Functions and Associated Variational Problems*, Comm. Pure Appl. Math. 42, 1989, pp577-685.
- [11] L. Rudin, S. Osher and E. Fatemi, *Nonlinear Total Variation Based Noise Removal Algorithms*, Physica D, Vol 60(1992), pp. 259-268.
- [12] David M. Strong, Peter Blomgren, and Tony F. Chan, *spatially Adaptive Local Feature-Driven Total Variation Minimizing Image Restoration*, Proceedings of SPIE, vol. 3167, 1997.
- [13] C. Vogel and M. Oman, *Iterative Methods for Total Variation Denoising*, SIAM J. Sci. Comput., Vol. 17, 227-238, 1996..