A VECTOR BIMEASURE INTEGRAL WITH SOME APPLICATIONS*

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Abstract A Fubini type theorem is obtained for a class of vector bimeasure integrals. Some applications to the theory of harmonizable processes are also considered.

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I. Introduction

A theory of integration with respect to a bimeasure was initiated by Morse and Transue [7–9] and further developed by Thomas [11]. For these authors, bimeasures are continuous bilinear functionals on $C_c(E_1) \times C_c(E_2)$, where $C_c(E_i)$, i=1,2, are the usual spaces of continuous functions with compact support on the locally compact Hausdorff spaces E_i, i=1,2. More recently, motivated by the problem of finding a Fourier representation for the covariance of a second order process, this theory has been expanded by Niemi [10] and Chang and Rao [2,3]. Along the bilinear functional approach, the set function approach has now also been studied as well as the Banach valued case developed by Ylinen [13].

In the works mentioned above the authors consistently impose, in their definition of integrability, a Fubini type condition which cannot usually be bypassed. In Section II we show that, a suitable and natural restriction of the definition of integrability implies a stronger Fubini type property. In Section III , we study the spectral domain of harmonizable processes.

II. Vector Bimeasure Integration

Let X be a Banach space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and let $(E, \mathcal{M})$ be a measurable space. A *vector measure* is a σ–additive set function $\mu: \mathcal{M} \to X$. Integration of functions $f: E \to \mathbb{F}$ with respect to vector measures is taken in the Bartle, Dunford and Schwartz [1] sense, and the reader is referred to Dunford and Schwartz [4,IV.10] for the properties of this vector integral.

Let $(E_1, \mathcal{M}_1)$ and $(E_2, \mathcal{M}_2)$ be two measurable spaces. A *vector bimeasure* (bimeasure when $X = \mathbb{F}$) is a separately σ–additive set function $\beta: \mathcal{M}_1 \times \mathcal{M}_2 \to X$, i.e., $\beta(\cdot, B)$ and $\beta(A, \cdot)$ are vector measures for all $A \in \mathcal{M}_1$, $B \in \mathcal{M}_2$.

The proof of our result as well as our definition of integrability rely on the following two lemmas. The first is classical and can be found in [4,p.323], while the second is in [13,p.122].

Lemma 1. Let f: $E \longrightarrow \mathbb{F}$ be μ–integrable. Then, the set function $\nu(A) = \int_A f d\mu$, $A \in \mathcal{M}$, is a vector measure.

Lemma 2. Let f: $E_1 \longrightarrow \mathbb{F}$ be $\beta(\cdot,B)$–integrable for all $B \in \mathcal{M}_2$. Then the set functions ${}_f\beta(A,\cdot)\colon \mathcal{M}_2 \longrightarrow X$, $B \longrightarrow {}_f\beta(A,B) = \int_A f(\cdot)d\beta(\cdot,B)$ are vector measures for all $A \in \mathcal{M}_1$.

The vector measures $\beta_g(\cdot,B)$ are defined in a completely symmetrical way for functions g: $E_2 \longrightarrow \mathbb{F}$ and $B \in \mathcal{M}_2$.

We can now define β–integrability.

Definition 3. A pair of functions (f,g), f: $E_1 \longrightarrow \mathbb{F}$, g: $E_2 \longrightarrow \mathbb{F}$ is said to be integrable with respect to the vector bimeasure $\beta\colon \mathcal{M}_1 \times \mathcal{M}_2 \longrightarrow X$ (β–integrable for short) if the following two conditions hold:

(i) f is $\beta(\cdot,B)$–integrable for all $B \in \mathcal{M}_2$ and g is $\beta(A,\cdot)$–integrable for all $A \in \mathcal{M}_1$,

(ii) f is $\beta_g(\cdot,B)$–integrable for all $B \in \mathcal{M}_2$ and g is ${}_f\beta(A,\cdot)$–integrable for all $A \in \mathcal{M}_1$.

For $X = \mathbb{F}$ this definition of integrability is stronger than that of Morse and Transue. For these authors, (f,g) is integrable if in (i) and (ii), A and B are replaced by E_1 and E_2 and if in addition

$$\int_{E_1} f(\cdot)d\beta_g(\cdot,E_2) = \int_{E_2} g(\cdot)d_f\beta(E_1,\cdot). \tag{1}$$

It is also more restrictive than the *strong integral* of Niemi [10] or the *β–integral* of Ylinen [13]. For both of them, a pair (f,g) is integrable if in (ii), A and B are respectively replaced by E_1 and E_2 and if in addition (1) is satisfied. However, our definition is weaker than the *strict β–integral* of Chang and Rao [2,3] (there is no additional Fubini condition).

With weaker definitions than Definition 3, the Fubini type property (1) cannot be obtained as a consequence of β–integrability (see [9], [13]). Nevertheless, by trading off these weaker definitions with the natural integrability conditions of definition 3, not only (1), but also the following stronger result holds.

Theorem 4. If the pair (f,g) is β–integrable, then for all $A \in \mathscr{M}_1$, $B \in \mathscr{M}_2$,

$$\int_A f(\cdot)d\beta_g(\cdot,B) = \int_B g(\cdot)d_f\beta(A,\cdot), \tag{2}$$

and the common value in (2) can thus be denoted by $\int_A \int_B fgd\beta$.

Proof. Let (f,g) be β–integrable. If both f and g are simple functions, then (2) is trivial. Let f and g be bounded (f and g are measurable since integrable in the Bartle, Dunford and Schwartz sense). Then f (resp. g) is the uniform limit of a sequence $\{f_n\}$ (resp. $\{g_m\}$) of simple functions such that $|f_n| \leq M = \sup |f|$ (resp. $|g_m| \leq N = \sup |g|$). Since (f,g) is β–integrable, it follows by the dominated convergence theorem for vector measures (see [4,p.328]) that,

$$\int_A f(\cdot)d\beta_g(\cdot,B) = \lim_{n\to+\infty} \int_A f_n(\cdot)d\beta_g(\cdot,B) = \lim_{n\to+\infty} \int_B g(\cdot)d_{f_n}\beta(A,\cdot), \tag{2'}$$

where for all $C \in \mathscr{M}_2$,

$$\lim_{n\to+\infty} \int_A f_n(\cdot)d\beta(\cdot,C) = \int_A f(\cdot)d\beta(\cdot,C) = {}_f\beta(A,C). \tag{3}$$

Furthermore, $\|{}_{f_n}\beta(A,\cdot)\|(B) \leq M\ \|\beta\|(A,B)$, where $\|\cdot\|(\cdot,\cdot)$ is the *Fréchet* (or *semi–variation* of the corresponding vector (bi–)measure (see [4,IV.10], [13]) and $\|{}_f\beta(A,\cdot)\|(B) \leq M\ \|\beta\|(A,B)$. But,

$$\left|\left|\int_B g(\cdot)d_f\beta(A,\cdot) - \int_B g(\cdot)d_{f_n}\beta(A,\cdot)\right|\right| \leq \left|\left|\int_B g(\cdot)d_f\beta(A,\cdot) - \int_B g_m(\cdot)d_f\beta(A,\cdot)\right|\right|$$

$$+ \left|\left|\int_B g_m(\cdot)d_f\beta(A,\cdot) - \int_B g_m(\cdot)d_{f_n}\beta(A,\cdot)\right|\right|$$

$$+ \left|\left|\int_B g_m(\cdot)d_{f_n}\beta(A,\cdot) - \int_B g(\cdot)d_{f_n}\beta(A,\cdot)\right|\right|$$

where $||\cdot||$ is the norm on X. Now, the first and the third term on the right hand side of the above inequality are both bounded by $\sup|g_m - g|\,\|\beta\|(A,B)$. By (3) the middle term converges to zero as $n \to +\infty$. Hence, since $\{g_m\}$ converges to g uniformly, the left hand side converges to zero, and using (2'), equality in (2) remains valid.

Next, let f be bounded and let $B_n = B \cap \{n \le |g| < n+1\}$. Then,

$$\int_B g(\cdot)d_f\beta(A,\cdot) = \sum_{n=0}^{\infty} \int_{B_n} g(\cdot)d_f\beta(A,\cdot) \quad \text{(Lemma 1)}$$

$$= \sum_{n=0}^{\infty} \int_A f(\cdot)d\beta_g(\cdot,B_n) \quad \text{(g is bounded on } B_n)$$

$$= \int_A f(\cdot)d\beta_g(\cdot,B) \quad \text{(Lemma 2).}$$

If f is not bounded, then with $A_n = A \cap \{n \le |f| < n+1\}$, we have

$$\int_A f(\cdot)d\beta_g(\cdot,B) = \sum_{n=0}^{\infty} \int_{A_n} f(\cdot)d\beta_g(\cdot,B) \quad \text{(Lemma 1)}$$

$$= \sum_{n=0}^{\infty} \int_B g(\cdot)d_f\beta(A_n,\cdot) \quad \text{(f is bounded on } A_n)$$

$$= \int_B g(\cdot)d_f\beta(A,\cdot) \quad \text{(Lemma 2),}$$

and the result is obtained. ∎

As given by Definition 3 and Theorem 4, the vector bimeasure integral shares familiar properties, such as bilinearity in (f,g), dominated convergence, etc. It is also absolute, namely, for measurable functions f and g, the β-integrability of (f,g) and of $(|f|,|g|)$ are equivalent. These, and further properties can be easily verified by using the techniques and results of [13] and [2].

III. Applications to Stochastic Analysis

We now provide some simple applications of the results of the previous section which are of particular interest in "stochastic integration".

Let $(\Omega,\mathscr{B},P)$ be a probability space and let $Y\colon \mathbb{R} \to L^p(\Omega,\mathscr{B},P)$, $Z\colon \mathbb{R} \to L^q(\Omega,\mathscr{B},P)$, $1 \le p < +\infty$, $1/p + 1/q = 1$, be two continuous and (weakly) harmonizable processes. Equivalently, (see [4,VI.7], [10] or [2]), let $Y_t = \int_{\mathbb{R}} e^{ity} d\mu(y)$ and $Z_t = \int_{\mathbb{R}} e^{itz} d\nu(z)$, $t \in \mathbb{R}$, for two vector (random) measures $\mu\colon \mathscr{B}(\mathbb{R}) \to L^p(\Omega,\mathscr{B},P)$ and $\nu\colon \mathscr{B}(\mathbb{R}) \to L^q(\Omega,\mathscr{B},P)$, ($\mathscr{B}(\mathbb{R})$ is the Borel σ-algebra of $\mathbb{R}$). Then, if $\mathscr{E}$ denotes expectation, it readily follows from Hölder's inequality that $\beta(A,B) = \mathscr{E}\{\mu(A)\overline{\nu(B)}\}$, is a bimeasure on $\mathscr{B}(\mathbb{R}) \times \mathscr{B}(\mathbb{R})$. We show that for such bimeasures, i.e., "induced" by random measures, the β-integrability of pairs of functions f and $g\colon \mathbb{R} \to \mathbb{C}$, follows from the μ-integrability of f and the ν-integrability of g.

<u>Theorem 5</u>. If f is μ-integrable and g is ν-integrable, then the pair (f,g) is β-integrable, and for all $A, B \in \mathscr{B}(\mathbb{R})$,

$$\mathscr{E}\{\int_A f d\mu \overline{\int_B g d\nu}\} = \int_A \int_B f\bar{g}\, d\beta. \tag{4}$$

<u>Proof</u>. Let M be a μ-null set, i.e., M is a subset of a Borel set N such that $\|\mu\|(N) = 0$ where $\|\mu\|(\cdot)$ denotes the *Fréchet* (or *semi-*) *variation* of μ (see [4,IV.10], [13]). Then, $\beta(\cdot,\cdot) = \mathscr{E}\{\mu(\cdot)\overline{\nu(\cdot)}\}$ and from Hölder's inequality,

$$\begin{aligned}\|\beta(\cdot,B)\|(N) &= \sup\{|\Sigma_{i=1}^N a_i\, \beta(\cdot,B)(N_i)|;\ \{N_i\}_{i=1}^N \text{ Borel partition of } N,\ a_i \in \mathbb{C},\ |a_i| \le 1\}\\ &\le \sup\{\|\Sigma_{i=1}^N a_i\, \mu(N_i)\|_{L^p(P)}\}\, \|\nu(B)\|_{L^q(P)}\\ &= \|\mu\|(N)\|\nu(B)\|_{L^q(P)}\\ &= 0,\end{aligned}$$

and M is a $\beta(\cdot,B)$-null set for all $B \in \mathscr{B}(\mathbb{R})$. In fact, since $\|\nu(B)\|_{L^q(P)} \le \|\nu\|(B) \le \|\nu\|(\mathbb{R})$, M is a $\beta(\cdot,B)$ null set uniformly in $B \in \mathscr{B}(\mathbb{R})$.

Now, let f be μ–integrable. Then there exists a sequence of simple functions $f_n \to f$ μ–a.e., hence $\beta(\cdot,B)$–a.e. for all B, and such that for all $A \in \mathscr{B}(\mathbb{R})$, $\{\int_A f_n d\mu\}$ is a Cauchy sequence in $L^p(P)$. To prove that for all $B \in \mathscr{B}(\mathbb{R})$, f is $\beta(\cdot,B)$-integrable, it is thus enough to show that for all A, $B \in \mathscr{B}(\mathbb{R})$, $\{\int_A f_n(\cdot)d\beta(\cdot,B)\}$ is a Cauchy sequence in $\mathbb{C}$. But since the f_n's are simple functions, a direct application of Hölder's inequality gives

$$\left|\int_A (f_n - f_m)(\cdot)d\beta(\cdot,B)\right| = \left|\mathscr{E}\{\int_A (f_n - f_m)d\mu\,\overline{\nu(B)}\}\right|$$

$$\leq \left\|\int_A (f_n - f_m)d\mu\right\|_{L^p(P)} \|\nu(B)\|.$$

Hence, f is $\beta(\cdot,B)$–integrable for all $B \in \mathscr{B}(\mathbb{R})$, and it follows that

$$\int_A f(\cdot)d\beta(\cdot,B) = \mathscr{E}\{\int_A fd\mu\,\overline{\nu(B)}\}. \tag{5}$$

Similarly, a ν–null set is a $\beta(A,\cdot)$–null set for all $A \in \mathscr{B}(\mathbb{R})$, and if g is ν–integrable, it is also $\beta(A,\cdot)$–integrable for all $A \in \mathscr{B}(\mathbb{R})$, and for all A, $B \in \mathscr{B}(\mathbb{R})$,

$$\int_B g(\cdot)d\beta(A,\cdot) = \mathscr{E}\{\mu(A)\int_B gd\overline{\nu}\}. \tag{6}$$

Hence, for μ–integrable f and ν–integrable g, (i) in Definition 3 is satisfied. Next, we show that (ii) is also satisfied.

First, note that using Hölder's inequality, it easily follows from (6) (resp. (5)) that a μ–null set (resp. a ν–null set) is also a $\beta_g(\cdot,B)$–null set for all $B \in \mathscr{B}(\mathbb{R})$ (resp. a ${}_f\beta(A,\cdot)$–null set for all $A \in \mathscr{B}(\mathbb{R})$). Then, using (6) and since the f_n are simple functions, it follows that $\int_A (f_n - f_m)(\cdot)d\beta_g(\cdot,B) = \mathscr{E}\{\int_A f_n - f_m)d\mu \int_B gd\overline{\nu}\}$. Again, by Hölder's inequality, the function f is $\beta_g(\cdot,B)$–integrable for all $B \in \mathscr{B}(\mathbb{R})$, and $\int_A f(\cdot)d\beta_g(\cdot,B) =$

$\mathcal{E}\{\int_A f d\mu \int_B g d\bar{\nu}\}$, for all A, B $\in \mathcal{B}(\mathbb{R})$. Similarly, g is ${}_f\beta(A,\cdot)$ integrable for all A $\in \mathcal{B}(\mathbb{R})$, and $\int_B g(\cdot) d_f\beta(A,\cdot) = \mathcal{E}\{\int_A f d\mu \int_B g d\bar{\nu}\}$, for all A, B $\in \mathcal{B}(\mathbb{R})$. Hence (f,g) is β–integrable, and so is $(f,\bar{g})$ and (4) also follows. ∎

Let $\mathcal{X}_Y$ denote the closure in $L^p(\Omega,\mathcal{B},P)$ of the linear span of the process Y, i.e., $\mathcal{X}_Y = \overline{sp}\{ Y_t : t \in \mathbb{R} \}$, and let $\mathcal{X}_Z$ be defined similarly. Let $\mathcal{X}_\mu = \overline{sp}\{ \mu(A) : A \in \mathcal{B}(\mathbb{R}) \}$. Then, since the exponentials are dense in $\mathcal{X}_\mu$, it follows that $\mathcal{X}_\mu = \mathcal{X}_Y$ (this can be obtained by simple modifications of the arguments in [12,p.40]), and similarly, $\mathcal{X}_Z = \mathcal{X}_\nu$. Hence for any $U \in \mathcal{X}_Y$, $V \in \mathcal{X}_Z$, we have $U = \int_{\mathbb{R}} f d\mu$, $V = \int_{\mathbb{R}} g d\nu$ for some μ–integrable f and some ν–integrable g. It therefore follows from Theorem 5 that $\mathcal{E}\{U\bar{V}\} = \int_{\mathbb{R}}\int_{\mathbb{R}} f\bar{g} d\beta$. In particular, we have $R_{YZ}(s,t) = \mathcal{E}\{Y_s \bar{Z}_t\} = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{isy} e^{-itz} d\beta(y,z)$. Hence, the two processes Y and Z are *jointly stationary*, i.e., $R_{YZ}(s,t) = R_{YZ}(s-t)$, if and only if $\beta(A,B) = 0$ whenever $A \cap B = \phi$, in which case $\beta(\cdot,\cdot)$ uniquely determines a complex measure supported on the diagonal of $\mathbb{R} \times \mathbb{R}$.

It is readily verified (for example, by taking stationary processes X and Y such that X = Y) that, in general, the converse of Theorem 5 does not hold. However, when p = q = 2 and $\mu = \nu$, Theorem 5 admits a partial converse. This recovers some results obtained, with a different bimeasure integral, by Niemi [10,p.29] and by Chang and Rao [3,p.42]; and also extends the classical stationary case.

Corollary 6. Let p = q = 2 and $\mu = \nu$. Then a pair $(f,\bar{f})$ is β–integrable if and only if f is μ–integrable.

Proof. Theorem 5 gives the "if" part. For the "only if" part, we first show that a $\beta(\cdot,B)$–null set for all B $\in \mathcal{B}(\mathbb{R})$, is also a μ–null set. For N $\in \mathcal{B}(\mathbb{R})$, $\|\beta\|(N,B) = \sup\{|\Sigma_i \Sigma_j a_i \bar{b}_j \beta(N_i,B_j)|\} \le \sup\{\Sigma_j |b_j| |\Sigma_i a_i \beta(N_i,B_j)|\} \le \sup\{\Sigma_j |b_j| \|\beta(\cdot,B_j)\|(N)\} = 0$, when N is a $\beta(\cdot,B)$–null set for all B $\in \mathcal{B}(\mathbb{R})$. Hence, taking B = N, we get $\|\mu\|^2(N) =$

$\|\beta\|(N,N) = 0$.

Now, let $(f,\bar{f})$ be β–integrable. Then there exists a sequence $\{f_n\}$ of Borel simple functions such that $\beta(\cdot,B)$–a.e., $|f_n| \le |f|$ and $\lim_{n\to+\infty} f_n = f$. Next, since $(f,\bar{f})$ is β–integrable, by Theorem 4, and by the dominated convergence theorem for (vector) measures, we have for all $B \in \mathscr{B}(\mathbb{R})$,

$$\int_B\int_B f\bar{f}d\beta = \lim_{n\to+\infty}\int_B f_n(\cdot)d\beta_{\bar{f}}(\cdot,B) = \lim_{n\to+\infty}\int_B \bar{f}(\cdot)d_{f_n}\beta(B,\cdot)$$

$$= \lim_{n\to+\infty}\ \lim_{m\to+\infty}\int_B \bar{f}_m(\cdot)d_{f_n}\beta(B,\cdot) = \lim_{n\to+\infty}\ \lim_{m\to+\infty}\int_B\int_B f_n\bar{f}_m d\beta.$$

Similarily, $\int_B\int_B f\bar{f}d\beta = \lim_{m\to+\infty}\ \lim_{n\to+\infty}\int_B\int_B f_n\bar{f}_m d\beta$. Finally, since a $\beta(\cdot,B)$–null set for all $B \in \mathscr{B}(\mathbb{R})$ is also a μ–null set, we get

$$\lim_{n,m\to+\infty} \mathscr{E}\left|\int_B (f_n-f_m)d\mu\right|^2 = 0$$

and f is μ–integrable. ∎

In view of Theorem 4 as well as the above results, Definition 3 appears to provide (at least in a stochastic framework) the appropriate conditions for bimeasure integration. In fact, these results can be extended to matrix bimeasures, as shown in Houdré [5]. The reader is also referred to Kluvánek [6] for illuminating remarks on bimeasures.

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