An Existence and Uniqueness Theorem for Roulettes

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Abstract: For a pair of plane curves β and γ, we give a sufficient and necessary condition for the existence of a unique plane curve α that rolls on β, while a reference point P traces γ. This study was motivated by rolling curve solutions to a few classical problems of the calculus of variations.

1. Introduction

A *roulette* γ is traced out by a reference point P of a plane curve α, that rolls without slipping on a second co-planar curve β. We will only consider rolling while the unit tangent vectors of α and β agree at a unique point of contact.

2. An Equation for γ

Let $\alpha : U \subseteq \mathbb{R} \to \mathbb{R}^2; t \mapsto (\alpha_1(t), \alpha_2(t))$, $\beta : V \subseteq \mathbb{R} \to \mathbb{R}^2; u \mapsto (\beta_1(u), \beta_2(u))$, $r = P - \alpha$, $P = (a, b)$, and $\varphi = \cos^{-1}(r \cdot \alpha'/|r||\alpha'|)$. We will assume that α and β are differentiable and non-singular ($\alpha'(t), \beta'(u) \neq 0$) on U and V, respectively. The normals to α and β are given by $\pi/2$, counterclockwise rotations of each curves unit tangent.

Now, imagine placing α on β so that $\alpha(t_0)$ is in contact with $\beta(u_0)$ and both curves are tangent at the point of contact. Then envision rolling α on β for a length of arc s. If ϑ is the angle between $r(t)$ and $e_1 = (1, 0)$, and if $(\beta_1(u_c), \beta_2(u_c))$ is the new point of contact between α and β, the coordinates for P are

$$\gamma(t) = (\beta_1(u_c) + |r(t)| \cos(\vartheta), \beta_2(u_c) + |r(t)| \sin(\vartheta)). \tag{1}$$

The new point of contact can be obtained by solving

$$s = \int_{t_o}^{t} |\alpha'(\tau)| d\tau = \int_{u_o}^{u_c} |\beta'(\zeta)| d\zeta \tag{2}$$

for u_c. Observe that $u_c : U \to V$ is a monotone function of t with derivative $u_c'(t) = |\alpha'(t)|/|\beta'(u_c(t))|$. We'll write $\beta_c = \beta \circ u_c$.

It is not difficult to see that $\vartheta = \varphi + \xi$, where $(\beta'(u_c(t)) \cdot e_1, \beta'(u_c(t)) \cdot e_2)/|\beta'(u_c(t))| = (\cos(\xi), \sin(\xi))$. Hence,

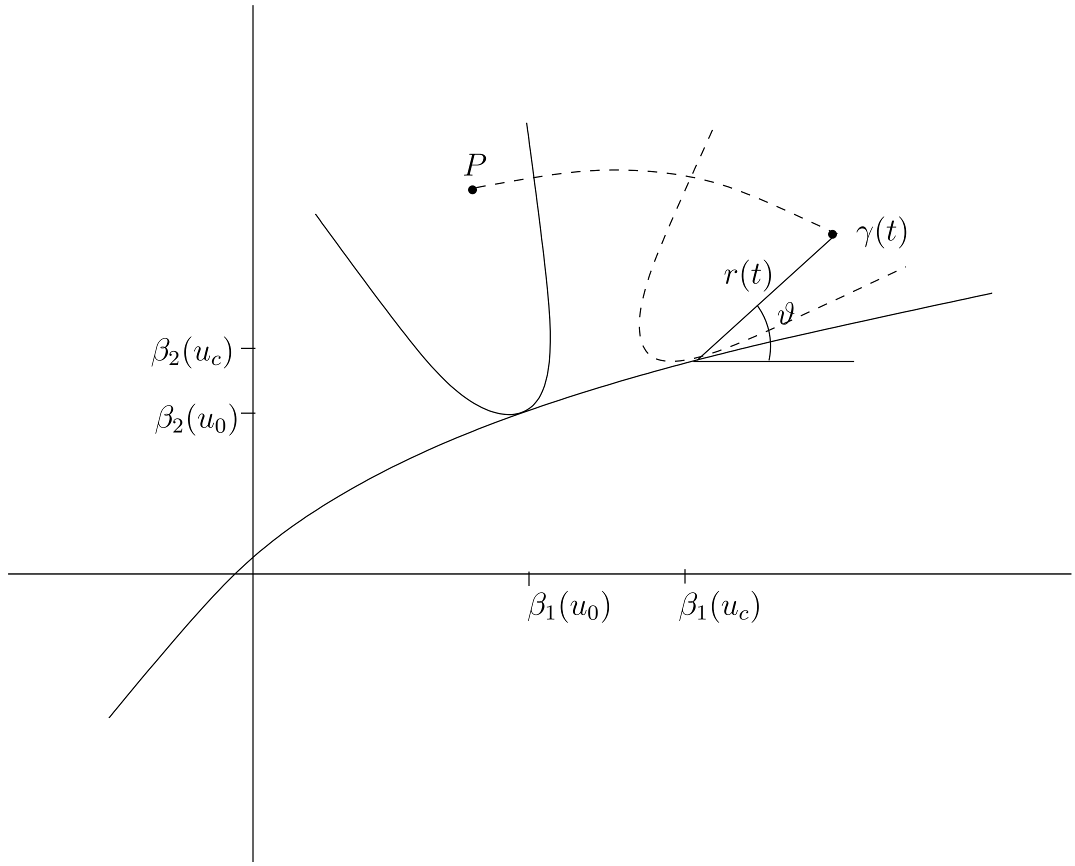

Figure 1: Rolling α on β; the path that P traces is γ.

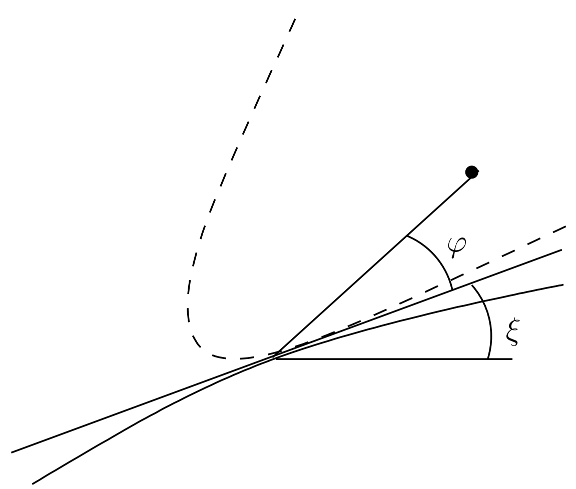

Figure 2: $\vartheta = \varphi + \xi$

$$\gamma(t) = \beta_c(t) + \begin{pmatrix} |r(t)|\cos(\varphi+\xi) \\ |r(t)|\sin(\varphi+\xi) \end{pmatrix} = \beta_c(t) + \begin{pmatrix} \cos(\xi) & -\sin(\xi) \\ \sin(\xi) & \cos(\xi) \end{pmatrix} \begin{pmatrix} |r(t)|\cos(\varphi) \\ |r(t)|\sin(\varphi) \end{pmatrix}. \quad (3)$$

Using the definition of φ, we have

$$\begin{pmatrix} |r(t)|\cos(\varphi) \\ |r(t)|\sin(\varphi) \end{pmatrix} = \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} a-\alpha_1(t) \\ b-\alpha_2(t) \end{pmatrix},$$

with $(\alpha'(t)\cdot e_1, \alpha'(t)\cdot e_2)/|\alpha'(t)| = (\cos(\phi), \sin(\phi))$. By defining

$$Q = \begin{pmatrix} \cos(\xi) & -\sin(\xi) \\ \sin(\xi) & \cos(\xi) \end{pmatrix} \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix} = \begin{pmatrix} \cos(\xi-\phi) & -\sin(\xi-\phi) \\ \sin(\xi-\phi) & \cos(\xi-\phi) \end{pmatrix},$$

(3) becomes

$$\gamma = Q(P-\alpha) + \beta_c \tag{4}$$

This leads us to our first theorem.

Theorem 2.1 *Let P, α and β be defined as above, and suppose γ is traced out by P as α rolls on β. Then the coordinates of γ are given by a translation by β_c of the radial vector $r = P-\alpha$ that has been rotated by the difference of inclination angles of α and β_c.*

Example 1
We'll show that if $P=(a,b)$ is a point inside the circumference of a circle α of radius R, then P traces out an ellipse, as α rolls in another circle β of radius $2R$. This result is due to Besant [1].

Let

$$\alpha(s) = R(\cos(s/R), \sin(s/R)), \quad s\in[0,2\pi R),$$

and

$$\beta(u) = 2R(\cos(u/2R), \sin(u/2R)), \quad u\in[0,4\pi R).$$

$s = \int_0^{u_c} |\beta'(\zeta)|d\zeta = u_c$; $\alpha'(s)/|\alpha'(s)| = (\cos(s/R+\pi/2), \sin(s/R+\pi/2))$ and $\beta'(s_c)/|\beta'(s_c)| = (\cos(s/2R+\pi/2), \sin(s/2R+\pi/2)$, so $\xi-\phi = -s/2R$. It follows that

$$\gamma(s) = Q(P-\alpha(s)) + \beta_c(s) = \begin{pmatrix} (a+R)\cos(s/2R) + b\sin(s/2R) \\ b\cos(s/2R) + (-a+R)\sin(s/2R) \end{pmatrix}. \tag{5}$$

Without loss of generality we can suppose that $a,b>0$. Doing so, and substituting

$$(\cos\tau(s), \sin\tau(s)) = \left(\frac{|\gamma(s)|}{\tilde{a}}\cos(\eta(s)-\theta), \frac{|\gamma(s)|}{\tilde{b}}\sin(\eta(s)-\theta)\right)$$

in

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \tilde{a}\cos\tau(s) \\ \tilde{b}\sin\tau(s) \end{pmatrix}$$

gives (5), where $\eta(s)$ satisfies $\gamma(s) = |\gamma(s)|(\cos\eta(s), \sin\eta(s))$, $\theta = \tan^{-1}\left(\sqrt{\frac{|P|-a}{|P|+a}}\right)$, $\tilde{a} = R+|P|$ and $\tilde{b} = R-|P|$. $\tau'(s) = (R^2-|P|^2)/2R|\gamma(s)|^2 > 0$, so τ is *monotone* on $[0,4\pi R)$. Consequently, the trace of γ is an ellipse centered at the origin.

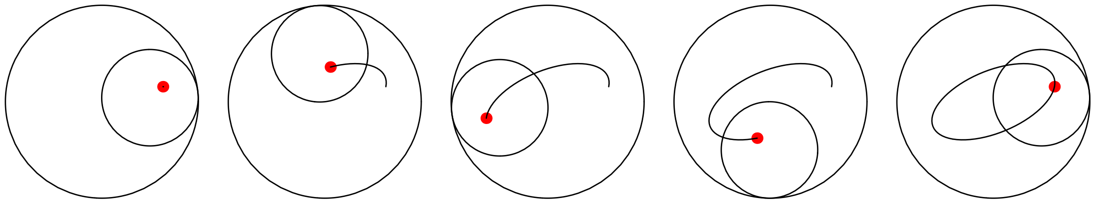

Figure 3: A circle of radius R rolling in a circle of radius $2R$ with $|P| < R$.

Example 2
In this example, we'll roll a logarithmic spiral on a straight line and see what curve the origin ($P = (0,0)$) traces out. Let

$$\alpha(\theta) = e^{\theta}(\cos(\theta), \sin(\theta)), \quad \beta(t) = (0, t) \quad \theta, t \in \mathbb{R}.$$

$s = \int_{-\infty}^{\theta_c} |\alpha'(\phi)|d\phi = \sqrt{2}e^{\theta_c}$; $\alpha'(\theta)/|\alpha'(\theta)| = (\cos(\theta+\pi/4), \sin(\theta+\pi/4))$ and $\beta'(\theta_c)/|\beta'(\theta_c)| = (1,0)$, so $\phi = \theta + \pi/4$ and $\xi = 0$. A straightforward calculation shows

$$\gamma(\theta) = Q(P - \alpha(\theta)) + \beta_c(\theta) = \frac{e^{\theta}}{\sqrt{2}}(1,1).$$

That is, γ is *linear*. Therefore, we have just shown that the origin traces out the line $y = x$ as the logarithmic spiral rolls on $y = 0$.

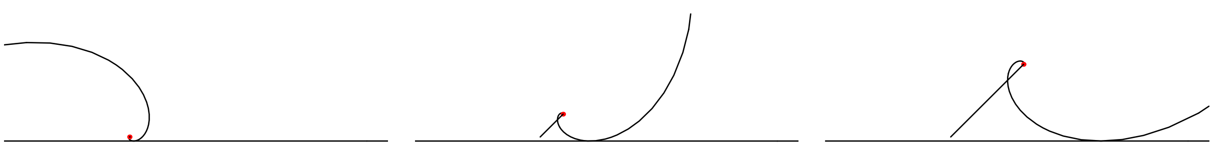

Figure 4: A logarithmic rolls on $y = 0$, as the origin traces $y = x$.

3. The Inverse Problem

In this section, we'll consider the possibility of recovering α and P, for a given pair β and γ. The following lemma states a fundamental property of roulettes.

Lemma 3.1 *Let α and β be differentiable, non-singular plane curves. Suppose that α, with reference point P, rolls on β to trace γ. Then the radial vector from the point of contact β_c to the roulette γ is always in the direction normal to the roulette.*

PROOF: It suffices to show that $\gamma'(t) \cdot (\gamma(t) - \beta_c(t)) = 0$.

$$\gamma'(t) = Q'(P - \alpha(t)) - Q\alpha'(t) + \beta_c'(t).$$

By equation (4), $\gamma(t) - \beta_c(t) = Q(P - \alpha(t))$, and since the dot product is invariant under rotations

$$-Q(P - \alpha(t)) \cdot Q\alpha'(t) = -(P - \alpha(t)) \cdot \alpha'(t) = -|P - \alpha(t)||\alpha'(t)|cos(\varphi).$$

$$Q' = (\phi' - \xi') \left(\begin{array}{cc} \cos(\xi - \phi - \pi/2) & -\sin(\xi - \phi - \pi/2) \\ \sin(\xi - \phi - \pi/2) & \cos(\xi - \phi - \pi/2) \end{array} \right),$$

so

$$Q(P - \alpha(t)) \cdot Q'(P - \alpha(t)) = 0.$$

The tangents to $\alpha(t)$ and $\beta_c(t)$ coincide at each point of contact; thus,

$$(\gamma(t) - \beta_c(t)) \cdot \beta'(u_c(t)) = |P - \alpha(t)||\alpha'(t)|cos(\varphi),$$

which completes the proof. □

For any two differentiable, non-singular plane curves β and γ, the following theorem gives a sufficient and necessary condition for the existence of a plane curve α with reference point P such that α rolls on β, while P traces γ. Furthermore α is unique, up to a Euclidean motion (a shift and a rotation). This theorem extends the result in [2], which was restricted to the case where β is a line and γ is periodic with respect to β.

Theorem 3.2 *Let $\gamma : U \subseteq \mathbb{R} \to \mathbb{R}^2$ and $\beta : V \subseteq \mathbb{R} \to \mathbb{R}^2$ be differentiable and non-singular. There exists a differentiable, non-singular plane curve α, with reference point $P \in \mathbb{R}^2$, such that P traces out γ, as α rolls on β if, and only if, there is a bijection $\Lambda : \gamma(U) \to \beta(V); \gamma(t) \mapsto \beta(u)$, where u is the unique number in V that satisfies*

$$(\gamma(t) - \beta(u)) \cdot \gamma'(t) = 0. \tag{6}$$

α is unique up to a Euclidean motion.

PROOF: (Existence) $\Rightarrow$ Suppose there exists a plane curve α, with reference point P that traces out γ, as α rolls on β. Lemma 3.1 asserts that $\gamma(t) - \beta_c(t)$ is always along the direction normal to $\gamma'(t)$, so there is a unique solution in V to (6) for all $t \in U$. We take $\Lambda(\gamma(t)) = \beta_c(t) = (\beta \circ u_c)(t)$. Λ is a composition of monotone functions and so is bijective.

$\Leftarrow$ Conversely, suppose that there is a bijection Λ as described in the statement. We define $\tilde{u} : U \to V$ as the solution to (6). The monotonicity of $\tilde{u}$ follows from the monotonicity of of Λ. We'll write $\tilde{\beta} = \beta \circ \tilde{u}$.

Let

$$\alpha(t) = \rho(t) \left(\cos\theta(t), \sin\theta(t)\right), \quad t \in U; \tag{7}$$

where $\rho(t) = |\gamma(t) - \tilde{\beta}(t)|$ and

$$\theta(t) = \theta(t_0) + \int_{t_0}^{t} \frac{\sin(-\psi(\tau))}{\rho(\tau)} |\tilde{\beta}'(\tau)| \, d\tau;$$

$(\cos\psi(t), \sin\psi(t)) = \left((\tilde{\beta}(t) - \gamma(t)) \cdot \beta'(\tilde{u}(t)), (\tilde{\beta}(t) - \gamma(t)) \cdot \mathcal{J}\beta'(\tilde{u}(t)) \right) / \rho|\beta'(\tilde{u}(t))|$ with $\mathcal{J}(x, y) = (-y, x)$.

$|\alpha'(t)| = \sqrt{\rho'(t)^2 + (\rho\theta'(t))^2} = |\tilde{\beta}'(t)| = |\beta'(\tilde{u}(t))||\tilde{u}'(t)| \neq 0$, so α is differentiable and non-singular. If $\tilde{u}'(t) = |\alpha(t)|/|\beta(\tilde{u}(t))|$, then $\tilde{u} = u_c$. If $\tilde{u}'(t) = -|\alpha(t)|/|\beta(\tilde{u}(t))|$ we simply reparameterize $\beta(V)$ so that it's traced out in the opposite direction to get $\tilde{u} = u_c$. Now we'll use equation (4) to determine the roulette obtained from rolling α on β.

Letting $(\alpha'(t) \cdot e_1, \alpha'(t) \cdot e_2)/|\alpha'(t)| = (\cos(\phi), \sin(\phi))$ and
$(\beta'(\tilde{u}(t)) \cdot e_1, \beta'(\tilde{u}(t)) \cdot e_2)/|\beta'(\tilde{u}(t))| = (\cos(\xi), \sin(\xi))$, we have

$$\cos(\theta(t) - \phi + \xi) = \frac{(\tilde{\beta}(t) - \gamma(t)) \cdot e_1}{\rho} \quad \text{and} \quad \sin(\theta(t) - \phi + \xi) = \frac{(\tilde{\beta}(t) - \gamma(t)) \cdot e_2}{\rho}. \tag{8}$$

Hence,

$$\begin{pmatrix} \cos(\xi - \phi) & -\sin(\xi - \phi) \\ \sin(\xi - \phi) & \cos(\xi - \phi) \end{pmatrix} \begin{pmatrix} 0 - \rho(t)\cos\theta(t) \\ 0 - \rho(t)\sin\theta(t) \end{pmatrix} + \tilde{\beta}(t) = \gamma(t),$$

and, thus, α rolls on $\tilde{\beta}$, while $P = (0, 0)$ traces γ.

(Uniqueness) Let α be defined as in (7), and suppose there exists another curve χ that rolls on β while the origin traces γ. We parameterize χ as follows

$$\chi(t) = \eta(t) (\cos\vartheta(t), \sin\vartheta(t)), \quad t \in U.$$

The existence of χ requires that there is a unique line segment between $\gamma(t)$ and $\beta(u)$ along the direction normal to $\gamma'(t)$ for each $\gamma(t) \in \gamma(U)$ with corresponding $\beta(u) \in \beta(V)$. Therefore, $\eta(t) = |\gamma(t) - \beta(u)|$. By defining ϱ and ζ implicitly as $((\beta(u) - \gamma(t)) \cdot e_1$, $(\beta(u) - \gamma(t)) \cdot e_2) = \rho(\cos(\varrho), \sin(\varrho))$ and $(\chi'(t) \cdot e_1, \chi'(t) \cdot e_2)/|\chi'(t)| = (\cos(\zeta), \sin(\zeta))$, from (8) we have

$$\varrho = \theta(t) - \phi + \xi + j\pi = \vartheta(t) - \zeta + \xi + k\pi,$$

for some $j, k \in 2\mathbb{Z}$. Since,

$$\tan(\theta(t) - \phi + j\pi) = \tan(\theta(t) - \phi) = -\frac{\rho(t)\theta'(t)}{\rho'(t)},$$

$$\tan(\vartheta(t) - \zeta + k\pi) = \tan(\vartheta(t) - \zeta) = -\frac{\eta(t)\vartheta'(t)}{\eta'(t)},$$

and $\rho(t) = \eta(t)$, $\theta'(t) = \vartheta'(t)$. Therefore, α and χ differ by a rotation. Furthermore, if we choose *any* $\tilde{P} \in \mathbb{R}^2$, and if we let $\alpha = \tilde{\alpha} - P$, $\tilde{P}$ traces γ, while $\tilde{\alpha}$ rolls on $\tilde{\beta}$. Thus, α is unique up to a Euclidean motion. □

References

[1] W. H. Besant, *Notes on Roulettes and Glissettes,* Deighton, Bell, and Co. Cambridge, 1890.
[2] J. Bloom & L. Whitt, *The Geometry of Rolling Curves*, American Mathematical Monthly, Vol. 88 #6 (1981) 420 - 426.