Quiz 2 for Math 2605A1-2, Fall 2004

Name:

This quiz is to be taken without notes of any sorts. The allowed time is 20 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$.

Consider the function

$$f(x,y) = 3 - x^2 - \frac{y^2}{4} \ .$$

Think of the graph of f as a a mountain, so that $z = f(x,y)$ is your altitude at the point (x,y).

I: (4 points) Find an equation for the tangent plane to the graph of $z = f(x,y)$ at at the point $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The gradient is

$$\nabla f = \begin{bmatrix} -2x \\ -y/2 \end{bmatrix}$$

The normal vector at the point $(1,2)$ is given by

$$\mathbf{N} = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}$$

and since $f(1,2) = 1$, the plane by the equation

$$-2(x-1) - (y-2) - (z-1) = 0 \ .$$

II: (3 points) Find the tangent line to the level curve of f that passes through the point $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Give the line in parametric form.

The point $(1,2)$ is on the level curve. The gradient at this point is given by

$$\begin{bmatrix} -2 \\ -1 \end{bmatrix} ,$$

and hence the vector

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

is a direction vector. Thus, the line is given by

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \end{bmatrix} .$$

III: (3 points) You stand at the point $\mathbf{x}_0$, and walk due East with a speed of 2 meters per second, so that your velocity vector is $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$. What is the rate of change of your altitude?

The rate of change is given by

$$\frac{d}{dt} f(\mathbf{x_0} + t\mathbf{v})|_{t=0} = \mathbf{v} \cdot \nabla f(\mathbf{x_0}) \ ,$$

where $\mathbf{x_0} = (1, 2)$ and $\mathbf{v} = (2, 0)$. Hence the rate of change is -4.

Extra credit: The tangent that you computed in problem I intersects the plane $z = 1$ in a line. Give the line in parametric form.

We have to intersect the plane

$$2(x-1) + (y-2) + (z-1) = 0$$

with the plane $z = 1$. The resulting line is given by the equation

$$2(x-1) + (y-2) = 0$$

and hence in parametric form by

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \end{bmatrix} .$$