Quiz 3 for Calculus ++, Math 2605A1-2, October 12, 2004

Name:

This quiz is to be taken without calculators and notes of any sorts. The allowed time is 20 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$.

I: Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 6 & 1 \\ 3 & 1 & 2 \end{bmatrix} .$$

a) (3 points) Using the largest off-diagonal elements for the first step in the Jacobi Algorithm write down the Givens rotation G and calculate the matrix $A^{(1)} = G^T AG$.

The 2×2 submatrix is

$$\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} .$$

The eigenvalues are $5, -1$ and the corresponding eigenvectors are

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} .$$

Thus, the Givens rotation is

$$G = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Finally

$$G^T AG = \begin{bmatrix} 5 & \sqrt{2} & 0 \\ \sqrt{2} & 6 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

b) (1 point) Find an exact eigenvalue of the matrix A.

-1 is obviously and exact eigenvalues of A because it is an eigenvalue of $G^T AG$.

II: (3 points) Recall that $\mathrm{Off}(A)$ denotes the sum of the squares of the off-diagonal elements of A and $A^{(k)}$ is the matrix after the k-th step in the Jacobi algorithm. Which of the statments are true and which are false:

a) $\mathrm{Off}(A^{(12)}) \geq \mathrm{Off}(A^{(13)})$ FALSE it has to decrease.

b) $\mathrm{Off}(A^{(k)}) \geq \frac{1}{1+k^2}$ FALSE it tends to zero exponentially fast.

c) $\mathrm{Off}(A^{(13)}) \leq \frac{2}{3}\mathrm{Off}(A^{(12)})$ TRUE.

III: (3 points) Consider the matrix

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} + t \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} .$$

Find all the eigenvalues for small values of t, i.e., in the forms $\mu_i(0)+\mu_i'(0)t+o(t), i = 1, 2, 3$ with explicit values for $\mu_i(0)$ and $\mu_i'(0)$.

The eigenvalues and normalized eigenvectors of the first matrix are

$$4\ ,\ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\ ,\ 2\ ,\ \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\ ,\ 5\ ,\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} .$$

Recall that the formula for the eigenvalues of $A + tB$ for small values of t is

$$\mu(t) = \mu(0) + v \cdot Bvt + O(t^2)$$

we get

$$4 + 4t + O(t^2)\ , 2 + O(t^2)\ ,\ 5 + 5t + O(t^2)\ .$$