Quiz 3 for Calculus ++, Math 2605A1-2, February 19, 2004

Name:

This quiz is to be taken without calculators and notes of any sorts. The allowed time is 25 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$.

I: (5 points) Maximize the function $f(x, y) = x^2 + 4xy + y^2$ subject to the constraint $x^2 + y^2 = 1$.

Using Lagrange multiplier we get

$$\begin{bmatrix} 2x + 4y \\ 2y + 4x \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

which amounts to solving the equations

$$(x + 2y)y = (y + 2x)x \ , x^2 + y^2 - 1 = 0 \ ,$$

or what is the same

$$x^2 = y^2 \ , x^2 + y^2 = 1 \ .$$

The solutions are $x^2 = y^2 = 1/2$ or

$$\frac{1}{\sqrt{2}}(\pm 1, \pm 1)$$

where the $\pm$ s are independent, there are four pairs of solutions.

$$f(\frac{1}{\sqrt{2}}(1, 1)) = f(\frac{1}{\sqrt{2}}(-1, -1)) = 3$$

and

$$f(\frac{1}{\sqrt{2}}(-1, 1)) = f(\frac{1}{\sqrt{2}}(1, -1)) = -1 \ .$$

Thus the maximum is attained at

$$\frac{1}{\sqrt{2}}(1, 1) \ , \ \frac{1}{\sqrt{2}}(-1, -1)$$

and the value of the function there is 3.

II: (5 points) Using the point $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as an initial value, apply one step of Newtons method to calculate an approximate value for the solution of the system

$$xy - 1 = 0 \ , x^2 - y^2 - 2 = 0 \ .$$

$$x_1 = x_0 - J_F(x_0)^{-1}F(x_0)$$

$$J_F(x) = \begin{bmatrix} y & x \\ 2x & -2y \end{bmatrix}$$

$$J_F(x_0) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$$J_F(x_0)^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$$F(x_0) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

and hence

$$x_1 = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

III: (Additional 3 points credit) Calculate the Givens rotation in the first step of the Jacobi algorithm for the matrix

$$\begin{bmatrix} 4 & 1 & 4 \\ 1 & 3 & 2 \\ 4 & 2 & -2 \end{bmatrix} .$$

You only have to calculate the Givens rotation.

The submatrix with the largest off-diagonal elements is

$$\begin{bmatrix} 4 & 4 \\ 4 & -2 \end{bmatrix}$$

The eigenvalues and eigenvectors are

$$6 \; , \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \; , \quad -4 \; , \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

and hence the Givens rotation is

$$\frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 0 & -1 \\ 0 & \sqrt{5} & 0 \\ 1 & 0 & 2 \end{bmatrix}$$