Practice–quiz 4 for Math 2605J1-J2, Fall 2007

Name:

This quiz is to be taken without notes of any sorts. The allowed time is 20 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write 1.414....

I: (3 points) Find the length of the curve

$$\mathbf{x}(t) = \begin{bmatrix} t \\ t^2 \\ \frac{2t^3}{3} \end{bmatrix} \ , \ 0 \le t \le 2 \ .$$

The speed is

$$|d\mathbf{x}(t)/dt| = [1 + 4t^2 + 4t^4]^{1/2} = (1 + 2t^2) \ .$$

Hence the length is

$$\int_0^2 |d\mathbf{x}(t)/dt| dt = \int_0^2 (1 + 2t^2) dt = 2 + \frac{2}{3} 2^3 = \frac{22}{3} \ .$$

II: (3 points) Find the solution of the differential equation

$$x' = x^2 \ , \ x(0) = 1 \ .$$

This equation can be written as

$$\frac{x'}{x^2} = 1$$

or

$$-\frac{d}{dt} \frac{1}{x(t)} = 1$$

which means that

$$-\frac{1}{x(t)} = t + C$$

where C is some constant. Hence

$$x(t) = -\frac{1}{t + C}$$

Since $x(0) = 1$ we get that $C = -1$ and therefore

$$x(t) = \frac{1}{1 - t}$$

Note that the solution ceases to exist at $t = 1$.

III: (4 points) Using Householder reflections, find the QR factorization of the matrix

$$\begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} .$$

The vector $\mathbf{u}$ in the Householder reflection is

$$\mathbf{u} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

and therefore

$$M_1 = I - 2\mathbf{u}\mathbf{u}^T = \frac{1}{3} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ 2 & -2 & -1 \end{bmatrix}$$

Next

$$M_1 A = \begin{bmatrix} 3 & 2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The Housoholder reflection that maps the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and hence

$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$R = M_2 M_1 A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$Q = \frac{1}{3} \begin{bmatrix} 2 & 2 & 1 \\ 1 & -2 & 2 \\ 2 & -1 & -2 \end{bmatrix}$$

IV: (3 points) Draw in a qualitative way the vector field given by

$$\mathbf{F}(\mathbf{x}) = \begin{bmatrix} x \\ -y \end{bmatrix} .$$