Quiz 4 for Calculus ++, Math 2605A1-2, April 1, 2004

Name:

This quiz is to be taken without calculators and notes of any sorts. The allowed time is 25 minutes. Provide exact answers; not decimal approximations! For example, if you mean $\sqrt{2}$ do not write $1.414\ldots$.

I: Consider the matrix $A = \begin{bmatrix} 6 & 8 \\ -8 & 6 \end{bmatrix}$.

a) (2 points) Find the Householder reflection that maps the vector $\begin{bmatrix} 6 \\ -8 \end{bmatrix}$ to a multiple of $\mathbf{e_1}$.

$$\mathbf{u} = \frac{1}{4\sqrt{5}} \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$Q = \frac{1}{5} \begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix}$$

b) (2 points) Find the QR decomposition of the matrix A.

$$R = QA = \begin{bmatrix} 10 & 0 \\ 0 & -10 \end{bmatrix}$$

II: (3 points) Compute e^{tA} where

$$A = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix} .$$

$$e^{At} = \begin{bmatrix} e^{3t} & a(t) \\ 0 & e^{2t} \end{bmatrix}$$

$$e^{At}A = \begin{bmatrix} 3e^{3t} & 4e^{3t} + 3a(t) \\ 0 & 2e^{2t} \end{bmatrix}$$

and

$$Ae^{At} = \begin{bmatrix} 3e^{3t} & 4e^{2t} + 2a(t) \\ 0 & 2e^{2t} \end{bmatrix}$$

These two matrices must be the same and hence

$$a(t) = 4(e^{2t} - e^{3t}) .$$

Thus

$$e^{At} = \begin{bmatrix} e^{3t} & 4(e^{2t} - e^{3t}) \\ 0 & e^{2t} \end{bmatrix}$$

III: (3 points) The Householder reflection that maps the vector $\begin{bmatrix} 1+i \\ 4 \end{bmatrix}$ to a multiple of $\mathbf{e_1}$ is of the form $\mathbf{I} - 2\mathbf{u}\mathbf{u}^*$. Find $\mathbf{u}$. Call

$$\mathbf{w} = \begin{bmatrix} 1+i \\ 4 \end{bmatrix}$$

$$\langle \mathbf{w}, \mathbf{w} \rangle = 18$$

So we pick

$$\mathbf{z} = \begin{bmatrix} \sqrt{18}c \\ 0 \end{bmatrix}$$

where c is a complex number of magnitude 1 chosen in such a way that

$$\langle \mathbf{z}, \mathbf{w} \rangle = \sqrt{18}\overline{c}(1+i)$$

is real. Hence we choose

$$c = \frac{1+i}{\sqrt{2}}$$

and we get

$$\mathbf{z} = \begin{bmatrix} 3(1+i) \\ 0 \end{bmatrix}$$

Hence

$$\mathbf{u} = \frac{1}{2\sqrt{6}} \begin{bmatrix} 2(1+i) \\ -4 \end{bmatrix}$$

Extra credit: (3 points) Find the Schur factorization of the matrix in **Problem I**. The vector

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

is an eigenvector with eigenvalue $6+8i$. Hence

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

and

$$U^* A U = \begin{bmatrix} 6+8i & 0 \\ 0 & 6-8i \end{bmatrix}$$