MATH 3406

1. **Learning Goals**

After taking this course you should have some understanding of advanced topics of linear algebra. Besides the usual techniques like multiplying matrices and solving equations, you should understand the Ranges and Nullspaces of matrices and their transposed as well as their mutual relationships. You are expected to understand in a conceptual way least square approximations, eigenvalues and eigenvectors as well as the singular value decomposition. These topics are not isolated from applications and hence some understanding of those will be expected.

2. Text

'Linear Algebra and its Applications' by Gil Strang, Hardcover: 487 pages, Publisher: Cengage Learning; 4th edition (2006), ISBN-10: 0030105676, ISBN-13: 978-0030105678

3. Topics

To say that linear algebra is important is an understatement. There are a number of obvious reasons. Linear problems are the only ones that can be solved in a systematic way. Even more obvious is the fact that solving equations involve only the fundamental four operations. Linear algebra is crucial in virtual all technological applications. It is used in filters, in statistics and especially in computer science. Manipulation of images is essentially the same as dealing with matrices. Image compression is closely related to the singular value decomposition and even wavelet theory requires a substantial amount of linear algebra. In short, linear algebra is one of the most important devices to formulate problems in mathematics and engineering. Much deeper is the fact that some of the fundamental laws of nature, quantum mechanics, is linear. This very fact has enormous consequences, in fact without quantum mechanics we would not be able to talk about semi conductors, hence not about integrated circuits and ultimately not about computers.

Needless to say, such a wide range of applicability requires a certain conceptual apparatus. You had a first course in linear algebra which gave you some taste of what is involved. In Math 3406 we enter deeper into this wonderful field.

Vectors and Matrices, Subspaces You already know about vectors and matrices and maybe you remember about subspaces which is one of the key concepts for solving linear equations. You may remember linear combinations and linear independence. The beauty about this particular concept is that one can think of it in geometrical terms. For representational purposes one has to introduce the notion of a basis, i.e., a linearly independent set that generates the space. This leads naturally to the notion of dimension.

Solving a linear system of equations Solutions can be achieved in an efficient through factorization of matrices, like the LU factorization where L is lower triangular and U is upper triangular. This allows to solve systems of equation through backward substitution. This will lead us to one of the main results in linear algebra, namely that for any $m \times n$ matrix the sum of the dimension of the column space and the dimension of the kernel equals n.

Orthogonality and least squares Likewise, the notion of orthogonality can be generalized to arbitrary dimensions and allows the formulation of interesting and important problems, like least square fitting. The key to the understanding of matrices is that there are relations between certain subspace, e.g., like the nullspace or kernel of a matrix A and the range or column space of its transposed A^T. An important problem is to find an orthonormal basis for a subspace. This can be achieved either by the Gram-Schmidt method or by the Householder reflection method.

Determinants The analog of volumes in high dimensions can be understood through the notion of determinants. While determinants are *not* amenable to numerical computations they are important in view of their geometrical interpretation.

Eigenvalues It is one of the key ideas in mathematics to change a representation of an object so that it looks simple. For matrices this leads to the eigenvalue problem, i.e., the question whether a matrix can be diagonalized or not. An important subclass of matrices that can always be diagonalized are the symmetric matrices. For non-symmetric matrices that cannot be diagonalized there is a replacement, the Jordan canonical form.

Singular value decomposition This is one of the most important factorization of matrices. This factorization is widely used to analyze data (principal component analysis), in face recognition and compression of images.

Linear transformations This notion will be used throughout the course. It is a handy way of thinking about matrices, they are in one to one correspondence. Moreover, it gives a better understanding of the underlying geometry, in particular about the meaning of basis.

Fast Fourier transform This is one of the key algorithm to compute the Fourier transform numerically. Of course it requirese the understanding of complex numbers.

Application Throughout the course we will look at applications described in Chapter 10 of GS.

4. A few words about proofs and understanding mathematics

For most of the students, the mathematical ideas related to applications of linear algebra are the important aspects of linear algebra and we will talk about those. In order to successfully apply these techniques one has to understand them thoroughly; one needs to know why they work. E.g., in face recognition algorithms one needs the singular value decomposition and one has to understand what that is and why it works. This is what proofs do for you. They should give you an understanding how things work. Thus, you will have to understand certain proofs! I will not be formal, this is not the point and I will skip certain proofs which pertain to topics less useful for applications. But those I present I expect you to understand. Moreover, it is important for everybody to learn some logical reasoning. You will have to know why certain procedures work, e.g., why is every symmetric matrix diagonalizable. These ideas are crucial for understanding and applying these tools.

The best way to understand mathematics is to SEE it. Thus, linear algebra has to go along with geometry. This is the point about introducing various geometric concepts such as sub spaces and orthogonality. Please pay attention to this. It is plain that not all facts of linear algebra can be understood in such a simple way, but wherever possible one should do that. For example, diagonalizing a symmetric matrix is equivalent to rotating a symmetric ellipsoid so that its symmetry axes line up with the coordinate axes.