# HOMEWORK 3 , DUE THURSDAY JANUARY 30

**Problem 1, (5 points):** Please solve problem 2.2.32 in Heil
**Solution:** Write $A = (A \setminus B) \cup (A \cap B)$ and $B = (B \setminus A) \cup (A \cap B)$. Then

$$|A| + |B| = |A \setminus B| + |A \cap B| + |B \setminus A| + 2|A \cap B|$$

and

$$|A \cup B| = |A \setminus B| + |A \cap B| + |B \setminus A| + |A \cap B|$$

**Problem 2, (5 points):** Please do problem 2.2.37 in Heil.
**Solution:** That a) is equivalent to b) follows from the definition of measurability and taking complements. If $E$ is measurable, for any $k = 1, 2, \dots$ there exists $U_k$ open with $E \subset U_k$ and $|U_k \setminus E|_e < 1/k$. Hence $E \subset G$ where $G = \cap_{k=1}^{\infty} U_k$ and $|G_\delta \setminus E|_e < 1/k$ for any $k$ and hence it is zero. Thus, $G = E \cup Z_1$ where $G$ is $G_\delta$ and $|Z_1| = 0$. Similarly, there exists $H$, $F_\sigma$ such that $E = H \cup Z_2$ and $|Z_2| = 0$. Hence $H \subset E \subset G$ and $|G \setminus H| = |Z_1| + |Z_2| = 0$.

**Problem 3, (5 points):** Please do problem 2.2.40 in Heil.
**Solution:** We use Caratheodory's criterion: Since $E$ is measurable, we have for any set $B$

$$|B|_e = |B \cap E|_e + |B \setminus E|_e$$

and if we choose $B = E \cup A$ we have that

$$|E \cup A|_e = |(E \cup A) \cap E|_e + |(E \cup A) \setminus E|_e \ .$$

But $E \cap A = \emptyset$ and therefore

$$(E \cup A) \cap E = E \text{ and } (E \cup A) \setminus E = A \ .$$

**Problem 4, (5 points):** Please do problem 2.2.36 in Heil.
**Solution:**The first statement means that for every $x \in E$ there exists a set $Z_x \subset F$ of zero measure so that if $y \in Z_x$ the statement $P(x, y)$ might be false. Thus, for $y \in L = \cup_x Z_x$ there exists an $x \in E$ such that the statement might be false. The second statement says there is a set $Z$ of zero measure such that for $y \in Z$ the statement $P(x, y)$ might be false for every $x \in E$. Thus, these two statements are not the same.

**Problem 5, (5 points):** Please do problem 2.2.33 in Heil.
**Solution:**Consider the set

$$Z = \cup_{n,m,n \neq m} E_n \cap E_m \ .$$

This set is a countable union of sets of measure zero and hence $|Z| = 0$. Now we consider the sets

$$F_n = E_n \setminus Z$$

and note that $F_n \cap F_m = \emptyset$ if $m \neq n$. By countable additivity

$$\sum_n |E_n| = \sum_n |F_n| = | \cup_{n=1}^{\infty} F_n| = | \cup_{n=1}^{\infty} E_n|$$