HOMEWORK 5, DUE THURSDAY FEBRUARY 13

Problem 1, (5 points): Please do problem 3.1.14 in Heil
Solution: Since f is monoton increasing the set $\{f > a\}$ is of the form $(c, \infty) \cap E$ or of the form $[c, \infty) \cap E$. These sets are measurable.

Problem 2, (5 points): Please do problem 3.1.17 in Heil
Solution: The set

$$\{f = a\} = \{f \geq a\} \cap \{f \leq a\}$$

both of which are measurable.

Now take a non-measurable set $A \subset [0, 1]$ and define $f(x) = x$ and $x \in A$ and $f(x) = x + 1$ for $x \in [0, 1] \setminus A$. The set $\{f \geq 1\} = [0, 1] \setminus A$ and $\{f < 1\} = A$ and f is not measurable. The set $f^{-1}(c)$ is either empty or a single point and these sets are measurable.

Problem 3, (5 points): Please work problem 3.2.9 a) and b)
Solution: The set $\{\phi_n > a\}$ is measurable since it is disjoint union of intervals. Suppose now that f is continuous at the point x_0. This means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon$$

whenever $|x - x_0| < \delta$. For every n there exists k so that $\frac{k}{n} \leq x_0 < \frac{k+1}{n}$. Pick N such $1/N < \delta$. Then for all $n > N$ we have that

$$|\frac{k}{n} - x_0| < \delta$$

for some k and hence

$$|\phi_n(x_0) - f(x_0)| = |\sum_{k \in \mathbb{Z}} [f(\frac{k}{n}) - f(x_0)] \chi_{[\frac{k}{n}, \frac{k+1}{n})}(x_0)| = |f(\frac{k}{n}) - f(x_0)| < \varepsilon$$

which means that $\phi_n(x_0)$ converges to $f(x_0)$. If f is continuous at almost every point, then ϕ_n converges a.e. to f and since the pointwise limit of measurable functions is measurable, f is measurable.

Problem 4, (5 points): Please do problem 3.2.21 a) in Heil
Solution: Consider the sets

$$E_n = \{|f| > n\}$$

where $n = 1, 2, 3 \ldots$. Since $|E| < \infty$, $|E_n| < \infty$. Moreover, $E_1 \supset E_2 \supset E_3 \cdots$. The set

$$\cap_n E_n$$

must be a set of measure zero, because f is everywhere defined and finite. By continuity

$$\lim_{n \to \infty} |E_n| = 0 \ .$$

Pick and $\varepsilon > 0$. There exists N such that $|E_N| < \varepsilon/2$. Moreover, since E_N is measurable, there exists a closed set $F \subset E \setminus E_N$ such that $|(E \setminus E_N) \setminus F| < \varepsilon/2$. Hence

$$E \setminus F = [(E \setminus E_N) \setminus F] \cup [E_N \setminus F]$$

and

$$|E \setminus F| \leq \varepsilon$$

On F the function is a.e. bounded by N.

Problem 5, (5 points): Please work problem 3.3.8 in Heil

Solution: Suppose that $f_n \to f$ in $L^\infty(E)$. This means that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. For each n there exists a set Z_n with $|Z_n| = 0$ such that

$$\sup_{x \notin Z_n} |f(x) - f_n(x)| = \|f - f_n\|_\infty \ .$$

Set $Z = \cup Z_N$ and note that $|Z| = 0$. Moreover,

$$\sup_{x \notin Z} |f(x) - f_n(x)| = \|f - f_n\|_\infty \to 0$$

as $n \to \infty$. This means that f_n converges uniformly to f on $E \setminus Z$. Conversely, if f_n converges uniformly fo f on $E \setminus Z$ with $|Z| = 0$, then

$$\sup_{x \in E \setminus Z} |f(x) - f_n(x)| \to 0 \ .$$

However,

$$\sup_{x \in E \setminus Z} |f(x) - f_n(x)| = \|f - f_n\|_\infty \ .$$