HOMEWORK 6 , DUE THURSDAY FEBRUARY 20

Problem 1, (5 points): Please work problem 3.3.9 in Heil.
Solution: If $[a, a+1]$ and $[b, b+1]$ are disjoint then $\|f_a - f_b\|_\infty = 1$. If $[a, a+1]$ and $[b, b+1]$ intersect but $a \neq b$ then $\|f_a - f_b\|_\infty = 2$. In any case there is an uncountable set S of numbers such that if $a, b \in S$, $\|f_a - f_b\|_\infty = 1$.

Problem 2, (5 points): Please work problem 3.4.5 in Heil.
Solution: a) Take the sequence $f_n(x) = \max |x|, n$ on the real line. The point wise limit of this sequence is $f(x) = |x|$. If A is any set the uniform convergence on this set means that

$$\sup_{x \in A} |f_n(x) - f(x)| \to 0$$

as $n \to \infty$. If $\mathbb{R} \setminus A$ has finite measure, there exists a sequence of points $x_j \in A$ with $x_j \to \infty$ as $j \to \infty$. But then

$$\sup_A |f_n(x) - f(x)| = \infty \ ,$$

and the convergence is not uniform.
For b), take the interval $[-1, 1]$ and the sequence of measurable functions

$$f_n(x) = n\chi_{-1/2,1/2} \ .$$

The limit of this sequence is the function f that vanishes outside the interval $[-1/2, 1/2]$ and is identically $+\infty$ on the interval $[-1/2, 1/2]$. For $\varepsilon < 1/2$ any subset $A \subset [-1, 1]$ with $|[-1, 1] \setminus A| < \varepsilon$ must intersect the set $[-1/2, 1/2]$ in a set with positive measure. On this intersection

$$\sup_A |f_n(x) - f(x)| = \infty$$

and hence Egorov's theorem does not hold.

Problem 3, (5 points): Please solve problem 3.5.12 in Heil.
Solution: The pointwise limit of the sequence is the function which is 1 for $|x| < 1$, 0 for $|x| = 1$ and -1 for $|x| > 1$. The sequence f_n is continuous, the limiting function is not continuous and hence the convergence cannot be uniform. In the region $|x| < 1$ we have that

$$\left| \frac{1 - |x|^n}{1 + |x|^n} - 1 \right| > \varepsilon$$

implies

$$1 > |x| > \left(\frac{\varepsilon}{2 - \varepsilon} \right)^{1/n} \ .$$

A similar estimate holds for $|x| > 1$ from which we see that the measure of the region where $|f_n - f| > \varepsilon$ converges to zero as $n \to \infty$.

Problem 4, (5 points): Please work problem 3.6.2 a) and b) (not c)) in Heil.

Solution: a) implies b) is Luzin's theorem. To see the converse, consider the set $A = \{f \geq a\}$. Pick any $\varepsilon > 0$. There exists a closed set $F \subset E$ such that $|E \setminus F| < \varepsilon$. The set $G = \{x \in F : f(x) \geq a\}$ is closed and hence measurable since f is continuous on F. Since $A \setminus G \subset E \setminus F$ we also have that $|A \setminus G|_e < \varepsilon$. Since ε is arbitrary, A is measurable.

Problem 5, (5 points): Please solve problem 4.1. 3 d) and e) only, in Heil.
Solution: Since ϕ is simple and non-negative, it is of the form

$$\phi = \sum_{j=1}^{N} c_j \chi_{E_j}$$

where $c_j > 0$ and E_j measurable.

a) Since

$$\sum_{n=1}^{\infty} \int_{A_n} \phi = \sum_{j=1}^{N} c_j |E_j \cap A_n|$$

it follows by countable additivity that this equals

$$\sum_{j=1}^{N} c_j |E_j \cap (\cup_{n=1}^{\infty} A_n)| = \int_{\cup_{n=1}^{\infty} A_n} \phi \ .$$

b) We have that

$$\lim_{n \to \infty} \int \phi = \lim_{n \to \infty} \sum_{j=1}^{N} c_j |A_n \cap E_j|$$

Now $A_n \cap E_j \subset A_{n+1} \cap E_j$ all n and j and hence

$$\lim_{n \to \infty} \sum_{j=1}^{N} c_j |A_n \cap E_j| = \sum_{j=1}^{N} c_j |A \cap E_j| = \int_A \phi \ .$$