HOMEWORK 7 , DUE THURSDAY MARCH 5

Problem 1, (5 points): Please do problem 4.4.15 in Heil
Solution: Consider the function $x^\alpha \chi_{[\varepsilon,1]}$ where $\varepsilon > 0$. This function is certainly integrable and since it is piecewise continuous we know that its Lebesgue integral is the same as its Riemann integral. Hence for $\alpha \neq 1$

$$\int x^\alpha \chi_{[\varepsilon,1]} dx = \frac{1-\varepsilon^{\alpha+1}}{\alpha+1} \ .$$

If $\alpha > -1$ the right side converges to $1/(\alpha+1)$ and since $x^\alpha \chi_{[\varepsilon,1]}$ converges monotonically to $x^\alpha \chi_{[0,1]}$ this function is integrable by monotone convergence. If $\alpha > 1$ the right side diverges as $\varepsilon \to 0$ and once more by monotone convergence the function $x^\alpha \chi_{[\varepsilon,1]}$ is not integrable. As similar argument works for $\alpha = 1$ using the logarithm.

For the function $g_\beta(x) = x^\beta \chi_{[1,\infty)}$ is the monotone limit of the sequence $x^\beta \chi_{[1,n)}$ as $n \to \infty$. By the argument above we see that the function g_β is integrable if $\beta < -1$ and otherwise not.

Problem 2, (5 points): Please work problem 4.4.22 in Heil
Solution: Since for $a < b$

$$\int f\chi_{[0,b]} - \int f\chi_{[0,a]} = \int f\chi_{(a,b]} = 0$$

we have that the integral vanishes for every interval in $[0,1]$. In particular for any open interval. Every open set $U \subset [0,1]$ can be written as a countable number of disjoint open intervals I_k (note: $[0,a)$ and $(b,1]$ are open relative to the interval $[0,1]$. Set

$$\phi_N(x) = \sum_{k=1}^{N} \chi_{I_k}(x) \ .$$

We have that ϕ_N converges to $\chi_U(x)$ pointwise and since $f \in L^1$ we have that

$$|\phi_N f| \leq |f|$$

and by the dominated convergence theorem

$$\int \phi_N f \to \int \chi_U f = \int_U f \ .$$

For each k we have that $\int_{I_k} f = 0$ and therefore $\int_U f = 0$. Pick and G_δ set H. There exists a decreasing sequence of open sets U_k so that $\cap U_k = h$ and in particular $\chi_{U_k} \to \chi_H$ pointwise. Again by dominated convergence

$$\int_{U_k} f \to \int_H f$$

and hence $\int_H f = 0$. Moreover any measurable set A can be written as $A = H \setminus Z$ where $|Z| = 0$ and hence

$$\int_A f = 0$$

for any measurable set. Now for $t > 0$ we have that

$$0 = \int_{\{f>t\}} f \geq t|\{f > t\}|$$

and hence $|\{f > t\}| = 0$. Similarly

$$0 = \int_{\{f<-t\}} f \leq -t|\{f < -t\}|$$

and hence $|\{f < -t\}| = 0$. Thus $f = 0$ a.e.

Problem 3, (5 points): Please do problem 4.4.23 in Heil
Solution: By Fatou's lemma

$$C \geq \liminf_{n\to\infty} \int |f_n| \geq \int |f|$$

and hence f is integrable.

Now

$$\Big||f_n| - |f - f_n| - |f|\Big| \leq \Big||f_n| - |f - f_n|\Big| + |f| \leq 2|f|$$

and hence by dominated convergence

$$\lim_{n\to\infty} \int \Big||f_n| - |f - f_n| - |f|\Big| = 0$$

which is a slightly stronger statement than required. We need $\|f_n\|_1 < C$. The sequence $\chi_{[-n,n]}$ provides a counterexample. It converges pointwise to the function 1 which is not intergable on the real line.

Problem 4, (5 points): Please do problem 4.5.15 a) in Heil
Solution: On the interval $[1, 2]$ we have that

$$\Big|\frac{n^2 \sin(x/n)}{1 + nx^2}\Big| \leq \frac{n^2 \frac{x}{n}}{1 + nx^2} = \frac{nx}{1 + nx^2} = \frac{x}{\frac{1}{n} + x^2} \leq \frac{1}{x}$$

using that $|\sin(x)| \leq |x|$. The function $1/x$ is integrable on the interval $[1, 2]$. Now, as $n \to \infty$ we find that

$$\frac{n^2 \sin(x/n)}{1 + nx^2} = \frac{n \sin(x/n)}{1/n + x^2} \to 1/x$$

and by dominated convergence

$$\lim_{n\to\infty} \int_1^2 \frac{n^2 \sin(x/n)}{1 + nx^2} dx = \ln(2)$$

Problem 5, (5 points): Please work problem 4.5.16 in Heil
Solution:

a) If $f = 0$ a.e., then obviously $\inf_A f = 0$. Consider the set $\{f > \alpha\}$ where α is real. The function is measurable and integrable and hence for $\alpha > 0$

$$0 = \int_{f>\alpha} f \geq \alpha|\{f > \alpha\}|$$

and hence $|\{f > 0\}| = 0$. Similarly, for $\alpha < 0$

$$0 = \int_{f<\alpha} f \leq \alpha|\{f < \alpha\}|$$

and since $\alpha < 0$, $|\{f < 0\}| = 0$.

b) Consider the set $\{|f| > n\}$. Since f is integrable we find that $|f|\chi_{|f|>n} \leq |f|$ and

$$\lim_{n\to\infty} \int_{|f|>n} |f| = 0$$

by dominated convergence. For any $\varepsilon > 0$ the exists N such that $\lim_{n\to\infty} \int_{|f|>N} |f| < \varepsilon$. Now set

$$A = E \setminus \{|f| > N\} \ .$$

On A the function $|f|$ is bounded by N and $\int_{E\setminus A} |f| < \varepsilon$.