0.1 Stability of matter interacting with magnetic fields

In this section we give the prove of the following stability result

Theorem 1 *The ground state energy of N electrons interacting with K fixed nuclei each having charge Z, i.e., the ground state energy $E_0(Z,B,N,K)$ of the Hamiltonian*

$$H = \sum_{j=1}^{N} [\sigma \cdot (p_j + A(x_j))]^2 + V_c \tag{0.1}$$

acting on antisymmetric functions in $\wedge_{j=1}^{N} L^2(\mathbb{R}^3;\mathbb{C}^2)$ satisfies the estimate

$$E_0(Z,B,N,K) + \frac{1}{8\pi e^2}\int B(x)^2 dx \geq -C(N+K) \tag{0.2}$$

provided that

$$\frac{\sqrt{3}}{8\alpha^2} > \frac{\pi}{2}Z + 2.2159 q^{1/3} Z^{2/3} + 1.0307 q^{1/3} \ . \tag{0.3}$$

In particular for $\alpha = 1/137$ there is stability for $Z < 2586$. for Z close to zero there is stability provided that $\alpha < 0.458 q^{-1/6}$ which yields for $q = 2$ the bound $\alpha < 0.408$.

As we have seen, the kinetic energy and the Coulomb as well as the field energy scale in a different way. Thus, one expects that the minimizer will 'choose' a certain length scale. As we already know this can only happen when the various conditions on the fine structure constant that are necessary for stability are satisfied. If these conditions are not satisfied we must have collapse, i.e., there must be an approximate scale invariance that drives the energy towards $-\infty$.

Since the Coulomb potential is a complicated many body interaction we eliminate it using the result on relativistic stability of matter, i.e.,

$$H \geq \sum_{j=1}^{N} \left\{ [\sigma_j \cdot (p_j + A(x_j))]^2 - D|p_j + A(x_j)| \right\} \tag{0.4}$$

where D satisfies the estimate

$$D \geq \frac{\pi}{2}Z + 2.2159 q^{1/3} Z^{2/3} + 1.0307 q^{1/3} \ . \tag{0.5}$$

Thus we have reduced the problem to one of a single particle operator. Denote by $-e_1 \leq -e_2 \leq$ the negative eigenvalues of

$$h = [\sigma \cdot (p + A(x))]^2 - D|p + A(x)| \ , \tag{0.6}$$

then we have that

$$H \geq -q \sum_{i=1}^{[\frac{N}{q}]} e_i \tag{0.7}$$

where $[\frac{N}{q}]$ denote the smallest integer greater or equal to $\frac{N}{q}$. As before

$$e = \int_0^\infty \chi_{e>e}(e)de \tag{0.8}$$

and hence

$$q\sum_{i=1}^{[\frac{N}{q}]} e_i = q\int_0^\infty \sum_{i=1}^{[\frac{N}{q}]} \chi_{e_i>e}(e)de \tag{0.9}$$

$$= q\int_0^\infty \min\left(N_e(h), [\frac{N}{q}]\right) de \ , \tag{0.10}$$

where $N_e(h)$ is the number of eigenvalues of h below $-e$.

Now, recalling that the system must have a certain natural energy scale if it is stable at all, we denote this unknown energy scale by μ. Thus, we rewrite (**??**) as

$$q\int_0^\mu \min\left(N_e(h), [\frac{N}{q}]\right) de + q\int_\mu^\infty \min\left(N_e(h), [\frac{N}{q}]\right) de \ , \tag{0.11}$$

and estimate it from above by

$$q \quad [\frac{N}{q}]\mu + q\int_\mu^\infty N_e(h)de \tag{0.12}$$

$$\leq \quad (N+q)\mu + q\int_\mu^\infty N_e(h)de \tag{0.13}$$

The first term should estimate the energy of the system, while the second term should be compensated by the field energy. Thus, we have to relate this term to something that has the same dimensionality as the field energy. This is achieved by a sliding energy scale; for all $e \geq \mu$ we have

$$[\sigma\cdot(p+A(x))]^2 \geq \frac{\mu}{e}[\sigma\cdot(p+A(x))]^2 \geq \frac{\mu}{e}(p+A(x))^2 - \frac{\mu}{e}\sigma\cdot B \ . \tag{0.14}$$

Further

$$|p+A(x)| \leq \frac{1}{2ue}|p+A(x)|^2 + \frac{ue}{2} \ , \tag{0.15}$$

where $u>0$ to be determined later. Hence,

$$h \geq \left(\frac{\mu}{e} - \frac{D}{2ue}\right)|p+A(x)|^2 - \frac{\mu}{e}\sigma\cdot B - \frac{Due}{2} =: h'_e =: \frac{1}{e}k - \frac{Due}{2} \ . \tag{0.16}$$

Hence,

$$N_e(h) \leq N_e(h'_e) \leq N_{(1-Du/2)e^2}(k) \ . \tag{0.17}$$

Now,

$$\int_\mu^\infty N_e(h)de \leq \int_0^\infty N_{(1-Du/2)e^2}(k)de \quad = \quad \frac{1}{\sqrt{1-\frac{Du}{2}}}\int_0^\infty N_{e^2}(k)de \tag{0.18}$$

$$= \quad \frac{1}{\sqrt{1-\frac{Du}{2}}}\sum_j \lambda_j^{1/2} \ , \tag{0.19}$$

where $-\lambda_j$ are the negative eigenvalues of the operator k. Needless to say that u has to be chosen such that

$$\frac{Du}{2} < 1 \ . \tag{0.20}$$

Using the Lieb-Thirring inequality for the square root of the eigenvalues leads to the estimate

$$\sum_j \lambda_j^{1/2} \le \mu^2 \left(\mu - \frac{D}{2u}\right)^{-3/2} \frac{1}{8\pi} \int B(x)^2 dx \ . \tag{0.21}$$

A naive application of the Lieb-Thirring inequality would lead to a constant that is by a factor of two larger. This is due to the spin which forces to count each eigenvalue twice. That the factor of two can be omitted is shown in [**?**]. Optimizing the above expression over μ yields

$$\mu = 2\frac{D}{u}$$

which yields the bound

$$H \ge -(N+q)\frac{2D}{u} - \frac{1}{\sqrt{1-\frac{Du}{2}}} \left(\frac{D}{2u}\right)^{1/2} \frac{2}{\pi\sqrt{3}} \int B(x)^2 d^3x \ . \tag{0.22}$$

Optimizing this result over u, recalling that $Du < 2$, yields

$$H \ge -(N+q)2D^2 - \frac{8D}{\sqrt{3}} 8\pi \int B(x)^2 d^3x \ . \tag{0.23}$$

Thus, we get the stability condition $1/\alpha^2 > 8D/\sqrt{3}$, i.e.,

$$\frac{\sqrt{3}}{8\alpha^2} > \frac{\pi}{2} Z + 2.2159 q^{1/3} Z^{2/3} + 1.0307 q^{1/3} \ . \tag{0.24}$$

Another proof that yields slightly worse results but is very simple and flexible follows using the BKS inequality. Recall that

$$h = [\sigma \cdot (p + A(x))]^2 - D|p + A(x)| \tag{0.25}$$

and estimate it below using the arthmetic – geormetric mean inequality

$$DR|[\sigma \cdot (p + A(x))]| - \frac{R^2 D}{4} - D|p + A(x)| \ , \tag{0.26}$$

for some number R which we shall choose later. Now

$$H \ge -D\mathrm{Tr}\,[R|[\sigma \cdot (p + A(x))]| - |p + A(x)|]_- - \frac{NR^2D^2}{4} \ , \tag{0.27}$$

where $[\cdots]_-$ denotes the negative part of $\cdots$ and using the BKS inequality yields

$$\mathrm{Tr}\,[R|[\sigma \cdot (p + A(x))]| - |p + A(x)|]_- \le \mathrm{Tr}\left[R^2[\sigma \cdot (p + A(x))]^2 - |p + A(x)|^2\right]_-^{1/2} \ . \tag{0.28}$$

The right side can be brought into the form

$$\sqrt{R^2-1}\mathrm{Tr}\left[(p+A(x))^2+\frac{R^2}{R^2-1}\sigma\cdot B\right]_-^{1/2} \tag{0.29}$$

which is again in a form suitable for applying the Lieb – Thirring inequality which yields the bound

$$\frac{R^4}{(R^2-1)^{3/2}}\frac{1}{8\pi}\int B^2(x)d^3x\ . \tag{0.30}$$

Optimizing this expression over R yields as an upper bound

$$\frac{16}{\sqrt{3}}\frac{1}{8\pi}\int B^2(x)d^3x\ , \tag{0.31}$$

and hence there is stability provided that

$$\frac{\sqrt{3}}{16\alpha^2}>\frac{\pi}{2}Z+2.2159q^{1/3}Z^{2/3}+1.0307q^{1/3}\ . \tag{0.32}$$