November 7, 1999

Additional Exercises

Add to Chapter 2 the following Exercise 20:

20. Find a sequence of bounded, measurable sets in $\mathbf{R}$ whose characteristic functions converge weakly in $L^2(\mathbf{R})$ to a function f with the property that $2f$ is a characteristic function. How about the possibility that $f/2$ is a characteristic function?

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Add to Chapter 3 the following Exercise 9:

9. Prove assertion (iii) in Sect. 3.3, namely that $f^*(x)$ is lower semicontinuous. Consequently, $\{x : f^*(x) > t\} = \{x : |f(x)| > t\}^*$, as in assertion (iv).

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Add to Chapter 4 the following Exercise 8:

8. Prove the equivalence of the two definitions of weak L^q given in Section 4.3. That is, if $\langle f\rangle_{p,w}$ denotes the left side of 4.3(3) then

$$C_1\langle f\rangle_{p,w} \leq \|f\|_{p,w} \leq C_2\langle f\rangle_{p,w} \ ,$$

where $\|f\|_{p,w}$ is given by 4.3(5) and C_1 and C_2 are two universal constants independent of f. Find explicit values for these constants.

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Add to Chapter 6 the following Exercise:

10. Consider the function $f(x) = |x|^{-n}$ on $\mathbf{R}^n$. Although this function is not in $L^1_{loc}(\mathbf{R}^n)$, it is defined as a distribution for test functions on $\mathbf{R}^n$ that vanish at the origin, by

$$T_f(\phi) = \int_{\mathbf{R}^n} |x|^{-n}\phi(x)\mathrm{d}x \ .$$

a) Show that there is a distribution $T \in \mathcal{D}'(\mathbf{R}^n)$ that agrees with T_f for functions that vanish at the origin. Give an explicit formula for one such T.

b) Characterize all such T's. Theorem 6.14 may be helpful here.

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Add to Chapter 7 the following Exercises 6 and 7.

6. Suppose that $\Omega_1,\ \Omega_2,\ \Omega_3, \ldots$ are disjoint, bounded, measurable subsets of $\mathbf{R}^n$. Denote by D_j the diameter of Ω_j (i.e., $\sup\{|x-y| : x \in \Omega_j,\ y \in \Omega_j\}$) and by $|\Omega_j|$ its volume. Let $f \in H^{1/2}(\mathbf{R}^n)$ and define the average of f in Ω_j to be

$$\overline{f} := \int_{\Omega_j} f \ / \ |\Omega_j| \ .$$

Prove the strict inequality

$$(f, |p|f) > \frac{\Gamma(\frac{n+1}{2})}{2\pi^{(n+1)/2}} \sum_j \frac{|\Omega_j|}{D_j^{n+1}} \int_{\Omega_j} |f - \overline{f}|^2 \ .$$

This is an example of a type of inequality known as a **Poincaré inequality**.

7. Use the result of Exercise 6 to show that for functions $f_1, f_2, \ldots f_N$, each in $H^{1/2}(\mathbf{R}^n)$, and any measurable set Ω with diameter D,

$$\sum_{j=1}^{N}(f_j, |p| f_j) \geq \frac{\Gamma(\frac{n+1}{2})}{2\pi^{(n+1)/2}} \frac{|\Omega|}{D^{n+1}} \left[\sum_{j=1}^{N} \int_\Omega |f_j(x)|^2 \mathrm{d}x - \lambda \right] ,$$

where λ is the largest eigenvalue of the **Gram matrix**

$$G_{i,j} = \int_\Omega \overline{f_i(x)} f_j(x) \underset{\cdot}{\mathrm{x}} \ .$$

Hint: This is a problem in linear algebra.

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Add to Chapter 9 the following Exercise 5:

5. Consider the distribution in $\mathcal{D}'(\mathbf{R}^n)$ given by

$$T_R(\phi) := |S^{n-1}|^{-1} \int_{S^{n-1}} \phi(R\omega) \mathrm{d}\omega \ .$$

By Theorem 6.22 there exists a unique, regular Borel measure μ such that $T_R(\phi) = \int \phi(x) \mu(\mathrm{d}x)$.

a) Compute 9.7(1) for this measure μ and compute $D(\mu, \mu)$. You have to show that $|x - y|^{2-n}$ is measurable with respect to $\mu(\mathrm{d}x) \times \mu(\mathrm{d}y)$.

b) Prove that with $\nu = \mu - \rho \mathrm{d}x$, and $\rho \in L^1(\mathbf{R}^n)$ nonnegative, $D(\nu, \nu) \geq 0$.

c) Use the above to compute

$$\inf \left\{ D(\rho, \rho) : \rho(x) \geq 0, \int \rho = 1, \ \rho(x) = 0 \text{ for } |x| > R \right\} .$$

Is the infimum attained?

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Add to Chapter 2 the following exercise 21:

21. At the end of the proof of Theorem 2.6 (Differentiability of norms) there is a displayed pair of inequalities, valid for $|t| \leq 1$:

$$|f|^p - |f - g|^p \leq \frac{1}{t} \{ |f + tg|^p - |f|^p \} \leq |f + g|^p - |f|^p$$

Write out a complete proof of these two inequalities.