Additional Exercises for "Analysis"

Additional Exercise for Chapter 2:

Lemma 2.8 (Projection on convex sets) is not stated for $p = 1$ for good reason. Here is an example of an L^1 space, a set K, and a function f such that the conclusion of Lemma 2.8 fails. Show that K is convex and norm closed and that there is no $g \in K$ that minimizes the distance to f. The space is $L^1(\mathbb{R}; (1 + x^2)dx)$, $K = \{g : \int g dx = 1\}$ and $f = 0$.

Additional Exercise for Chapter 4:

One can rewrite the HLS inequality 4.3(1) as

$$\int_{\mathbb{R}^n} |x|^{-\lambda}(f * h)(x)\mathrm{d}x \leq C(n, \lambda, p)\|f\|_p\|h\|_r$$

with $1/p + 1/r = 2 - \lambda/n$. The function $|x|^{-\lambda}$ is in weak L^q with $q = n/\lambda$, so $1/p + 1/r = 1 + 1/q'$. We know from Young's inequality that $f * h$ is in $L^{q'}$ and we might be tempted to conclude from 4.3(1) that we can replace $f * h$ in the integral above by *any* $L^{q'}$ function (with a possible readjustment of the constant C). This is not so; the $L^{q'}$ functions obtained by convolution of an L^p and an L^r function are special and one of the ways in which they are special is that the integral above is finite.

Find a function $g \in L^{q'}(R^n)$ so that the integral above is infinite when $f * h$ is replaced by g.