Chapter 11

Introduction to the Calculus of Variations

If we now define

$$B = \{x \in A \ : \ \phi(x) < 1\},$$

we have that $B \subset \bigcup_{k=1}^{\infty} B_{1/k}$. But, it is easy to see directly from 11.15(8) that capacity is countably subbadditive (cf. Exercise 11). Therefore

$$\mathrm{Cap}(B) \leq \sum_{k=1}^{\infty} \mathrm{Cap}(B_{1/k}) = 0,$$

$\mathrm{Cap}(A) = \mathrm{Cap}(A \sim B)$, and f is a true minimizer of (8) for the set $A \sim B$.

Our next goal is to deduce properties c)–e) of ϕ, as well as $\phi \leq 1$. Item c) is proved as follows. Let η be any *nonnegative* function in $C_c^\infty(\mathbb{R}^n)$, in which case $\Delta\eta$ is also in $C_c^\infty(\mathbb{R}^n)$. For $\varepsilon > 0$ let $f_\varepsilon := f - \varepsilon g$ with $g = |x|^{1-n} * \Delta\eta$. Correspondingly,

$$\phi_\varepsilon := |x|^{1-n} * f_\varepsilon = \phi - \varepsilon(|x|^{1-n} * |x|^{1-n}) * \Delta\eta.$$

(We are using Fubini's theorem here to exchange the order of integration in the repeated convolution.) By Theorem 6.21 (solution of Poisson's equation) and the fact that $|x|^{1-n} * |x|^{1-n} = C_n|x|^{2-n}$ we have that

$$-(|x|^{1-n} * |x|^{1-n}) * \Delta\eta = C_n'\eta,$$

with $C_n' > 0$. Therefore, f_ε is an admissible function for $A \sim B$ and every $\varepsilon > 0$, because $\phi_\varepsilon \geq \phi$. Since f is a minimizer of (8) for the set $A \sim B$,

$$0 \leq -2\varepsilon \int_{\mathbb{R}^n} fg + \varepsilon^2 \int_{\mathbb{R}^n} g^2.$$

This holds for all $\varepsilon > 0$, so $\int_{\mathbb{R}^n} fg \leq 0$. In other words,

$$0 \geq \int_{\mathbb{R}^n} f|x|^{1-n} * \Delta\eta = \int_{\mathbb{R}^n} \phi\Delta\eta$$

for every nonnegative $\eta \in C_c^\infty(\mathbb{R}^n)$. (Fubini's theorem has been used again.) This means, by definition, that $\Delta\phi \leq 0$ in the distributional sense, and c) is proved.

A similar argument, but now without imposing the condition that $\eta \geq 0$, proves d).

Item e) is left to the reader as an exercise with Fourier transforms.

The proof that $\phi \leq 1$ is a bit involved. Since ϕ is superharmonic, and since ϕ vanishes at infinity, Theorem 9.6 (subharmonic functions are potentials) shows that $\phi = |x|^{2-n} * \mathrm{d}\mu$, where μ is a positive measure. Therefore, by Fubini's theorem,

$$|x|^{2-n} * |x|^{1-n} * \mathrm{d}\mu = |x|^{1-n} * \phi = C_n|x|^{2-n} * f.$$

Taking the Laplacian of both sides we conclude, by Theorem 6.21, that $C_n f = |x|^{1-n} * \mathrm{d}\mu$ as distributions, and hence as functions by Theorem 6.5 (functions are uniquely determined by distributions). We conclude, therefore, that

$$\mathrm{Cap}(A \sim B) = \mathrm{Cap}(A) = C_n \int_{\mathbb{R}^n} f^2 = 2\mathcal{E}(\mu) = \int_{\mathbb{R}^n} \phi \, \mathrm{d}\mu.$$

Now, let $\phi_0(x) := \min\{1, \phi(x)\}$, which is also superharmonic. (Why?) Again, by Theorem 9.6, $\phi_0 = |x|^{2-n} * \mathrm{d}\mu_0$. Then

$$\int_{\mathbb{R}^n} \phi \, \mathrm{d}\mu \geq \int_{\mathbb{R}^n} \phi_0 \, \mathrm{d}\mu = \int_{\mathbb{R}^n} \phi \, \mathrm{d}\mu_0 \quad \text{[by Fubini]} \quad \geq \int_{\mathbb{R}^n} \phi_0 \, \mathrm{d}\mu_0.$$

Thus, if we define $f_0 = |x|^{1-n} * \mathrm{d}\mu_0$, we see that f_0 satisfies the correct conditions and gives us a lower value for $\mathrm{Cap}(A \sim B) = \mathrm{Cap}(A)$, which is a contradiction unless $\phi = \phi_0$. ■