Commentaries

The subject of 'inequalities' was first systematically established by Hardy, Littlewood and Pólya in their book of the same name. The goal, loosely speaking, is to search for an inequality between algebraic or analytic expressions of certain variables that becomes an equality in certain (possibly limiting) cases. The reader may think of Hölder's inequality as an example, but also such things as the dependence upon its shape of the lowest frequency of a drum.

The usefulness and importance of this area can hardly be overstated. Many of the results have entered the 'mathematical subconcious' while others have to be considered as deep contributions to mathematical culture such as the isoperimetric inequality that goes back to ancient times.

What makes this area especially attractive is that it cuts across boundaries of mathematical and scientific disciplines. A case in point is Sobolev's inequality, an indispensable tool in partial differential equations. When looked at from a different angle it is very closely related to the Yamabe problem in differential geometry and, finally, it is the deepest formulation of the uncertainty principle in quantum mechanics.

In physical applications inequalities occur naturally through the principle of least energy. The ultimate goal is to find the lowest value of the energy and to describe the state of the system at that energy, usually as a solution to some partial differential equation. Famous examples are soap bubbles, elasticity theory and the Schrödinger atom. The unifying mathematical theory is known as the calculus of variations, which gave the impetus to much of what is now known as Analysis and Functional Analysis.

The physicists computational approach to such models is often hampered by the fact that they are not exactly solvable and, as often happens in such cases, they resort to uncontrolled approximations. Inequalities that establish bounds on physical quantities are very useful in such circumstances. One example that illustrates this point nicely is the paper of G. Baym and A.J. Leggett [BL] in which they give rigorous upper bounds to barrier penetration probabilities in many body quantum systems and are able to rule out cold fusion. They use a number of inequalities for Schrödinger operators and well established experimental bounds. Another example that comes to mind is the use of unitarity in estimating scattering cross sections. Often some cross section can be measured and some not. With the help of unitarity one can derive bounds on the latter. One of the most famous inequalities in physics is what is now called Bell's inequality. It is an inequality about classical independent events and delineates in a quantitative way the difference between quantum mechanics and classical mechanics. There have been experiments that violate Bell's inequality [Bel] and give some credence to the opinion that quantum mechanical correlations are classically not explicable without action at a distance.

Although there is no all encompassing theory of inequalities, many of them have common underlying features. One senses this strongly in Lieb's work which, superficially spans a variety of seemingly unrelated problems. The unity of the whole work is provided by his methods of investigation. As an example, many of Lieb's inequalities that concern physical quantities, such as energy and entropy have a common mathematical source, in that they all can be expressed in one way or another as convexity inequalities. Convexity is at the root of his work on entropy, but it also resurfaces again in the form of the Brunn-Minkowsi inequality, applied to the one-dimensional plasma and rearrangements.

A Selecta gives us the chance to trace some of the historical development of Lieb's work. It is therefore natural to start with his work in statistical mechanics and condensed matter and to see how his interests expanded. It is also interesting to see how Lieb acquired the formidable knowledge of mathematics he is known for; how he summons it all in pursuit of a problem and how it grows with every solved one.

Part I: Inequalities related to statistical mechanics and condensed matter

One of the major themes in Lieb's work has been condensed matter physics beginning with one-dimensional models in statistical mechanics, and highlighted by his exact solution of the two-dimensional six-vertex model. His papers in this area could easily form another volume of *Selecta*. In the present volume, we include only those papers in which his analysis led directly to results on inequalities.

I.1, I.2: Both papers are joint work with Mattis with whom he eventually wrote the well-known book *Mathematical Physics in One Dimension* [LM]. They contribute to the understanding of the nature of ferromagnetism. For a many electron system, the total spin value is a good quantum number, provided the interactions are not spin dependent. It is a long-standing problem to understand why the ground state for certain systems has large total spin. Its manifestation is a permanent magnetization of the system as can happen, e.g., in iron and hence the appellation 'ferromagnetism'. There is an old idea of Heisenberg that the sign of a certain integral, called the exchange integral, determines whether a system is ferromagnetic or not. In the first paper it is shown that for one-dimensional quantum systems the ground state energy $E(S)$ with total spin S is monotone increasing as a function of S. Hence, the absolute ground state has spin zero and there is never ferromagnetism in one-dimensional systems even though the exchange integral can have either sign, depending on the potential. Similar ideas were then used by Lieb and Mattis for interacting spin systems in the second paper.

I.9: The first paper, **I.1** has a generalization to positive temperature, that is to an average over all eigenfunctions of the Schrödinger equation, not just the lowest. This was finally achieved 28 years later in a work with Aizenman **I.9** using the Wiener integral representation of the positive temperature state. The use of Wiener integrals to derive rigorous bounds on quantum-mechanical quantities also appears in **III.7**.

I.3-I.6: We describe the history leading up to these papers on quantum mechanical entropy by quoting from the review of Wehrl [We, p. 249],

> "Strong subadditivity was known for many years in information theory but not generally called that. For statistical mechanics, at least, it was Robinson and Ruelle (1967) [RR] who coined the word and who first realized that it was important. They proved it in the classical case. Then Lanford and Robinson (1968) [LR] conjectured it in quantum mechanics"

and the notes in Bratteli and Robinson (vol. II, p. 435)[BR], “Despite attempts by many authors, this conjecture remained unverified for 6 years until Lieb and Ruskai (1973) gave a proof based upon a result of Lieb’s” [in **II.3**].

For the statement of the inequality, consider three Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ and their tensor products. These spaces correspond to subsystems of a composite system $\mathcal{H}_{123} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$. Let ρ_{123} be a density matrix on $\mathcal{H}_{123}$, i.e., a positive semi-definite matrix with trace equal to one. By taking partial traces in the obvious fashion density matrices describing the subsystems are obtained, such as ρ_{12} on $\mathcal{H}_{12}$. For any of these density matrices, the quantum mechanical entropy can be defined, following von Neumann, as

$$S(\rho) = -\mathrm{Tr}\rho \ln \rho \ .$$

The *strong subadditivity* (SSA) of the quantum entropy can then be stated as

$$S(\rho_{123}) + S(\rho_2) \leq S(\rho_{12}) + S(\rho_{23}). \tag{1}$$

In **I.3**, which is joint work with Araki, Lieb took a first step and proved the weaker inequality

$$S(\rho_{123}) \leq S(\rho_{12}) + S(\rho_{23}). \tag{2}$$

This sufficed to establish the existence of an infinite volume limit for entropy. In this paper, they also proved the so-called triangle inequality

$$|S(\rho_1) - S(\rho_2)| \leq S(\rho_{12}) \leq S(\rho_1) + S(\rho_2)$$

which gives a lower bound and complements the (weak) subadditivity upper bound $S(\rho_{12}) \leq S(\rho_1) + S(\rho_2)$.

Papers **I.4, I.5** with Ruskai contain the proof of SSA. The key was the realization that SSA could be reformulated as a statement about the convexity of the so-called conditional entropy $S(\rho_{12}) - S(\rho_1)$ of a two-component system. This reformulation led Lieb to the convexity of a quite different quantity called the Wigner-Yanase-Dyson entropy [WY] and the notion of convex trace function, which will be discussed later (see **II.3**). (Subsequently, it was realized that the convexity of the conditional entropy was equivalent to the more general notion of convexity of relative entropy. Although that term is not used, eq. (3.2) of **I.4** states that the relative entropy is monotone under partial traces.)

Paper **I.6** contains a summary of these results, comparing the differences between classical discrete, classical continuous and quantum entropy. In addition, Lieb considers, and provides counterexamples to, a number of natural generalizations of subadditivity inequalities.

I.7: Simon proved an inequality for the Ising model [Si] which has been used often to prove exponential decay of correlation functions, among other things. It is a special case of an inequality of Boel and Kasteleyn [BK].

Lieb improved this inequality significantly (it is now often referred to as the Simon-Lieb inequality) with several consequences. These include a more effective method for bounding the phase transition temperature and an algorithm which, in principle, can compute the exact transition temperature to any accuracy (although it is unwieldy in practice). In another direction, Lieb proposed a conjecture which, if true, would extend the results from the Ising model to the plane-rotor model. This conjecture was verified by Rivasseau [Ri].

I.8: Lieb has had a long-standing interest in the Hubbard model, which is an important model in condensed matter physics for describing interacting electrons. With Wu, Lieb showed [LW] that this model is exactly solvable in one-dimension; however, it is much harder in two- and three-dimensions where little is known about its ground state or its behavior at positive temperatures. In the paper under discussion, Lieb proves a result for this model that is analogous to that of **I.1**, namely, that for an even number of electrons and an attractive on-site interaction among them, the ground state has spin zero. This result is independent of the dimension of the system or the structure of the underlying lattice.

If the interaction is repulsive and the underlying lattice is bipartite, then the total spin of the ground state is $S = ||A| - |B||/2$ for the half filled band case. Here $|A|$ and $|B|$ denote the number of vertices in the two components defining the bipartite lattice. One may speak here of itinerant ferromagnetism since the magnetization is created by electrons that are not attached to any specific vertex.

Part II. Inequalities in algebra and combinatorics

Lieb's work in statistical mechanics often led to related problems in combinatorics, linear algebra, and maps of operator algebras. Much of this work is of interest in its own right.

It has long been realized that trace-preserving, completely positive maps seem to be the appropriate mathematical structure needed to model noise in quantum communication channels and quantum computers. With recent advances in the development of algorithms that use quantum particles for data encryption, computation and communication, as well as promising experiments toward their implementation, Lieb's work in this area has taken on renewed importance. His work on quantum mechanical entropy, discussed in the previous section has had a particularly strong impact, e.g., Shor [Sh] recently used SSA to prove some important results on channel capacity. In addition, Lieb's later work with Solovej **IV.3** on quantum coherent operators for spin systems and his work with Carlen on Clifford algebras in **II.7, II.9** is likely to have further impact on problems in quantum information theory.

II.1: Schur defined certain functions of $n \times n$ matrices, called "immanants", of which the determinant is a special case. For each irreducible representation λ of the symmetric group S_n, the corresponding immanant of a matrix A is

$$I_\lambda(A) = \frac{1}{m_\lambda} \sum_{P \in S_n} \chi_\lambda(P) A_{1,P1} \cdots A_{n,Pn} \ ,$$

where m_λ is the degree of the representation and $\chi_\lambda(P)$ is the character of the permutation P. For positive semidefinite matrices Schur proved that the determinant (alternating representation) is the smallest immanant. The permanent corresponds to the identity representation.

In this paper Lieb derives lower bounds for the permanent and conjectures that the permanent is the largest immanant (for positive semidefinite matrices). He goes further and considers generalized immanants in which S_n is replaced by some subgroup of S_n. Schur's proof generalizes to this case, too, i.e., the determinant is the smallest one. Lieb's full conjecture is that the permanent is the largest. In this paper he proves this for a special case of the generalized immanant.

This conjecture has become known as "permanent dominance" and there are many papers devoted to it. For Schur- immanants it has been proved for up to 9×9 matrices and for some special classes of matrices. The permanent is important in several combinatorial and algebraic situations, hence the significance of this conjecture (see [Hai], for example). Only the determinant, however, is invariant under unitary transformations and that is one reason the problem has defied solution up to now.

II.2: Many important coefficients in combinatorics (e.g., the binomial coefficients) are unimodal, meaning that one (or possibly two adjacent) coefficient is the largest and the coefficients drop off monotonically on either side of the maximum.

Harper [Har] showed that the Stirling numbers of the second kind are unimodal. Lieb proved more, namely that these coefficients are log-concave (for Stirling numbers of the first *and* second kind). His technique was to show that the polynomial whose coefficients are Stirling numbers times binomial coefficients times k has only real zeros. Then, by Newton's inequality, the coefficients in the polynomial, when divided by the binomial coefficients, are log concave. This technique has been used many times since. It seems that Lieb was among the the first to employ it. See [St] for a review.

II.3: This paper considers functions that map matrices to complex numbers via the trace of more complex objects involving several matrices. In addition to proving a number of inequalities, it also uncovers surprising relations among them. The heart of the paper is a series of convexity results for functions that are homogeneous and some of them become linear if restricted to commuting matrices. Thus, one could not even have predicted, from examining the special case in which the matrix is a complex number, whether these trace functions would be convex or concave. Proving such results required considerable ingenuity and the use of new techniques. Two of these results merit further discussion.

For positive semidefinite A and B the map

$$A, B \to \mathrm{Tr} A^p K^* B^{1-p} K$$

is jointly concave for any K and for $0 \leq p \leq 1$. Wigner and Yanase [WY] proved this for $p = 1/2$ in connection with a theory about entropy, and briefly discussed the case $p \neq 1/2$, introduced by Dyson (but they did not actually make the explicit concavity conjecture and considered the $p \neq 1/2$ case to be uninteresting for their purposes). The concavity conjecture for $p \neq 1/2$ seems first to have been made by Baumann in his Ph.D. thesis, where he proved it for 2×2 matrices [Bau].

For positive semidefinite A and Hermitian K the map

$$A \to \mathrm{Tr}\exp\{K + \log A\}$$

is concave. This second inequality was an essential ingredient in the original proof of strong subadditivity of entropy. One route to SSA uses the following consequence of this second concavity theorem, which is interesting in its own right,

$$\operatorname{Tr} e^{\log R - \log S + \log T} \leq \operatorname{Tr} \int_0^\infty R(S+uI)^{-1}T(S+uI)^{-1}du.$$

This can be regarded as a generalization of the inequality $\operatorname{Tr} e^{A+B} \leq \operatorname{Tr} e^A e^B$ to three matrices; it is both significant and surprising because the generalization $\operatorname{Tr} e^{A+B+C} \leq |\operatorname{Tr} e^A e^B e^C|$ is known to be false.

Another proof of SSA, in connection with Minkowski's inequality for matrices, appears in **II.9**.

II.4: The main result of this paper (which is joint with Ruskai) is that for an operator map ϕ of the type known as "completely positive" the inequality

$$\phi(A^*A) > \phi(A^*B)[\phi(B^*B)]^{-1}\phi(B^*A)$$

holds. Moreover, it is not necessary that $\phi(B^*B)$ be invertible, as the proof shows that whenever the left side is finite, the right side can be defined as a suitable limit. In the important special case in which ϕ is a partial trace, it implies that the map

$$A, K \to K^*A^{-1}K$$

is jointly convex, i.e, the appropriate operator inequality holds.

Completely positive maps play an important role in the theory of operator algebras and arise naturally in the theory of open quantum systems, where the maps are also trace-preserving. Subsequently, Choi [Ch] showed that the requirement of complete positivity can be reduced to two-positivity. Although other proofs have been found, the original argument in **II.4** is extremely simple and the formulation given there seems to have been a breakthrough.

II.5: Lieb introduces complex-valued functions, F, of matrices (which Simon, in his book [Sim] calls "Liebian") satisfying two conditions, namely, that F is positivity preserving. i.e., $B \geq A \geq 0 \Rightarrow F(B) \geq F(A)$, and satisfies the Schwarz-like condition $F(X^*X)F(Y^*Y) \geq |F(X^*Y)|^2$. Examples of such functions are: immanants, k^{th} elementary symmetric functions of the singular values, and the norm. Additional examples were found by Merris [Me] and others later. Lieb uses the results of **II.4** to show, among other things, that $F(\sum_j X_j^*X_j)F(\sum_j Y_j^*Y_j) \geq |F(\sum_j X_j^*Y_j)|^2$, which is analogous to a log-convexity statement. As a non-obvious special case, he obtains a simple new proof of the inequality (due to Rotfel'd and Seiler-Simon) [Ro][SS]

$$|\det(I + A + B)| \leq \det(I + |A|)\det(I + |B|).$$

II.7: In the early 70's, the same time period as Lieb's work on convex trace functions and entropy, Gross [Gr] developed hypercontractivity estimates and log Sobolev inequalities — tools that proved invaluable in the study of ground states in quantum field theory of bosonic systems. These inequalities are, in fact, special cases of the sharp form of Young's inequality (see **V.3** and **VI.7**). At that time Gross also further developed the non-commutative integration needed for fermionic systems and extended some of his hypercontractivity estimates to this framework. However, these results were not optimal and many open conjectures remained. They were not resolved until 20 years later when Lieb, jointly with Carlen proved optimal hypercontractivity bounds for the fermion oscillator semigroup, and an optimal fermion logarithmic Sobolev inequality. These results, required the following optimal convexity inequality for matrices

$$2^{-2/p}\left(Tr|A+B|^p + Tr|A-B|^p\right)^{2/p} \geq \left(Tr|A|^p\right)^{2/p} + \left(Tr|B|^p\right)^{2/p} \ .$$

Although this sufficed for the fermionic work, characteristically, Lieb did not stop there but collaborated with Ball and Carlen to prove a host of sharp convexity and smoothness inequalities (see **II.8**). In essence, Ball, Carlen and Lieb extended to the non-commutative C^p ideals almost everything that was known about convexity and smoothness of L^p spaces. Their paper also contains a comprehensive summary of the corresponding L^p inequalities, including new proofs and even a few new inequalities. Some idea of the scope of this work can be seen in the diagram on p. 471 of **II.8**.

II.8: The unit ball in an L^p-space of functions is, of course, convex. One of the important developments in that subject was a quantification of this convexity and also of the smoothness of the balls when $p > 1$, by Clarkson [Cl] and later Hanner [Ha]. The question addressed in this paper is whether these inequalities extend to the trace ideals for matrices, $X \to \{\mathrm{Tr}|X|^p\}^{1/p}$? Hanner's inequality is proved for $1 \leq p \leq 4/3$, for $p = 2$ and for $4 \leq p \leq \infty$. The other cases are still open!

II.9: Minkowski's inequality for functions on the product of two measure spaces states that for $p \geq 1$ and $f \geq 0$, $\left(\int_X \left(\int_Y f(x,y)d\nu\right)^p d\mu\right)^{1/p} \leq \int_Y \left(\int_X f(x,y)^p d\mu\right)^{1/p} d\nu$. The reverse inequality holds for $0 < p \leq 1$. Does this important inequality extend to partial traces of matrices defined on the tensor product of two Hilbert spaces? The answer, proved here jointly with Carlen, is yes.

More interestingly, one can trivially extend Minkowski's inequality to functions of three variables, $f(x,y,z)$ by applying Minkowski's inequality with z fixed, and then integrating over z. The corresponding inequality for matrices defined on the tensor product of three Hilbert spaces is far from trivial. It is proved here for $0 < p \leq 1$ and $p = 2$. Surprisingly, it is shown to be false for $p > 2$. The case $1 < p < 2$ is still open! However, this three Hilbert space theorem for $0 < p \leq 1$ yields one more proof of the strong subadditivity of entropy.

Part III. Inequalities related to the stability of matter

Inequalities related to the stability of matter form an important part of Lieb's work. The constitution of matter, i.e., properties of a system consisting of a large number of electrons interacting with nuclei via electrostatic forces, was one impetus for the development of quantum mechanics. Very early on this theory was successful in explaining the size of the hydrogen atom and of atoms in general. The understanding of its implications for bulk matter is relatively recent and due to Dyson and Lenard [DL]. In the mid-seventies a completely new and much improved version of the Stability of Matter theorem was proved by Lieb and Thirring [LT]. Stability of Matter means that the ground state energy has a lower bound that is proportional to the number of particles involved and, in a nutshell, this is the case because of the Schrödinger equation and the Pauli exclusion principle.

III.1, III.8: The key ingredients of the Lieb-Thirring work on stability of matter are the Lieb-Thirring inequalities. These are generalizations of Sobolev's inequality to orthogonal systems of functions, incorporating in this way the Pauli exclusion principle. These two papers give a detailed account of these results. The key estimate is a bound on the moments of eigenvalues of a Schrödinger operator on $L^2(\mathbb{R}^n)$,

$$-\Delta + V \ ,$$

where $V(x)$ is a potential and Δ is the Laplace operator. Denote by $\lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$ the negative eigenvalues of this operator. The following inequality was proved in **III.1**:

$$\sum_j |\lambda_j|^\gamma \le C(\gamma) \int V_-(x)^{\gamma+n/2} \mathrm{d}x \ , \tag{3}$$

where γ is a positive constant whose allowed values, meaning the cases for which $C(\gamma)$ is finite, depends on the dimension n. Here, $V_-(x)$ denotes the negative part of $V(x)$.

The particular case $\gamma = 1$ for $n = 3$ is used in the Lieb-Thirring proof of stability of matter. One of the crucial inequalities is a lower bound on the kinetic energy of N fermions, i.e., the Sobolev-type inequality alluded to before. Consider an N-electron wave function $\psi(x_1, \ldots, x_N)$ normalized by $\int |\psi|^2 \mathrm{d}x_1 \cdots \mathrm{d}x_N = 1$. (We neglect spin here for simplicity.) The Pauli exclusion principle states that any wave function describing fermions must be antisymmetric under interchange of the particle labels. The kinetic energy of such a function is defined to be

$$T(\psi) = \sum_{i=1}^N \int_{\mathbb{R}^3} |\nabla_i \psi(x_1, \ldots, x_N)|^2 \mathrm{d}x_1 \cdots \mathrm{d}x_N \ .$$

The one-particle density $\rho_\psi(x)$ which is a function on $\mathbb{R}^3$ is defined by

$$\rho_\psi(x) = N \int |\psi(x, x_2 \ldots, x_N)|^2 \mathrm{d}x_2 \cdots \mathrm{d}x_N \ .$$

(Since $|\psi|^2$ is symmetric in its N variables, it does not matter which $N-1$ variables we integrate over.) The following inequality is not hard to deduce from (3) setting $n = 3$ and $\gamma = 1$,

$$T(\psi) \ge c \int_{\mathbb{R}^3} \rho_\psi(x)^{5/3} \mathrm{d}x \ ,$$

for some explicit constant c. Except for an altered value of the constant c, the right side is precisely the kinetic energy in Thomas-Fermi theory — an approximate theory in which all the physical quantities are expressed in terms of the one particle density ρ_ψ. Without the antisymmetry (Pauli principle), the best lower bound we would be able to derive is $T(\psi) \geq cN^{-2/3} \int_{\mathbb{R}^3} \rho_\psi(x)^{5/3}\mathrm{d}x$. An account of this theory and its relation to the stability of matter and thermodynamics can be found in [Li1]. Also mentioned in this paper is the "cheese" theorem which Lieb proved jointly with Lebowitz in [LiLe] and which, according to Mandelbrot [Ma], is a contribution to fractal geometry. The "cheese" theorem is about efficient packing of balls in $\mathbb{R}^n$, whose radii decrease geometrically and which cover $\mathbb{R}^n$ up to a set of measure zero.

III.3: The deepest and, in some sense, most useful form of the Lieb-Thirring inequality is the case $\gamma = 0$ which simply counts the number of eigenvalues of this Schrödinger operator. This bound, which works for all dimension $n \geq 3$ is the celebrated Cwikel-Lieb-Rozenblum bound. It says that the number of negative energy bound states in a Schrödinger operator is measured by the volume of the region in phase space where the energy $p^2+V(x)$ is negative. This volume is given by $c_n \int V_-(x)^{n/2}\mathrm{d}x$ where c_n is the volume of the n-dimensional unit ball. Three very different proofs were given, one by Rozenblum, one by Cwikel and the one by Lieb, almost at the same time. While Rozenblum's proof [Roz] was first and Cwikel's proof [Cw] is the most general, Lieb's proof furnished the best constant among the three. Further improvements on the constant were given by Lieb in **III.6** and also in [BS]. The sharp constant is not known. All the other relevant Lieb-Thirring bounds (not necessarily with the best constant) can be deduced from this one.

III.4: What makes atomic physics difficult is, besides the kinetic energy, the presence of the Coulomb repulsion among electrons. If we are given a N-electron wave function $\psi(x_1, \ldots, x_N)$ (we neglect the spin), then the Coulomb repulsive energy is given by the expectation value

$$C(\psi) := \sum_{i<j} (\psi, |x_i - x_j|^{-1}\psi) \ .$$

In the spirit of Thomas-Fermi theory one would like to replace this energy by the repulsive energy of the charge density ρ_ψ, i.e.,

$$D(\rho_\psi) := \frac{1}{2} \int \frac{\rho_\psi(x)\rho_\psi(y)}{|x-y|} \mathrm{d}x\mathrm{d}y \ .$$

The difference $C - D$ is called the exchange energy and in this paper, which is jointly with Oxford, an effective bound on the exchange energy was calculated.

In particular it is shown that the inequality

$$C(\psi) - D(\rho_\psi) \geq -1.68 \int \rho_\psi(x)^{4/3}\mathrm{d}x \tag{4}$$

holds for all wave functions ψ (irrespective of their symmetry properties). A similar estimate with 1.68 replaced by 8.4 was proved by Lieb earlier [Li2]. Inequality (4) is often used by quantum chemists. It also leads, together with the Lieb-Thirring inequality above, to a lower bound for the true quantum mechanical energy by Thomas-Fermi theory, which, in turn, leads to the celebrated Lieb-Thirring proof of the stability of matter.

Actually, in the original Lieb-Thirring proof of stability of matter [LT], which was much earlier than [Li2], another bound, $C\sqrt{N}\left(\int \rho^{5/3}\right)^{1/2}$, was used in place of $C\int \rho^{4/3}$ on the right side of (4).

III.5: This paper deals with general density functional theory rather than a particular theory, such as the Thomas-Fermi theory. Consider a many-body Hamiltonian describing an atom or, more ambitiously a solid, in which the electrons are the only quantum mechanical particles and the nuclei are fixed. Typically, such a Hamiltonian is made up of two types of interactions: single particle interactions such as the interaction of the electrons with the potential V due to the nuclei, and two-body interactions such as the Coulomb repulsion among the electrons. The latter is considered fixed, but the effective one-body potential V will depend upon the particular system under consideration. In a famous 1964 paper, Hohenberg and Kohn [HK] showed that if the one-body potentials for two systems differ by more than a constant, then the single-particle electron densities associated with their respective ground states must also be different. (Since, in general, a given one-particle density can be obtained from many different many-body wave functions, this result was regarded as somewhat surprising.) Thus, knowing the ground state electron density (a function of three space variables) determines the potential V which, in turn, determines the ground state, a function on the high dimensional configuration space.

As a consequence, there must be a universal functional $F(\rho)$ of the density ρ that should describe the ground state energy of atoms and maybe their low lying excited states. Despite the obvious difficulty of finding the universal functional $F(\rho)$, Hohenberg and Kohn's result attracted a good deal of attention and was used as the basis for a number of approximation schemes.

In **III.5** Lieb formulated an explicit variational principle for a related functional that does not require knowledge of the mysterious "universal functional" and showed its connection to Hohenberg-Kohn theory and a number of other density functional theories.

Much of this paper may seem rather technical. However, it also establishes the importance of such technicalities for real physical systems, particularly when one wants to use a functional so complicated that one cannot expect every smooth density to arise from a "nice" potential or vice versa.

This important paper attempts to provide a firm mathematical foundation for density functional theory. It proves many results, both fundamental and technical, and raises a number of open question. After first appearing in a collection of papers in honor of Tisza in 1982, it was reprinted (with minor revisions) twice — first in the *Int. J. Quant. Chem.* (the version which appears here) in 1983 and again in a collection of papers on density functional theory in 1985.

III.6: This paper is an unexpected application of Lieb-Thirring inequalities to the computation of the Hausdorff dimension of the attractor for the three dimensional Navier-Stokes equations (modulo the annoying fact that the global solutions of the Navier-Stokes equations are not known to exist). It was realized by Ruelle [Ru] that the Hausdorff dimension can be bounded by sums of eigenvalues of a Schrödinger operator. His estimates used Weyl asymptotics and hence did not give any bounds on the Hausdorff dimension of the attractor for a fluid in a finite volume. It also relied on some unproven conjectures about certain constants. In **III.6**, Lieb implemented Ruelle's idea. Among other results an improvement on the constant of the Cwikel-Lieb-Rozenblum inequality was obtained.

III.7: Consider a Hamiltonian of three nonrelativistic particles interacting with a pair potential of the form $V(x-y)$. The three particles have mass m_1, m_2, m_3 not necessarily the same. Denote the ground state energy by $E(m_1, m_2, m_3)$. This ground state energy can be thought of as the mass of a baryon created by the quarks of mass m_1, m_2, m_3. It was conjectured by particle physicists that the inequality

$$E(m,m,m) + E(m,M,M) \leq 2E(m,m,M) \ ,$$

should hold quite generally for all masses m and M and potentials. Lieb proves this for a certain class of potentials, but also gives some potentials and masses $m > M$ for which this is not true, thus destroying the hope for a potential and mass independent inequality. The proof, characteristically, proceeds via the Trotter formula using the positive definiteness of certain kernels and also works for relativistic forms of the kinetic energies. The condition on the potential V for which the inequality holds appears to be close to optimal.

III.9, III.2: Independent of what has been said in the previous paragraphs, sharp constants in Lieb-Thirring inequalities provide an interesting playing field for analysis. There are a number of conjectures, some of them due to Lieb and Thirring that are open to this day. Already the one-dimensional case ($n = 1$) offers a host of challenging problems. It was relatively recently that Weidl [Wei] was able to establish a Lieb-Thirring inequality in one dimension for $\gamma = 1/2$. Relatively soon after that Lieb, jointly with Hundertmark and Thomas [**III.9**] were able to show that 1/2 is the sharp constant for this case and that the optimizing potential is a delta function and supports only one bound state. This phenomenon, namely that the optimizing potential should support only one bound state, is believed to hold for all $\gamma < 3/2$ for $n = 1$. It is known that for $\gamma = 3/2$ the classical phase space average also gives the sharp constant. This result, which can be found in **III.1** for the one dimensional case was recently extended by Laptev and Weidl [LaWei] to any dimension $n \geq 1$. In all dimensions and for $\gamma \geq 3/2$ the sharp constant in the Lieb-Thirring inequality is given by the classical constant. Indeed, it is a fairly general argument due to Aizenman and Lieb (see **III.2**) that if the classical constant is sharp for the Lieb-Thirring inequality for some γ it remains the sharp constant for all larger values of γ. Thus the classical constant is sharp for all $\gamma \geq 3/2$.

Part IV. Coherent states

Coherent states, originally discovered by Schrödinger, were fashioned into an incisive tool by Bargmann, Segal and Glauber, which is useful whenever one compares a quantum mechanical model with its classical counterpart.

IV.1: Here Lieb uses the less familiar Bloch coherent states to bound quantum mechanical partition functions Z in terms of classical ones. More precisely, with $f = -\beta^{-1} \ln Z$ denoting the free energy, he proves that

$$f^{\text{classical}}(J, \beta) \geq f^{\text{quantum}}(J, \beta) \geq f^{\text{classical}}(J+1, \beta)$$

where J is the spin of the quantum model, e.g., a Heisenberg ferro- or antiferromagnet consisting of spin J operators, and β denotes the inverse temperature.

The main tool, which is the general idea of coherent states, is to represent (or in some cases approximate) operators as superpositions of projections onto coherent states. For spin systems one can represent any operator on spin space by

$$\int_{\mathbb{S}^2} f(\Omega)\Pi(\Omega)\mathrm{d}\Omega$$

where f is a function on the unit sphere that can be explicitly calculated for a given operator. The projection $\Pi(\Omega)$ is the projection onto the eigenvector of $\Omega \cdot S$ with the largest eigenvalue J. Here Ω is a unit vector in $\mathbb{R}^3$ and $S = (S_1, S_2, S_3)$ are the spin matrices. Upper bounds and lower bounds of this type – obtained with the aid of coherent states– are not restricted to spin systems and are known as Berezin - Lieb inequalities (see also [Ber]).

IV.2: In quantum mechanics the entropy of a state is defined (von Neumann) in terms of the state's nonnegative density matrix ρ by $S_{\text{quantum}} = -\text{Trace}\rho \log \rho = -\sum_j \lambda_j \log \lambda_j$, where the λ_j are the eigenvalues of ρ and whose sum is one. (The right side of this formula is reminiscent of Shannon's definition of entropy with λ_j being probabilities in his case.) Clearly, $S \geq 0$. The density matrix is $\rho = Z^{-1} \exp[-\beta H]$, where H is the Hamiltonian, Z^{-1} is a normalization constant, and β is the reciprocal of the temperature. Usually, $H = p^2/2m + V(x)$.

What has been done since the time of Boltzmann is to approximate the quantum entropy by the "classical entropy", $S_{\text{classical}} = -\int \rho_{\text{class}}(p,x) \log \rho_{\text{class}}(p,x) dpdx$, where $\rho_{\text{class}}(p,x) = N' \exp[-(p^2/2m + V(x)/T]$, but this quantity does not have to be positive. Indeed, it tends to $-\infty$ as T tends to zero.

What is a good classical approximation to the quantum entropy that does not have this defect? Wehrl's idea (which also remedies some other defects of the Boltzmann definition) is to take the expectation value of ρ in a (Schrödinger- Bargmann- Segal- Glauber) coherent state $\psi_{p,x}$, namely, $\rho_{\text{Wehrl}}(p,x) = (\psi_{p,x}, \rho\psi_{p,x})$, and then use Boltzmann's formula to define

$$S_{\text{Wehrl}} = -\int \rho_{\text{Wehrl}}(p,x) \log \rho_{\text{Wehrl}}(p,x) \mathrm{d}p\mathrm{d}x \ .$$

Wehrl proved that $S_{\text{Wehrl}} > 0$ and conjectured that its minimum value was the number 1, which occurred when ρ is itself a one-dimensional projector onto a coherent state.

This was proved by Lieb in this paper by combining the knowledge of two sharp constants: in Young's inequality and in the Hausdorff-Young inequality. Moreover, Lieb went on to propose a similar entropy construction for the coherent states appropriate to representations of $SU(2)$ as in **IV.1** and conjectured that one-dimensional projectors would again give the minimum. These conjectures are still open after almost a quarter century, but two special cases have been verified recently by Schupp [Schu].

IV.3: In this paper, which is joint work with Solovej, the ideas of **IV.1** are generalized in that one does not compare the quantum model with spin J with the classical one, but with another quantum spin K model. A consequence is the following inequality between the free energy of the Heisenberg ferromagnet $f_f(J,\beta)$ and the Heisenberg antiferromagnet $f_a(J,\beta)$:

$$f_a\left(J, \frac{J}{J+1}\beta\right) \leq f_f(J,\beta) \leq f_a\left(J, \frac{J+1}{J}\beta\right)$$

and

$$f_f\left(J, \frac{J}{J+1}\beta\right) \leq f_a(J,\beta) \leq f_f\left(J, \frac{J+1}{J}\beta\right) \ .$$

This is remarkable since on the classical level the two models, the ferromagnet and the antiferromagnet are the same while quantum mechanically they are different. The above estimates delineate the difference. The paper also contains inequalities between f_f and f_f and between f_a and f_a for different J-values.

Part V. Brunn-Minkowski inequality and rearrangements

Rearrangement inequalities have a fairly short history starting with the work of Hardy, Littlewood, Polya, Riesz, Faber and Krahn in the thirties. The definition can easily be understood by visualizing a function f of two variables in the plane as a mountain range (without overhangs of course). Cutting this mountain range at a height h displays a level set of the function,

$$S_f(t) := \{x : f(x) > h\} \ .$$

This will be now 'rearranged' into a disk at the same height with the same Lebesgue measure and centered at the origin. Doing this with all level sets of this function yields level sets of a new 'rearranged' function f^*. This function will be obviously radial and nonincreasing. The value of this operation lies in the fact that it yields various very useful inequalities. A particularly important one is Riesz's inequality, which states that for three integrable nonnegative functions f, g, h

$$\int f(x)g(x-y)h(y)\mathrm{d}x\mathrm{d}y \leq \int f^*(x)g^*(x-y)h^*(y)\mathrm{d}x\mathrm{d}y \ .$$

Another consequence of rearrangements is the isoperimetric inequality in the form

$$\|\nabla f\|_p \geq \|\nabla f^*\|_p \ . \tag{5}$$

The case $p = 2$ leads to the famous Faber-Krahn inequality which says that the lowest frequency among drums of equal area is achieved for the disk. The case $p = 1$ can be "understood" using the co-area formula

$$\int |\nabla f(x)|\mathrm{d}x = \int_0^\infty \int_{\{f(x)=t\}} \mathrm{d}\mathcal{H}^{n-1}\mathrm{d}t \ ,$$

where $\mathcal{H}^{n-1}$ is the $n-1$ dimensional Hausdorff measure. Since the right is given by the 'surface area' of the level set at height t it follows from the isoperimetric inequality that this integral gets smaller when f is replaced by f^*. The proof of this fact is non-trivial and is due to Hilden [Hi]. In general, the level surfaces, of such a function, i.e., the boundary of the level sets, are not surfaces in the usual sense of differential geometry - which is the difficulty with this approach.

V.1: Jointly with Brascamp and Luttinger, Lieb showed that the Riesz rearrangement inequality can be extended to any number of functions. It is interesting to note that the proof of this general rearrangement inequality proceeds via the Brunn-Minkowski inequality. This inequality, in its simplest form, says that the n-th root of the volume of a slice of a convex body in $\mathbb{R}^{n+1}$ is concave as a function of the position of the slice. In particular this implies that the logarithm of this volume is concave. Let us also note that the Brunn-Minkowski inequality provides one of the standard proofs of the isoperimetric inequality.

V.2: This joint paper with Brascamp contains a further application of the Brunn-Minkowski inequality. In this paper they show that the marginals of log concave functions are log concave. Besides the application to rearrangements mentioned above, this innocuous statement has further interesting consequences for Wiener integrals. It is shown in **II.2** that the ground state wave function of a Schrödinger operator in a convex domain with Dirichlet boundary conditions and with a convex potential is itself a log concave function, thereby proving a long-standing conjecture of Payne. Subsequently this result was reproved by [SMYY] using the maximum principle and extended by Borell [Bo] who proved some concavity properties for Wiener integrals jointly in the space and time variables.

A particularly beautiful result related to log concavity is the existence of the quantum mechanical Wigner crystal in one dimension. Consider N particles moving inside an interval which interact among each other via a one - dimensional Coulomb repulsion $-|x|$. The interval also carries a uniform charge distribution of opposite sign. It is not hard to see, classically, that the lowest energy configuration is attained for evenly spaced particles, i.e., they form a periodic array. It was shown in **V.2** that this fact survives in quantum mechanics. Again, it is a consequence of the Brascamp-Lieb argument about log concave functions together with Trotter's formula. The quantum mechanical analog of a periodic array is that the one particle marginal of the ground state of these N particles is a periodic function of period L/N where L is the length of the box. The existence of a Wigner crystal in higher dimensions is still an open problem, even on the classical level.

V.3: This is again an application of rearrangements but with a surprising twist. The idea is to connect rearrangements with the tensor product. Young's inequality is

$$|\int f(x)g(x-y)h(y)\mathrm{d}x\mathrm{d}y| \le C\|f\|_p\|g\|_q\|h\|_r$$

where

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=2\ .$$

Although Riesz's rearrangement inequality applies here it does not lead to any standard variational problem. A crucial observation is that the inequality is preserved when the variables are doubled, except that C is then replaced by C^2. This leads to a natural iteration scheme. Take radial functions f, g, h, double their variables, i.e., consider $f^{(2)}(s,t) = f(s)f(t)$, etc. and rearrange these functions (which are functions of two variables now). It is clear that Gaussian functions are invariant under this operation but less clear that they are the only ones. The point, however, is that when this operation is repeated indefinitely then a central limit theorem holds, i.e., the sequence converges to 'Gaussian' functions. Loosely speaking, the only functions in infinite dimensions that are invariant under rotations are Gaussian functions. In this fashion it is established in **V.3** that Gaussians are optimizers in Young's inequality. There are many alternative proofs, one by Beckner [Bek] using an idea of Gross [Gr], which is also related to the central limit theorem. Certain special cases of this inequality were considered by Nelson [Ne], Glimm [Gl], Simon [Si2] and Segal [Se] who was the first to point out the fact concerning the doubling of variables for tensor products of positivity preserving kernels from L^p to L^q, p and q arbitrary. If the kernels are not positivity preserving then the doubling of variables works also provided that $q \geq p$. This is known as Beckner's lemma. The 'doubling of variables' idea resurfaces again in **VI.7**.

V.4: Another outcome of these ideas is the generalization of the Brunn-Minkowski inequality to measurable sets which leads among other things to generalizations of inequalities of Prekopa - Leindler. Consider two nonnegative functions f and g on $\mathbb{R}^n$ and pick any number $0 < \lambda < 1$. Consider the function h define by

$$h(z) := \operatorname{ess\,sup}\{f(x)^\lambda g(y)^{1-\lambda} : z = \lambda x + (1-\lambda)y\} \ .$$

Then

$$\int_{\mathbb{R}^n} h(x)\mathrm{d}x \geq \left(\int_{\mathbb{R}^n} f(x)\mathrm{d}x\right)^\lambda \left(\int_{\mathbb{R}^n} g(x)\mathrm{d}x\right)^{1-\lambda} \ .$$

This inequality is a generalization of the Brunn-Minkowski inequality. Pick any two *measurable* sets A and B and define their convex addition

$$\lambda A + ((1-\lambda)B := \{z : z = \lambda x + (1-\lambda)y, x \in A, y \in B\} \ ,$$

then the volumes satisfy the inequalities

$$|\lambda A + (1-\lambda)B| \geq |A|^\lambda |B|^{(1-\lambda)} \ .$$

When A is a ball and B a single point, then the right side vanishes while the left side is large. Note that the convex addition is all the points z in $\mathbb{R}^n$ such that $(z - \lambda A) \cap ((1-\lambda)B) \neq \emptyset$. In **V.4** the Brunn- Minkowski inequality is proved for the weaker notion of convex addition which is all points z in $\mathbb{R}^n$ such that $(z - \lambda A) \cap ((1-\lambda)B)$ has positive measure.

The Prekopa-Leindler inequality, and hence the Brunn-Minkowski inequality was generalized to Riemannian manifolds in [CMS] by using the Monge transportation problem in a surprising fashion. It seems that the transportation problem plays a role in all of this that has not been fully understood. E.g., the solution of the transportation problem together with the Henstock parametrization, has recently been used by Barthe [Bar] to give an alternative way to determine the sharp constant in Young's inequality.

V.5: In general, rearrangement inequalities are not strict; the L^p norms and Sobolev norms furnish examples. There are however exceptions and the paper **V.5** presents an important example. It says that when the middle function in Riesz's rearrangement inequality is strictly decreasing, then one can have equality in Riesz's rearrangement inequality only if the other two functions are symmetric decreasing about a common point in $\mathbb{R}^n$. This fact is used in **V.5** to prove uniqueness of solutions to Choquard's equation which is, by the way, the same as Pekar's equation that occurs in the study of the polaron model.

It is one of the few general methods to decide the cases of equality in variational problems. It is worth noting that this paper contains a very natural and simple proof of (5) for the case $p = 2$, as a consequence of Riesz's rearrangement inequality in which the middle function g is given by the heat kernel.

V.6, V.7: We have seen before that rearrangements decrease gradient norms. Thus, it is not unreasonable to expect that rearrangements act as a contraction on H^1, the space of square integrable functions whose gradient is also square integrable. It is a fact [CraTa, Chi, CaLo, LiLo] that rearrangement is a contraction on all L^p spaces with $p < \infty$. Simple arguments show that this cannot be true for H^1, however. A reasonable question is whether rearrangement is continuous in H^1. This question was affirmatively answered by Coron [Cor] in one-dimension. In higher dimensions Almgren and Lieb showed that it is wrong in general. The obstruction to this continuity is a peculiar property of functions. Intuitively, functions whose derivative vanish on a set of large measure should have large 'flat' spots. Flat spots reveal themselves in the distribution function as discontinuities. The distribution function is the measure of the level set at height t as a function of t. It is nonincreasing and, by choice, lower semicontinuous. Thus, if the function is constant on a set of positive measure, the distribution function should be discontinuous. Surprisingly, it was discovered in **V.6** that there exist functions in two or more variables whose derivatives vanish on sets of large Lebesgue measure, but that have absolutely continuous distribution functions. Almgren and Lieb construct explicit examples of such functions and call them co-area irregular functions. Returning to the question of the H^1-continuity of rearrangement, Almgren and Lieb pick such a co-area irregular function f and consider the sequence

$$f_j(x) = f(x) + \frac{1}{2j} W_j(x) \sin(jf(x)) \ .$$

Here, the function W_j is a smoothed characteristic function of the set where the derivative of f vanishes. The smoothing can be done in such a way that $\|W_j\|_\infty \leq \sqrt{j}$. It is straightforward to see that ∇f_j converges strongly to ∇f in L^p. Further, some elementary arguments show that the graphs of the rearranged functions f_j^* and f^*, when parametrized by height t intersect when $t = 2m\pi/j$ and the graph of f_j^* sits above the graph of f^* whenever $t = (2m + \sigma)(\pi/j)$, for $0 < \sigma < 1$. As j tends to infinity the points where the two graphs intersect move closer but the 'wiggles' in between persist. In fact it is shown in [**V.6, V.7**] that $\|\nabla f_j^* - \nabla f^*\| \geq c > 0$ for c independent of j.

Almgren and Lieb go far beyond this example by showing that the dichotomy of coarea regularity and irregularity of the limit function precisely characterizes the sequences for which rearrangement is H^1- continuous. All this is explained in great detail in the paper [AlLi], which due to its length, is not part of this Selecta. Instead we chose to reproduce the summary papers **V.6** and **V.7**. In **V.7** the notion of current is used while **V.6** and [AlLi] rely on more 'elementary' methods.

One inequality in the big paper that should be mentioned is what one could call the ultimate version of Riesz's rearrangement inequality. Consider a function $F(u,v)$ with $F(0,0)=0$, and assume that the mixed derivative $\partial_u\partial_v F$ is nonnegative. Then

$$\int\int F(f(x),g(x))W(ax+by)\mathrm{d}x\mathrm{d}y \leq \int\int F(f^*(x),g^*(x))W^*(ax+by)\mathrm{d}x\mathrm{d}y$$

where W is nonnegative and integrable and a and b are nonzero numbers.

Let us remark, finally, that in the case of Steiner symmetrization the situation is completely different. In fact, Steiner symmetrization is continuous in H^1, as was shown by Burchard [Bu].

Part VI. General analysis

VI.1: This paper deals with a similar issue as in the Lieb-Thirring inequalities. Given and orthonormal family of functions ψ_i, $i=1,\dots N$ in $L^2(\mathbb{R}^n)$ and define

$$u_i = \left(-\Delta + m^2\right)^{-1/2}\psi_i \quad \text{and} \quad \rho(x) = \sum_{i=1}^{N} |u_i(x)|^2 \ .$$

Among other results, Lieb, generalizing previous results by Battle-Federbush [BaFe] and Conlon [Co], proves that

$$\|\rho\|_p \leq A_n N^{1/p}, \ p = \frac{n}{n-2} \ ,$$

where A_n is a universal constant. The remarkable fact about this inequality is that due to the orthogonality of the ψ_i the constant is proportional to $N^{1/p}$. Without the orthogonality condition the best one could have is a factor N.

VI.2: Real analysis is an extremely well developed subject, with many textbooks devoted to it and one is led to believe that all the simple and useful theorems have been discovered. This paper, which is joint work with Brézis adds a further theorem, a simple, but nontrivial fact about pointwise convergence, to this list. It is the physicists way of looking at the problem of pointwise convergence, maybe a reason why it was missed by the masters of the subject. Fatou's lemma states that for any nonnegative sequence of integrable functions f_n that converges pointwise to an integrable function f,

$$\liminf_n \int f_n(x)\mathrm{d}x \geq \int f(x)\mathrm{d}x \ .$$

In general the inequality is strict. How can sequences of functions fail to converge? There are three intuitive ways: the sequence can oscillate to 'death', it can go 'up the spout' or it can escape to infinity. This is quantified by the correction term to Fatou's lemma, proved in **VI.2**, i.e.,

$$\int f_n(x)\mathrm{d}x = \int f(x)\mathrm{d}x + \int |f(x)-f_n(x)|\mathrm{d}x + o(1) \ .$$

The reader might feel that such a statement might not mean much but is convinced otherwise when he sees what kind of variational problems can be solved using that fact, examples of which can be found in **VI.2**. Certainly, such a theorem should be standard in any book on real analysis.

VI.3: A very influential paper of Lieb, instigated by Sokal in connection with certain problems of quantum field theory, is his work on the sharp version of the Hardy-Littlewood-Sobolev inequality. This inequality is well known and states that

$$\int_{\mathbb{R}^{2n}} \frac{f(x)g(y)}{|x-y|^\lambda} \mathrm{d}x\mathrm{d}y \le C(n,p,\lambda)\|f\|_p\|g\|_q$$

provided that

$$\frac{1}{p}+\frac{1}{q}+\frac{\lambda}{n}=2 \ , \qquad\qquad 1 < p,q < \infty \ .$$

The problem is to find the sharp constant $C(n,p,\lambda)$ and all the cases of equality. Merely proving the inequality itself with any constant is not easy. The problem was solved in one-dimension by Hardy and Littlewood in the early thirties and shortly thereafter Sobolev generalized it to arbitrary dimensions. One should also mention that Riesz, by an ingenious application of the inequality between the arithmetic and geometric mean derived the one in higher dimension from the case in one-dimension.

The search for the sharp constant is, however, considerably more difficult. This is due to the fact that the functional is invariant under translation and scaling and that makes it very likely that maximizing sequences converge weakly to zero. Note that the translation invariance can be broken by rearrangements but it is unclear, a priori, which mechanism breaks the scale invariance of the prospective optimizer. In **VI.3** Lieb shows quite generally that despite this difficulty, the optimizer always exists. The argument proceeds via rearrangements and using logarithmic variables, which display the scale invariance as translation invariance. He shows that the minimizing sequence cannot tend weakly to zero.

The paper is a real tour de force and contains many more interesting results. Chief among these is the actual evaluation of the sharp constant in the case where $p=q$, including all the cases of of equality. This case is particularly interesting since the functional is conformally invariant. An interesting point is that this large non compact symmetry group, which is generally an obstacle to proving the existence of optimizers, is now used to determine the explicit form of the optimizers themselves. This apparent dichotomy was overcome in [CaLo] to give a simple alternative proof (which uses "competing symmetries") of the sharp HLS inequality for the case $p=q$.

In some sense, the sharp HLS inequality for $p=q$ is the mother of all conformally invariant inequalities. Most of the conformally invariant inequalities, without any hard analysis, can be derived from the HLS inequality. To give one example, by differentiating the HLS inequality at $\lambda=0$, which can be done provided f,g are non- negative, one obtains the logarithmic HLS inequality. Dual to this is the Onofri-Beckner inequality, which gained great popularity recently [On, Bek2, CaLo2, OPS].

VI.4: It is intuitively clear that if the lowest Dirichlet eigenvalue of a domain is small then the set must be somehow 'fat'. It is surprisingly difficult to give a precise notion of this fact, one of which is given in this paper. Given two domains A and B, denote by $\lambda(A)$ the lowest eigenvalue of the Laplacian on this domain with Dirichlet boundary conditions. Then there exists always a translate B_x of B such that

$$\lambda(A \cap B_x) < \lambda(A) + \lambda(B) \ .$$

In particular if, for some given ball B, $\lambda(A \cap B_x)$ is large for all translations x, then $\lambda(A)$ must be large too. Thus, a set of a granular consistency must have a large Dirichlet eigenvalue. This intuition is also the basis of the following compactness result, which can also be found in **VI.4**. Consider a bounded sequence in $H^1(\mathbb{R}^n)$. In general one can extract a weakly convergent subsequence by the Banach-Alaoglou theorem. In many applications, however, one would like to ensure that the limit is not the zero function. One possibility is that the measure of the level set $\{x : f_n(x) > t\}$ (here t is some positive number) is uniformly bounded below by a constant ε. The reader might immediately object that such a condition is certainly not sufficient, since the sequence might successively split into smaller and smaller islands and, in this way, tend weakly to zero. This is precisely what the result explained above prevents from happening. Such a splitting would increase the $H^1(\mathbb{R}^n)$-norm beyond any bound. Thus the sequence cannot 'pulverize', it has to stay together. It is shown in this paper that there exists a sequence of translations y_n, so that the sequence $f_n(x - y_n)$ has a subsequence that has a nonzero weak limit.

There is a fairly extensive literature on the concentration phenomenon. It has been summarized in the papers [Lio], where it is shown that the limiting behavior of weak sequences can be reduced to studying a finite number of cases. In actual applications it is important to decide which of these cases is at hand and the convergence theorem mentioned above provides one of the tools for deciding that.

VI.5: This paper was the motivation of the above mentioned convergence theorem. The problem is to establish existence of solutions of the following systems of partial differential equations on $\mathbb{R}^n$:

$$-\Delta u_i(x) = g(u_1(x), \ldots, u_N(x)) \ , \ i = 1, \ldots, N \ .$$

Here $g_i = \partial_i G$ where G satisfies various assumptions. The solutions of these equations are critical points of the action functional

$$S(u) = \sum_i \int |\nabla u_i(x)|^2 \mathrm{d}x - \int G(u(x)) \mathrm{d}x \ ;$$

however, this functional is unbounded below. Using the idea of a constrained minimum by Coleman, Glaser and Martin [CGM] i.e., minimizing the first term in the action while keeping the second one fixed, Lieb, together with Brézis, establish existence of solutions of this system of equations and show that they have finite action. Moreover, they show that among all those solutions there is one that has minimal action. This problem is typical for not having obvious compactness properties. Weak sequences tend to leak out to infinity and **VI.4** provides the ideal tool for 'running after them'.

VI.6: In this paper, jointly with Brézis, Lieb improves on a Sobolev inequality with remainder, invented in [BN]. In that paper it was shown that

$$\|\nabla f\|^2 - S_n\|f\|_{2^*} \geq C_p(\Omega)\|f\|_p^2$$

holds for all functions $f \in H_0^1(\Omega)$ and for any $p < n/(n-2)$. The constant $S_n$is the sharp Sobolev constant. The improvement is that the right hand side can be replaced by $\|f\|_{p,w}$, the weak L^p norm, but with $p = n/(n-2)$. This improvement is used again in [DaLi] on the stability of relativistic molecules.

VI.7: Here, Lieb revisits a topic related to Young's inequality. The Fourier transform can be interpreted as a Gaussian integral kernel applied to a function. The Hausdorff-Young inequality says that

$$\|\widehat{f}\|_p \leq C_p\|f\|_q$$

where

$$\frac{1}{p} + \frac{1}{q} = 1 \ , 1 \leq q \leq 2 \ ,$$

and C_p is a universal constant. A famous result of Babenko and Beckner [Bab,Bek] states that the sharp constant in the Hausdorff-Young inequality is attained by Gaussian functions. In this paper Lieb establishes that for $1 < q < 2$ equality can occur only for Gaussian functions. One of the key tools, as in Beckner's lemma mentioned in **V.3**, is the sharp version of Minkowski's inequality. Its application is, however, quite different and allows Lieb to extract information of complex valued functions and not just nonnegative functions as is the case for Young's inequality.

Another high point of this paper is the ultimate version of Young's inequality. In essence it states that any integral of a product of L^p functions with various combination of arguments is maximized by Gaussian functions. This was recently reproved by Frank Barthe [Bar] using the Henstock parametrization and the solution to the Monge-Kantorovich mass transport problem.

VI.8: The following problem is related to **VI.7**. Consider the Woodward 'ambiguity function', which is very closely related to the Wigner transform,

$$A_{f,g}(\tau,\omega) = \int f(t-\tau/2)\overline{g(t+\tau/2)}e^{-2\pi i\omega\tau}\mathrm{d}t \ ,$$

where $f \in L^p$ and $g \in L^q$ with $1/p + 1/q = 1$. Lieb calculates sharp bounds on the magnitude of this ambiguity function. These bounds are saturated precisely when f as well as g are Gaussian functions. Depending on the values for p these bounds are upper or lower bounds. Although the paper is much in the spirit of **VI.7**, it is by no means a corollary of that paper.

Part VII. Inequalities related to harmonic maps

In the early eighties Lieb started to develop an interest in problems that have a geometric flavor to them, like liquid crystals or Stability of Matter with magnetic fields.

He was also very attracted by the Skyrme model that, although invented in the sixties, got a new lease on life, e.g., through the work of Adkinson, Nappi and Witten [ANW]. Lieb never wrote a research paper (apart from a review paper [Li3]) on that subject but made contributions to the theory of harmonic maps which is a closely related subject.

VII.1: The first paper devoted to this subject, which is joint with Brézis and Coron, is the problem of minimizing the functional

$$\int_D |\nabla\phi(x)|^2 \mathrm{d}x \tag{6}$$

where D is a domain in $\mathbb{R}^3$ and ϕ is a map from D into the sphere $\mathbb{S}^2$. Imagine now that the domain has small holes and specify the degree of the map near each hole and on the boundary. The problem is how to minimize the above functional over all those maps that have these specified degrees.

Pair the holes with positive degree with the holes that have negative degree and ignore the holes that have zero degree. Adding up the distances of all these pairs yields a length $L(C)$ that depends on this pairing C. It is shown in **VII.1** that the minimum of (6) equals $8\pi L$ where L is the minimum of $L(C)$ over all pairings. There is no minimizer for this problem. The intuition is that the $|\nabla\phi(x)|^2$ of an 'approximate minimizer' deteriorates into a measure that is concentrated on these lines connecting the paired holes.

VII.2, VII.4: The following question is now natural. Consider a map from a ball in $\mathbb{R}^3$ into $\mathbb{S}^2$ and fix the map on the boundary, say to be ψ. In many examples the minimizing map, which is called a harmonic map, develops singularities. This is at first surprising since one is dealing with an elliptic equation, but standard regularity results do not apply to elliptic *systems,* however.

Given this boundary data, can one estimate the number of singularities of the harmonic map in the interior? One knows by a result of Schoen and Uhlenbeck that the singularities must be isolated. In this paper with Almgren, a fairly complete solution of that question is given.

What determines the number of singularities is the Dirichlet integral of the boundary data and it does so in a linear fashion. In other words if we denote by $N(\psi)$ the number of singularities then there exists a constant C such that

$$N(\psi) \leq C \int_{\partial B} |\nabla\psi(x)|^2 \mathrm{d}S \ . \tag{7}$$

Based on **VII.1** one might guess that the modulus of the Jacobian of ψ integrated over the sphere is also a good candidate for estimating the number of singularities. This is completely false. Almgren and Lieb give an example of a boundary map where this quantity is zero but the corresponding harmonic map has many singularities.

The constant C in (7) is not known, not even remotely. This is due to the fact that the proof relies on compactness arguments. It should be said that there are no assumptions made about the minimizing map concerning its boundary regularity. In fact it is there where the interesting things are happening and where the singularities try to accumulate only to be stopped by the Dirichlet integral of the boundary data. Since the original paper is too long for this volume we reproduce here the summary papers, as we did for **V.6, V.7**

VII.3: How can one compute an area minimizing surface S that spans a closed curve C embedded in $\mathbb{R}^3$? One way, as shown in this paper, is to look at maps from $\mathbb{R}^3$ to the circle $\mathbb{S}^1$ with the property that at each point x of C, and for every little two-dimensional surface containing x and perpendicular to C, the map winds around $\mathbb{S}^1$ exactly once as we wind our way around x. If ϕ is such a map we can define its energy to be $\mathcal{E}[\phi] = \int_{\mathbb{R}^3} |\nabla\phi(x)|^2 dx$. The map that (almost) minimizes this energy (under the 'winding' condition mentioned above) is one that is constant everywhere in $\mathbb{R}^3$ except for a thin layer surrounding S. In this thin layer the map runs around $\mathbb{S}^1$ once as we go from the upper side of the layer to the lower side.

A similar construction can be made for 'minimizing' higher dimensional objects spanning lower dimensional objects in $\mathbb{R}^n$. The basic idea germinated from the paper **VII.1**, where it was shown that the geodesic between two points in $\mathbb{R}^3$? (namely, a straight line segment!) can be generated by minimizing the energy of maps from $\mathbf{R}^3$ to $\mathbb{S}^2$ with prescribed singularities (i.e., winding number 1) at each of the two points.

VII.5: A problem related to harmonic maps is the Ginzburg-Landau problem. In its simplest form it amounts to minimizing a certain energy functional over functions from the unit disc into the complex numbers having given boundary data. There are no general criteria for deciding whether a minimizer has certain symmetry properties of the problem (e.g., rotational symmetry) when the minimizer is a map or a vector field. For functions, there are useful devices like rearrangements but those techniques do not work for maps in general. In this paper it was shown that the minimizer in the radial class is weakly stable. There are better results available due to Mironescu [Mir] who proves that if the minimizer has a zero at the origin, then the minimizer is radial. It is surprising that even for this somewhat rudimentary model, the gap between intuition and proof is still large.

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