Stability of Matter

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Abstract

We review results concerning the problem of 'Stability of Matter'. Non-relativistic, relativistic quantum mechanics as well as matter interacting with classical fields is discussed and the strategies of the various proofs of these results is given in some detail. This is followed by a short discussion of the corresponding problem of matter interacting with the radiation field.

1 Introduction

With the term 'Stability of Matter' we summarize the simple observation that all material objects are extended in space and occupy a volume that is proportional to the mass of the object. The philosophical speculation as to why this is so goes back to the ancients and it became part of a serious scientific investigation as the atomic theory of matter took shape. With the discovery of the electron by J.J.Thomson (1899) and the experiments of Rutherford and his collaborators, Geiger and Marsden (1911), the following picture of atoms, the constituents of matter, emerged: An atom has a radius of about 10^{-10}m and consists of point electrons with charge $q = -e$ with $e = 1.60 \times 10^{-19}$ Coulombs whirling around a nucleus with positive charge $Q = Ze$ with $1 \leq Z \leq 92$, depending on which atom in the periodic table one is talking about. The nucleus has a radius which is several orders of magnitude smaller than the radius of an atom. The electrons and the nucleus interact with each other through electrostatic forces, i.e., Coulomb's law, Force $= qQ/r^2$ which is attractive for opposite charges and repulsive for like charges. Since bulk matter is always very nearly electrically neutral, the number of electrons, N, and the number of nuclei, K, are related by $N = ZK$

These facts made a qualitative understanding of matter very difficult. Not only does matter consist of hardly anything more than empty space, but the singularity of the attractive Coulomb interaction causes problems. Should it happen that an electron and a nucleus coalesce, there is no force strong enough to

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separate these two particles again. If this process gets repeated indefinitely, then, in the words of Jeans [26] ,"the matter in the universe would tend to shrink into nothing or to diminish indefinitely in size."

What is the mechanism that prevents this catastrophe, and keeps the electrons so far away from the nucleus ? One way to resolve this question, as Jeans speculated, would be to assume that the $1/r^2$ force law would not hold all the way down to the size of an atom. Rutherford and his collaborators, however, showed that this was unlikely, since the scattering cross section of the nucleus is, to a high degree of accuracy, that of a pointlike charge.

Going further, one may ask, how all these atoms conspire to give matter its bulkiness. It is our experience that two separate liters of a liquid, when poured together, give two liters of the same liquid, but it is not obvious why this is so since now twice as many molecules interact with each other. It is theoretically conceivable that the whole volume shrinks and we have less than two liters. This certainly happens when chemical reactions are involved, but even in that case the shrinking is small and of a different order of magnitude than the volume the liquid occupies. To bring this into sharper focus, let us concentrate on the internal energy instead. Suppose, for example, that the energy of a liter of water where proportional to $-N^{5/3}$ where N is the number of water molecules. The pouring of water would then have apocalyptic consequences. The energy difference before and after the pouring is $(2N)^{5/3} - 2N^{5/3}$ which has to be released in some form and is comparable to the energy set free in a big explosion. The only way to avoid such a catastrophe is that the **energy of bulk matter is proportional to the total number of particles**. This fact is what we call "Stability of Matter". The proportionality constant depends of course on the type of particles involved but not on their number.

The theory that finally shaped the above questions into a tractable mathematical problem is quantum physics. It was developed in a relatively short time span and culminated with the discovery of Schrödinger's equation in 1926. This theory showed us that the view of electrons and nuclei as very small particles that behave according to Newton's laws of classical mechanics is inadequate for understanding atoms and hence the constitution of matter.

In this article we review the progress that has been made so far starting with some elementary observations about the hydrogen atom, then giving the basic arguments for stability of non-relativistic and relativistic matter without and with magnetic fields. Finally we mention a few results concerning Quantum Electrodynamics. It is impossible to give all the details in such a condensed form and the rough outline of the proofs has to suffice.

2 The Hydrogenic atom

To describe quantum mechanics in a 'nutshell' let us start with Schrödinger's theory of a 'hydrogenic' atom. (In hydrogen, $Z = 1$ and $N = 1$; we want to consider $N = 1$ and arbitrary $Z > 0$.) We assume the nucleus to be a static, classical particle located at the origin, having charge Ze. (The nucleus can be considered fixed in space because its mass is by about a factor of 1800 greater than that of the electron, whose mass we denote by m.) Thus, it is not a serious approximation to assume that the only quantum mechanical particle is the electron. According to Born, the states of the electron are described by a complex valued **wave function** $\psi(x)$ with $|\psi(x)|^2$ being the probability density of finding the electron at the position x. Hence $|\psi(x)|^2$ must

be normalized, i.e.,

$$\|\psi\|_2^2 = \int_{\mathbb{R}^3} |\psi(x)|^2 dx = 1 \ . \tag{2.1}$$

The **quantum mechanical energy** $\mathcal{E}(\psi)$ of the atom in the state ψ is given by

$$\mathcal{E}(\psi) = T(\psi) + V(\psi) \ , \tag{2.2}$$

where

$$T(\psi) = \frac{\hbar^2}{2m} \int_{\mathbb{R}^3} |\nabla\psi(x)|^2 dx \ , \ V(\psi) = -(Ze)^2 \int_{\mathbb{R}^3} |x|^{-1} |\psi(x)|^2 dx \tag{2.3}$$

is the **kinetic energy** and **potential energy**.

Unless stated otherwise we shall adopt units such that the energy is measured in units of 4 Rydbergs. (1 Rydberg= $\frac{mc^2}{2}\alpha^2 = 13.56eV$, c= speed of light, and $\alpha = e^2/\hbar c = 1/137.04$ is the Sommerfeld fine structure constant). The length is measured in units of $\hbar^2/2me^2$, i.e., half the Bohr radius.

It is useful to note that the kinetic energy and the potential energy have the same dimensions as the 'square of an inverse length' and an 'inverse length' respectively.

The energy functional $\mathcal{E}(\psi)$ describes the atom in the following way: minimizing the energy over all normalized functions ψ yields the **ground state energy**, E_0, and the minimizer is the **ground state**, ψ_0. The fact that the energy functional (2.2) is bounded below is crucial of course. In fact for all functions ψ that satisfy (2.1) we have that

$$\mathcal{E}(\psi) \geq -Z^2 \frac{1}{4} = \mathcal{E}(\psi_0) \ . \tag{2.4}$$

where the unique normalized minimizer ψ_0 is given by

$$\psi_0(x) = (8\pi)^{-1/2} e^{-Z|x|/2} \ . \tag{2.5}$$

Although the theory of the Hydrogen atom can be found in any text book on quantum mechanics the above mentioned facts are by no means trivial and a proper proof uses a certain amount of machinery from the calculus of variations. It is, therefore, important to understand in simple terms the mechanism that prevents Jeans' catastrophe from happening.

Behind (2.4) is the simple intuition, that when we try to squeeze a normalized wave function into a small region, i.e., when we try to make the potential energy very negative then the variations in the function get big and the kinetic energy becomes large.

This general fact, that localizing an electron costs kinetic energy is called the **uncertainty principle**. There are many different mathematical relations expressing this idea, the most famous one being Heisenberg's uncertainty principle, which, however, is not very useful in our context.

The one that can be used for quantitative estimates is **Sobolev's inequality**

$$\int_{\mathbb{R}^3} |\nabla\psi|(x)^2 dx \geq S\big[\int_{\mathbb{R}^3} |\psi(x)|^6 dx\big]^{1/3} \ , \tag{2.6}$$

where $S = 3(\pi/2)^{4/3}$ is the sharp constant.

Applied to the hydrogenic atom we find a lower bound on the energy

$$S[\int \rho^3(x)dx]^{1/3} - \int \frac{Z}{|x|}\rho(x)dx \tag{2.7}$$

where $\rho = |\psi|^2$ satisfies $\int \rho dx = 1$. Simple minimization procedures lead to the minimizer

$$\rho = \left[\frac{Z}{|x|} - \lambda\right]_+ \tag{2.8}$$

where $[\cdots]_+$ denotes the positive part and λ is a Lagrange multiplier over which one optimizes. The ground state energy of our functional is

$$-\frac{Z^2}{S}\left(\frac{\pi}{2}\right)^{4/3} = -\frac{1}{3}Z^2 \ , \tag{2.9}$$

(see [30]). Note that this energy is surprisingly close to the actual value of the ground state energy of the hydrogenic atom. The point here is not the actual calculation but the fact that *the Sobolev inequality is a simple and versatile tool to obtain lower bounds for atomic energies.* We shall come back to this point when we talk about the many body problem.

3 Stability of non-relativistic matter

Schrödinger's theory goes far beyond a single atom, in fact it is believed to be the 'true' theory of non-relativistic matter if one ignores radiation processes.

We consider bulk matter composed of many nuclei and electrons. Unless mentioned otherwise, the nuclei will be treated as classical particles, or what is same, as particles that have an infinite mass. The state of the system is now described by a complex valued wave function Ψ associated with the electrons. It depends on all the electron coordinates, i.e.,

$$\Psi = \Psi(x_1, \sigma_1; \ldots; x_N, \sigma_N) \ , \tag{3.1}$$

where the space variables x_i range over $\mathbb{R}^3$ and the spin variables σ_i take values in $\{-1/2, 1/2\}$. A good way of thinking about the spin variables is to imagine 2 kinds of 'electrons', those with spin $-1/2$ and those with spin $1/2$. Mathematically, it is often convenient to allow the number of spin states to be an arbitrary number which we denote by q, then the σ variables can take q different values.

We denote by $\langle \Psi, \Phi \rangle$ the scalar product on the Hilbert-space $L^2(\mathbb{R}^{3N}; \mathbb{C}^{2N})$ of all these wave functions, i.e.,

$$\langle \Psi, \Phi \rangle = \sum_{\sigma_1, \ldots, \sigma_N} \int_{\mathbb{R}^3} \cdots \int_{\mathbb{R}^3} \overline{\Psi(x_1, \sigma_1; \ldots; x_N, \sigma_N)} \Phi(x_1, \sigma_1; \ldots; x_N, \sigma_N) d^3x_1 \cdots d^3x_N \tag{3.2}$$

We shall assume that Ψ is normalized, i.e.,

$$\|\Psi\|^2 = \sum_{\sigma_1, \ldots, \sigma_N} \int_{\mathbb{R}^3} \cdots \int_{\mathbb{R}^3} |\Psi(x_1, \sigma_1; \ldots; x_N, \sigma_N)|^2 d^3x_1 \cdots d^3x_N = 1 \ , \tag{3.3}$$

interpreting $|\Psi(x_1, \sigma_1; \ldots; x_N, \sigma_N)|^2$ as the probability density of finding N electrons at positions $x_1, \ldots, x_N$ having spins $\sigma_1, \ldots, \sigma_N$.

The dynamics of a system of electrons is governed by a **Hamiltonian**

$$H = \sum_{i=1}^{N} \tau_i + V_c \ , \tag{3.4}$$

which acts on wave functions. Here τ_i represents the kinetic energy operator acting on the $i-th$ variable. The standard non-relativistic choice is

$$\tau = -\Delta \ , \tag{3.5}$$

which is the kinetic energy operator of a non-relativistic electron. Different choices for τ will be discussed later. V_c is the Coulomb interaction among the particles and is given by

$$V_c = -\sum_{i=1,j=1}^{N,K} \frac{Z_j}{|x_i - R_j|} + \sum_{i<j}^{N} \frac{1}{|x_i - x_j|} + \sum_{i<j}^{K} \frac{Z_i Z_j}{|R_i - R_j|} \ . \tag{3.6}$$

The three terms describe, in succession, the attraction of the nuclei and electrons, the repulsion among the electrons and the repulsion among the nuclei.

The Schrödinger energy functional for a system of N electrons interacting with K nuclei with charges $Z_1, \ldots, Z_K$, which we assume to be fixed at the positions $R_1, \ldots, R_K$ is given by

$$\mathcal{E}(\Psi) = \langle \Psi, H\Psi \rangle \ . \tag{3.7}$$

Note that, so far, the spin variables do not play any dynamical role. They do not appear in the Hamiltonian.

As in the case of the 'hydrogenic' atom, minimizing $\mathcal{E}(\Psi)$ over all normalized wave functions Ψ yields the ground state energy E_0 of the system and the minimizer will be the ground state Ψ_0, i.e., the lowest energy state of this system of matter. Sometimes there may be no ground state, e.g., when there are too many electrons, but the ground state energy is always defined as the infimum of (3.7) over all normalized wave functions.

A further crucial ingredient for the description of matter is the **Pauli exclusion principle**, whose discovery in 1925 actually preceded quantum physics. It states that two electrons can never be in the same quantum state or, more precisely:

The wave function Ψ for N electrons (fermions) must be antisymmetric in the indices labeling the particles, i.e., the wave function changes sign if any two space-spin variables (x_j, σ_j) and (x_k, σ_k) are interchanged.

We denote the Hilbert-space of all such wave functions by

$$\mathcal{H} = \wedge^N L^2(\mathbb{R}^3; \mathbb{C}^q) \ , \tag{3.8}$$

the N-fold antisymmetric tensor product of $L^2(\mathbb{R}^3; \mathbb{C}^q)$.

Returning to (3.7) two different types of energies can be defined. One by minimizing over all normalized wave functions and one by minimizing over all normalized wave functions obeying the Pauli exclusion principle. The two a very different and the latter is the one realized in nature. It can be shown that the former is realized by minimizing (3.7) over the normalized *symmetric* functions (see [30]).

To understand a bit better the effect of the Pauli principle on energies consider a simple Hamiltonian of the form

$$H = \sum_{k=1}^{N} h_k \ , \tag{3.9}$$

where h_k denotes the operator h acting on the k-th particle coordinate. Denote by ϕ_j the normalized eigenfunctions of h and by λ_j the corresponding eigenvalues ordered according to size, $\lambda_1 \leq \lambda_2 \leq \cdots$. The smallest energy for the operator H, *ignoring the Pauli principle* is $N\lambda_1$ and the eigenfunction is $\Pi_i^N \phi_1(x_i)$. If *the Pauli principle is taken into account* we have to 'fill up the energy levels', only q electrons can be in the same state. The keep the energy as small as possible we put the first q elecrons electron into the ground state of h, ϕ_1. The Pauli principle then forces us to put the next q electron into a different state and the second eigenvalue is energywise the best. Continuing this way we obtain for the energy

$$q \sum_{j=1}^{A} \lambda_j + R\lambda_{A+1} \ . \tag{3.10}$$

where $N = qA + R$, $R < q$ and R, A are integers. The corresponding eigenfunctions must be linear combinations of $\Pi_{j=1}^N \phi_j(x_{\pi(j)}, \sigma_{\pi(j)})$ where π is any permutation of N objects and the antisymmetry forces the wave

function to be thus a constant times the determinant of the matrix $\phi_j(x_i)$ for $1 \leq i, j \leq N$. Normalizing this function yields the *Slater determinant*

$$\frac{1}{\sqrt{N!}} \det(\phi_j(x_i, \sigma_i)) \ . \tag{3.11}$$

Loosely speaking, the Pauli exclusion principle introduces a repulsive interaction between electrons in the sense that no two particles are allowed in the same state. The consequences of the exclusion principle are far reaching. The ground state will be degenerate in general and the ground state wave function will not be symmetric under rotations and as a consequence atoms and molecules and ultimately matter acquire shapes.

As an aside, note that, although the spin does not enter in a dynamical fashion it has a subtle effect. If we decompose the total wave function as a sum of products of functions that depend only on the spin and functions that only depend on the space variables then the symmetry type of the spin function will determine the symmetry type of the space function since each product has to be antisymmetric. That, in turn, will reflect the amount of energy 'stored' in the space function since the Hamiltonian acts only on the space part of the function. In particular, if we choose $q = N$ the spin has no effect on the energy since one can write the ground state as the product of an antisymmetric spin function and a symmetric function in the space variables. As we have noted before the minimization problem among the symmetric wave functions yields the same answer as the unrestricted minimization.

To summarize, the ground state energy of the system described by (3.4 - 3.6), taking the Pauli–principle into consideration can be found by the minimization problem

$$E_0 = \inf\{\langle \Psi, H\Psi \rangle : \|\Psi\| = 1, \Psi \in \mathcal{H}\} \ , \tag{3.12}$$

where $\mathcal{H}$ is given by (3.8). Finding E_0 and the corresponding minimizer is completely out of reach by any means. Instaed one concentrates on deriving some fundamental principles and features for the ground state energy and the corresponding wave function such as the Stability of Matter which we explain next.

We saw before that the ground state energy of hydrogen is a finite number. It is not very difficult to see that the ground state energy of a quantity of bulk matter is also finite, but nothing has been said about the dependence of the energy on the number of particles.

Following Lieb [30], it is convenient to distinguish between two notions of stability.

• A system is stable of the *first kind* if $E_0 > -\infty$.

• A system is stable of the *second kind* if $E_0 \geq C(N + K)$ where C is independent of N and K. Recall that N and K are the number of electrons and the number of nuclei.

THEOREM 3.1 (Stability of Matter)**.** *The Coulomb Hamiltonian (3.4, 3.5, 3.6) on the Hilbert space (3.8) is stable of the second kind. More precisely*

$$E_0 \geq -q^{2/3} 0.123 N \left(1 + 1.769 \sqrt{\frac{\sum_{k=1}^{K} Z_k^{7/3}}{N}} \right)^2 \ . \tag{3.13}$$

This theorem was first proved by Dyson and Lenard [11, 12]. and subsequently another proof was found by Federbush [15]. A completely new way of understanding this result was discovered by Lieb and Thirring which also led to a considerable improvement of the constant A [44]. The discoveries of Lieb and Thirring

are fundamental for the further development of the subject. Before doing that let us mention briefly two further implications of Theorem 3.1. It implies that matter (in its ground state) is bulky ,i.e.,

$$\sqrt{\langle |r^2| \rangle} \geq const.N^{1/3} \ , \tag{3.14}$$

where $\sqrt{\langle |r^2| \rangle}$ is the variance of the position operator,(see [45]). A second, highly nontrivial consequence, is that the thermodynamic functions exist. This was shown by Lieb and Lebowitz [35].

It has to be emphasized that the Pauli exclusion principle is crucial for stability of the second kind. In the absence of the Pauli principle the ground state energy $E_0 \leq -const.N^{5/3}$ (see [29]). The inclusion of dynamical nuclei does not change that picture in any essential way. The following theorem is due to Dyson [13].

THEOREM 3.2 (Bosonic matter is unstable)**.** *The ground state energy E_0 of $2N$ charged bosons, N with charge $+1$ and N with charge -1 satisfies*

$$E_0 \leq -CN^{7/5} \ . \tag{3.15}$$

The peculiar power 7/5 has been shown to be exact in (see [7]). Moreover, in [13], Dyson gives a precise value for the constant C which he conjectures to be the correct value. This has been recently proved in [43].

As was mentioned before, Theorem 3.1 has a surprisingly simple and conceptually appealing proof due to Lieb and Thirring [44] and since it is the starting point for further developments we reproduce the main ideas below. The key to understanding the theorem is the relation between this problem and Thomas - Fermi theory.

This theory was invented by Thomas and independently by Fermi [21], [50]. Its fundamental idea is to replace the high dimensional Schrödinger equation by a new, low dimensional, problem that involves only the **single particle density** $\rho_\Psi(x)$ defined by

$$\rho_\Psi(x) = \sum_{i=1}^{N} \sum_{\sigma_1,\ldots,\sigma_N} \int_{\mathbb{R}^3} \cdots \int_{\mathbb{R}^3} |\Psi(x_1,\sigma_1;\ldots;x,\sigma_i;x_N,\sigma_N)|^2 d^3x_1 \cdots d^3\hat{x}_i d^3x_N \ , \tag{3.16}$$

where $\hat{x}_i$ indicates that we do not integrate over the variable x_i. This function is now a function on $\mathbb{R}^3$ and can be thought of as the charge density of the N electrons. If there is only one electron, then $|\psi|^2 = \rho_\Psi$. Note that $\int \rho_\Psi = N$. The goal is to find a lower bound of the quantum mechanical energy (3.7) of the N-particle system in terms of a functional that has only the single particle density as an independent variable. In what follows, the function ρ is not necessarily the one particle density of an N particle wave function. The **Thomas Fermi functional** is given by

$$\mathcal{E}(\rho) = \gamma \frac{5}{3} q^{2/3} \int_{\mathbb{R}^3} \rho^{5/3}(x)dx - \sum_{j=1}^{K} Z_j \int_{\mathbb{R}^3} \frac{\rho(x)}{|x-R_j|}dx + \frac{1}{2} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\rho(x)\rho(y)}{|x-y|}dxdy + \sum_{i<j}^{K} \frac{Z_i Z_j}{|R_i - R_j|} \ . \tag{3.17}$$

We assume that ρ is nonnegative and

$$\int_{\mathbb{R}^3} \rho(x)dx = \lambda \tag{3.18}$$

where λ is a fixed number not necessarily an integer. We denote by

$$e_0(\lambda, R_1, \ldots, R_K) = \inf\{\mathcal{E}(\rho) : \rho(x) \geq 0, \int_{\mathbb{R}^3} \rho(x)dx = \lambda\} \tag{3.19}$$

the ground state energy of the Thomas-Fermi functional. Note that the positions of the nuclei are fixed. They just enter as parameters. The only unfamiliar term is the 'kinetic energy term'

$$\gamma \frac{5}{3} q^{2/3} \int_{\mathbb{R}^3} \rho^{5/3}(x) dx \tag{3.20}$$

where γ is some constant. To understand where this term comes from consider a cubic box of size L and calculate the ground state energy of the kinetic energy of N fermions in this box. The energy levels are given by the eigenvalues of the Laplace operator with Dirichlet boundary condition in that box which are

$$\frac{\pi^2(n_1^2 + n_2^2 + n_3^2)}{L^2} \tag{3.21}$$

Filling up the energy levels for N fermions having spin q yields for N large

$$const.q^{-2/3} \left(\frac{N}{L^3}\right) L^3 = const.q^{-2/3} \rho^{5/3} L^3 \ . \tag{3.22}$$

The functional (3.17) was completely investigated by Lieb and Simon [42] (see also [32] for a review). They proved among many other things that for fixed positions of the nuclei a minimizer exists provided $\lambda \leq \sum_{i=1}^K Z_i$. Otherwise there is no minimizer. If $\lambda \geq \sum_{i=1}^K Z_i$ then

$$e_0(\lambda, R_1, \ldots, R_K) = e_0(\sum_{i=1}^K Z_i, R_1, \ldots, R_K) \ . \tag{3.23}$$

Moreover, the energy e_0 is a monotone decreasing function of λ. Further, by numerical computations one finds that the energy of a single neutral atom with nuclear charge Z is given by

$$e_0(Z) = -3.678 \frac{Z^{7/3}}{\gamma} \ . \tag{3.24}$$

So far we kept the nuclei fixed. How do we have to choose the positions of the nuclei for the total energy to be minimal? The surprising answer is that there is no binding in T-F theory. This has been discovered by Teller and later proved rigorously by Lieb and Simon using potential theoretic arguments. In other words the energy of a system of electrons and nuclei is smallest if the atoms are infinitely far apart and neutral. This has the immediate consequence that matter in TF theory is stable,i.e.,

$$e_0(\lambda, R_1, \ldots, R_K) \geq -3.678 \sum_{j=1}^K Z_j^{7/3} / \gamma \ . \tag{3.25}$$

The next problem is to show that the Thomas - Fermi functional is indeed a lower bound to the true N-particle Schrödinger functional (3.7).

The first step is given by the following theorem first proved in [44]. It is a Sobolev type inequality for the electron wave function that takes into account the Pauli exclusion principle.

THEOREM 3.3 (Lieb-Thirring inequality)**.** *Let Ψ be the wave function of N fermions of spin q. Then*

$$T(\Psi) \geq \frac{3}{5} C q^{-2/3} \int_{\mathbb{R}^3} \rho_\Psi^{5/3}(x) dx \ , \tag{3.26}$$

where $C \geq (3\pi^2)^{2/3}$.

The constant given here has been found in [25] and is an improvement over the one given in [44]. The main feature of this inequality is that the right side is extensive, that is, if we think of having a wave function Ψ whose single particle density is distributed in N equal disjoint bumps across space then the right side of (3.26) is proportional to N.

This inequality can be derived from the following inequality, also proved for the first time in [44] and [45].

THEOREM 3.4. *Consider the Schrödinger operator*

$$-\Delta - U(x) \tag{3.27}$$

where U is a positive function. Denote the negative eigenvalues by $-\lambda_1 < -\lambda_2 \leq \dots$. Then

$$\sum_j \lambda_j \leq L_1 \int U(x)^{5/2} dx \ , \tag{3.28}$$

and

$$\sum_j \lambda_j^{1/2} \leq L_2 \int U(x)^2(x) dx \ . \tag{3.29}$$

The sharp constants for these two inequalities are not known but there exist the bounds

$$L_1 \leq \frac{2}{15\pi^2} \ , \ L_2 \leq \frac{1}{8\pi} \ . \tag{3.30}$$

Over the years, the constants have been improved notably by Lieb, [34]. The values given here are taken from [25] using the method of matrix valued potential pioneered by Laptev and Weidl [28],(see also [4]).

The other deeply quantum mechanical term in (3.7) is the Coulomb repulsion for the electrons,

$$C(\Psi) := \langle \Psi \sum_{1 \leq i < j \leq N} |x_i - x_j|^{-1} \Psi \rangle \ . \tag{3.31}$$

One would like to bound this term by the Coulomb repulsion of the one particle density

$$D(\rho_\Psi) := \frac{1}{2} \int_{R^3} \int_{R^3} \frac{\rho_\Psi(x) \rho_\Psi(y)}{|x-y|} dx dy \ . \tag{3.32}$$

This, however, is false. The quantum mechanical Coulomb repulsion is of the order of $(N^2 - N)/2$ while the term in (3.32) is of the order $N^2/2$.

The following theorem was shown in [40].

THEOREM 3.5. *For any normalized N-particle wave function Ψ the quantum mechanical Coulomb repulsion satisfies*

$$C(\Psi) \geq D(\rho_\Psi) - 1.68 \int_{\mathbb{R}^3} \rho_\Psi^{4/3}(x) dx \ . \tag{3.33}$$

This results holds irrespective of the symmetry type of Ψ.

Using the elementary inequality

$$a^{4/3} \leq \varepsilon a^{5/3} + \frac{1}{4\varepsilon} a$$

yields

$$\int_{\mathbb{R}^3} \rho_\Psi^{4/3}(x) dx \leq \varepsilon \int_{\mathbb{R}^3} \rho_\Psi^{5/3}(x) dx + \frac{1}{4\varepsilon} \int_{\mathbb{R}^3} \rho_\Psi(x) dx = \varepsilon \int_{\mathbb{R}^3} \rho_\Psi^{5/3}(x) dx + \frac{1}{4\varepsilon} N \tag{3.34}$$

which leads to a small change in the constant in front of the T - F kinetic term.

The energy of attraction between the electrons and the nuclei in (3.7) is given by

$$-\langle\Psi,\sum_{i=1}^{N}\sum_{j=1}^{K}\frac{Z_j}{|x_i-R_j|}\Psi\rangle=-\sum_{j=1}^{K}Z_j\int\limits_{\mathbb{R}^3}\frac{\rho_\Psi(x)}{|x-R_j|}dx\ . \tag{3.35}$$

Collecting all these terms one finds that the Thomas - Fermi energy functional is indeed a lower bound. Using the bound (3.25) and optimizing over the various constants one obtains the bound displayed in in Theorem 3.1.

In recent years the focus of this research area has expanded to incorporate more physical effects. Bulk matter interacts with magnetic fields and, if Z is large, binding energies can become large enough so that relativistic effects have to be included. In the next section we shall outline what is known about relativistic Coulomb systems.

4 Stability of relativistic matter

One of the important problems is the fusion of Einstein's special (and general relativity) with Quantum Mechanics. Up to now we do not have a mathematically rigorously formulated, relativistically invariant Quantum Theory of interacting particles. There exist very successful perturbative prescriptions to compute physical quantities, e.g., in Quantum Electrodynamics, but there is no theory that allows one to draw mathematically rigorous conclusions. The following is only a caricature of relativistic Quantum Mechanics but there is hope that it contains certain features that will persist in the theory to be. One of them is that in the ultra relativistic regime the kinetic energy and the potential energy scale the same way and hence are of the same order. Thus, the problem lies outside the realm of perturbation theory. Another good reason for considering this problem is that the relativistic stability estimate can be used to solve a host of other interesting problems.

The model consists of simply replacing the kinetic energy $\tau=-\Delta$ in (3.4) by its relativistic counterpart $\sqrt{-\Delta+1}-1$. Here we choose units in which the energy is measured in units of the rest energy of the electron mc^2, the length we measure in units of the Compton wave length divided by 2π, i.e., $\hbar/mc$. The Schrödinger Hamiltonian (3.4) is then changed to

$$\sum_{i=1}^{N}\tau_i+\alpha V_c \tag{4.1}$$

where, as usual $\alpha\approx 1/137.04$ is the fine structure constant and

$$\tau_i=\sqrt{-\Delta_i+1}-1 \tag{4.2}$$

acting on the coordinates of the i-th particle.

The formal definition of the kinetic energy is via the Fourier transform, i.e.,

$$\langle\psi,\sqrt{-\Delta+1}\psi\rangle=\int\limits_{\mathbb{R}^3}\sqrt{4\pi^2k^2+1}|\widehat{\psi}(k)|^2dk\ , \tag{4.3}$$

where the Fourier transform is given by

$$\widehat{\psi}(k)=\int\limits_{\mathbb{R}^3}e^{-2\pi ik\cdot x}\psi(x)dx\ . \tag{4.4}$$

To see the main effect of this new kinetic energy term, the case of the hydrogenic atom is instructive. We have to minimize

$$\langle\psi, (\sqrt{-\Delta+1}-1)\psi\rangle - Z\alpha\langle\psi, \frac{1}{|x|}\psi\rangle \tag{4.5}$$

over all normalized functions ψ. If we replace $\psi(x)$ by the scaled function $\psi_\lambda(x) = \lambda^{3/2}\psi_\lambda(x)$ a little computation shows that the normalization is unchanged and that

$$\langle\psi_\lambda, (\sqrt{-\Delta+1}-1)\psi_\lambda\rangle - Z\alpha\langle\psi_\lambda, \frac{1}{|x|}\psi_\lambda\rangle = \lambda\{\langle\psi, (\sqrt{-\Delta+(1/\lambda)^2}-1/\lambda)\psi\rangle - Z\alpha\langle\psi, \frac{1}{|x|}\psi\rangle\} \ . \tag{4.6}$$

From this we see that $Z\alpha$ cannot be too big for otherwise we could make the energy as negative as we please by letting λ go to ∞. Thus we are left with only two alternatives. For all ψ, either,

$$\langle\psi, \sqrt{-\Delta}\psi\rangle - Z\alpha\langle\psi, \frac{1}{|x|}\psi\rangle \geq 0 \ , \tag{4.7}$$

or otherwise the infimum of the energy is $-\infty$. Note that we lost the mass in the scaling procedure. It is irrelevant for the stability problem, but it will determine the energy once the problem is shown to be stable. We shall ignore it from now on.

In short, the essence of this relativistic theory is that the kinetic and the potential energy have the same dimensions. The whole stability problem of the relativistic atom is captured in the estimate (4.7) and leads immediately to the question for what values of $Z\alpha$ such an estimate holds. The following has been shown by various authors: [27],[24], [51]

THEOREM 4.1. *The estimate (4.7) holds if and only if $Z\alpha \leq 2/\pi$.*

Thus, recalling that $\alpha \approx 1/137$, atoms with a nuclear charge greater than 87 are not stable in this model. Inequality (4.7) is the relativistic version of the uncertainty principle

$$\int_{\mathbb{R}^3} |\nabla\psi(x)|^2 dx \geq \frac{1}{4}\int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|^2} dx \ . \tag{4.8}$$

Next, we consider the full theory. We return to the Hamiltonian H in (4.1) but we replace the kinetic energy τ by $\sqrt{-\Delta}$. For simplicity we set all Z_j's $= Z$. Let us emphasize that the stability problem for this Hamiltonian is simply to show that

$$\inf\langle\Psi, H\Psi\rangle \geq 0 \ , \tag{4.9}$$

where the infimum is taken over all $\Psi \in \mathcal{H}$, that is, wave functions that obey the Pauli principle. In relativistic theories there is no difference between stability of the first kind and stability of the second kind. The Hamiltonian of any relativistic system has to be positive. Clearly, a necessary condition for stability is that $Z\alpha \leq 2/\pi$. However, the following dimensional analysis shows that this is not the only condition that is needed for stability of relativistic systems. Consider, for simplicity, a single electron in the field of K nuclei of the same charge Z which we allow to be any nonnegative real number, not necessarily an integer. Imagine the electron and these nuclei to be confined to a box of size L. The kinetic energy will be of size $1/L$, the Coulomb attraction will be of size $-Z\alpha K/L$ and the repulsion of the nuclei will be of the order $Z^2K^2\alpha/2L$. If we set $Z = 1/K$ we get, by adding up these contributions for the total energy,

$$\frac{\alpha}{2L}(2/\alpha - 1) \ , \tag{4.10}$$

which can be made as negative as we please (by letting L go to zero) as soon as $\alpha > 1/2$. Thus, a necessary condition for stability to hold is that the fine structure constant be not too large. The above argument was made rigorous by Daubechies and Lieb [9] where stability for single electron systems was proved. Stability of the full relativistic system was shown for the first time by Conlon [6]. His result was subsequently improved by de la Llave-Fefferman [10]. A next big step is due to Lieb-Yau [46] who showed that stability holds for all $Z\alpha \leq 2/\pi$. Their theorem is:

THEOREM 4.2. *Fix $Z\alpha\pi \leq 2$. Then the relativistic Hamiltonian (4.1) (with $\tau = \sqrt{-\Delta}$) is stable provided that $\alpha \leq 1/94$.*

Note that the constants fall within the physical range. The following result is proved in [39]. Although it does not hold up to the critical values, it yields a somewhat better range of parameters for which stability holds. More importantly, it also holds for matter interacting with magnetic fields which will be important in the next section. Its proof proceeds via a Thomas - Fermi type theory, very much in line with the non - reltativistic case.

THEOREM 4.3 (Stability of relativistic systems)**.** *For all antisymmetric, normalized wave functions Ψ associated with particles having q spin states*

$$\sum_{j=1}^{N} (\Psi, \sqrt{-\Delta}\Psi) + \alpha(\Psi, V_c\Psi) \geq 0 \tag{4.11}$$

provided that

$$\frac{\pi}{2} Z + 2.2159 q^{1/3} Z^{2/3} + 1.0307 q^{1/3} \leq \frac{1}{\alpha} \ . \tag{4.12}$$

Naively, relativistic T-F theory would have the kinetic energy term

$$\frac{3}{4}\gamma \int \rho^{4/3}(x) dx \ , \tag{4.13}$$

which can be easily seen by a dimensional argument. Likewise, it is also very easy to see that such a term cannot bound the Coulomb singularity and hence the problem has to suitably modified. We shall consider instead a functional of the form

$$\mathcal{E}(\rho) = \beta(\sqrt{\rho}, \sqrt{-\Delta}\sqrt{\rho}) + \frac{3}{4}\gamma \int \rho^{4/3}(x) dx \tag{4.14}$$

$$+\alpha \left[-Z \sum_k \int \frac{\rho(x)}{|x - R_k|} dx + D(\rho, \rho) + Z^2 \sum_{k<l} \frac{1}{|R_k - R_l|} \right] \ . \tag{4.15}$$

where β and γ are constants. Again, we note that all the terms have the same dimension and hence the ground state energy of this theory is either minus infinity or zero, but for sufficiently large β and γ and sufficiently small α one expects stability. This is indeed the case as the following theorem taken from [39] shows.

THEOREM 4.4 (Relativistic T - F stability)**.** *The functional $\mathcal{E}$ is stable if $\beta \geq \pi Z\alpha/2$ and $\gamma \geq 4.8158 Z^{2/3}\alpha$.*

The size of the Coulomb potential of a collection of nuclei and electron has two sources. It is singular close to the nuclei and it can also be large because there are many nuclei. To disentangle these two issues the notion of Voronoi cells with respect to a collection of nuclei is very useful. Define

$$\Gamma_j = \{x \in \mathbb{R}^3 : |x - R_j| < |x - R_i|, i \neq j\} \ . \tag{4.16}$$

Clearly Γ_j is open and it easily seen to be convex. Further we define the nearest neighbor distance between the nuclei by

$$\min_{i\neq j} |R_i - R_j| \tag{4.17}$$

and set

$$D_j = \frac{\min_{i\neq j} |R_i - R_j|}{2} \ . \tag{4.18}$$

One can describe the Voronoi cell Γ_j in more geometric terms. Each plane P_k bisecting the segment connecting R_k and R_j defines an open half space H_k that contains the point R_j. The Voronoi cell Γ_j is then formed by the intersection of the half spaces H_k. In particular the ball B_j centered at R_j with radius D_j is a subset of Γ_j.

The following lemma is proved in [46].

LEMMA 4.5 (Removing the Coulomb singularity)**.**

$$\left(\sqrt{\rho}, \sqrt{-\Delta}\sqrt{\rho}\right) \geq \frac{2}{\pi} \sum_{j=1}^{K} \int_{B_j} \rho(x) \left(\frac{1}{|x-R_j|} - \frac{1}{D_j} Y(\frac{|x-R_j|}{D_j}) \right) dx \ . \tag{4.19}$$

The function $Y(r)$ is given by

$$Y(r) = \frac{2}{\pi(1+r)} + \frac{1+3r^2}{\pi(1+r^2)} \ln(1+r) - \frac{1-r^2}{\pi r(1+r^2)} \ln(1-r) - \frac{4r}{\pi(1+r^2)} \ln r \leq 1.56712 \ . \tag{4.20}$$

The important point for us is that

$$4\pi \int Y(r)^4 r^2 dr < 7.6245 \ . \tag{4.21}$$

This lemma says, in essence that the first term in the functional (4.14) can be used to remove the Coulomb singularity in each ball defining the Voronoi cell.

Next we have to deal with the Coulomb potential due to all the nuclei that an electron feels at the point x. It is given by

$$W(x) = Z \sum_k \frac{1}{|x-R_k|} \ . \tag{4.22}$$

Define

$$\delta(x) = \min\{|x-R_i| : 1 \leq i \leq K\} \ , \tag{4.23}$$

and set

$$\Phi(x) = W(x) - \frac{Z}{\delta(x)} \ . \tag{4.24}$$

Thus, for $x \in \Gamma_j$, $\Phi(x)$ is the potential at the point x due to all the nuclei outside of the Voronoi cell Γ_j. The nucleus at the point R_j does not contribute. Imagine now a charge distribution $\mu(dx)$ interacting with the potential $\Phi(x)$. This distribution can be a measure and does not have to be positive. Certainly, the *total* energy of all the charge distributions, i.e., including the one that generates the potential $\Phi(x)$ is nonnegative. (Note that there is no infinite self energy term from the nuclei.) Much more, however can be said and that is the content of the next lemma proven in [46].

LEMMA 4.6 (Electrostatic inequality)**.** *For any charge distribution μ*

$$D(\mu,\mu) - \int \Phi(x)\mu(dx) + Z^2 \sum_{k<l} \frac{1}{|R_k - R_l|} \geq \frac{Z^2}{8} \sum_j \frac{1}{D_j} \ . \tag{4.25}$$

The point about this lemma is the positive term on the right side of the inequality.

Proof of Theorem 4.4. Set $\beta = \pi Z\alpha/2$ and use Lemma 4.5 to 'pull the Coulomb tooth' and find that

$$\mathcal{E}(\rho) \geq \frac{3}{4}\gamma \int \rho^{4/3}(x)dx + \tag{4.26}$$

$$\alpha \left[- \int \rho(x)U(x)dx + D(\rho,\rho) + Z^2 \sum_{k<l} \frac{1}{|R_k - R_l|} \right] . \tag{4.27}$$

where

$$U(x) = Z \sum_k \frac{1}{|x-R_k|}(1-\chi_{B_k}) + \frac{\pi}{2}\chi_{B_k}\frac{1}{D_k}Y(\frac{|x-R_k|}{D_k}) . \tag{4.28}$$

Recall that

$$\Phi(x) = Z \sum_k \frac{1}{|x-R_k|} - \frac{Z}{\delta(x)} \tag{4.29}$$

which takes the value

$$Z \sum_{k\neq j} \frac{1}{|x-R_k|} \tag{4.30}$$

in the Voronoi cell Γ_j. Note that in the Voronoi cell Γ_k the difference

$$U(x) - \Phi(x) = \frac{Z}{|x-R_k|}(1-\chi_{B_k}) + \frac{\pi}{2}\chi_{B_k}\frac{1}{D_k}Y(\frac{|x-R_k|}{D_k}) \tag{4.31}$$

and hence we split $U(x)$ as

$$U(x) = [U(x) - \Phi(x)] + \Phi(x) \tag{4.32}$$

and the lower bound (4.26) takes the form

$$\mathcal{E}_1(\rho) + \alpha\mathcal{E}_2(\rho) \tag{4.33}$$

where

$$\mathcal{E}_1(\rho) = \frac{3}{4}\gamma \int \rho^{4/3}(x)dx - Z\alpha \int \rho(x)[U(x)-\Phi(x)]dx \tag{4.34}$$

and

$$\mathcal{E}_2(\rho) = D(\rho,\rho) - \int \Phi(x)\rho(x)dx + Z^2 \sum_{k<l} \frac{1}{|R_k - R_l|} . \tag{4.35}$$

The second functional is bounded below by

$$\frac{Z^2}{8} \sum_k \frac{1}{D_k} \tag{4.36}$$

by the electrostatic inequality Lemma 4.6. The first term we bound using Hölder's inequality by

$$\frac{3}{4}\gamma \|\rho\|_{4/3}^{4/3} - Z\alpha \|\rho\|_{4/3} \|U-\Phi\|_4 \tag{4.37}$$

and optimizing over $X = \|\rho\|_{4/3}$ yields

$$-\frac{(Z\alpha)^4}{4\gamma^3} \int [U(x)-\Phi(x)]^4 dx \tag{4.38}$$

$$= -\frac{(Z\alpha)^4}{4\gamma^3} \sum_k (\frac{\pi}{2})^4 \int_{B_k} D_k^{-4} Y(\frac{|x-R_k|}{D_k})^4 dx + \int_{\Gamma_k - B_k} \frac{1}{|x-R_k|^4} dx . \tag{4.39}$$

Since the Voronoi cell Γ_k lies on one side of the mid plane defined by the nearest neighbor nucleus we get an upper bound on the last term by integrating over the outside of the ball B_k and then subtract the integral of the half space whose z-coordinate is greater or equals D_k. Thus

$$\int_{\Gamma_k - B_k} \frac{1}{|x - R_k|^4} dx \leq \frac{4\pi}{D_k} - \frac{1}{D_k} \int_1^\infty dz \int_0^\infty \frac{2\pi r}{(r^2 + z^2)^2} dr = \frac{3\pi}{D_k} \ . \tag{4.40}$$

Hence we get that

$$\mathcal{E}_1(\rho) \geq -\frac{(Z\alpha)^4}{4\gamma^3} \left[(\frac{\pi}{2})^4 4\pi \int_0^1 Y(r)^4 r^2 dr + 3\pi \right] \sum_k \frac{1}{D_k} \tag{4.41}$$

$$= -\frac{(Z\alpha)^4}{4\gamma^3} \left[7.6245 (\frac{\pi}{2})^4 + 3\pi \right] \sum_k \frac{1}{D_k} \tag{4.42}$$

Adding the bounds yields in total

$$\mathcal{E}(\rho) \geq \left[-\frac{(Z\alpha)^4}{4\gamma^3} \left[7.6245 (\frac{\pi}{2})^4 + 3\pi \right] + \alpha \frac{Z^2}{8} \right] \sum_k \frac{1}{D_k} \tag{4.43}$$

and the condition on γ stated in the theorem yields the result. □

The next step is to relate the full quantum mechanical problem to the relativistic Thomas - Fermi problem. We need to explain how the first two kinetic energy terms in the relativistic Thomas - Fermi functional emerge. The first term follows from the inequality

$$\langle \Psi, \sum_{j=1}^N \sqrt{-\Delta} \Psi \rangle \leq (\sqrt{\rho_\Psi}, \sqrt{-\Delta} \sqrt{\rho_\Psi}) \ , \tag{4.44}$$

whose proof can be found in [6]. It has to emphasized that this inequality holds for any many-body wave function. The Pauli exclusion principle is not relevant in this case. The second term in (4.14) is the analog appearing in the Lieb - Thirring inequality but for the relativistic kinetic energy. This Lieb - Thirring type inequaltity is due to Daubechies [8]

LEMMA 4.7. *Assume that $U(x) \geq 0$ is in $L^4(\mathbb{R}^3)$. Then on $L^2(\mathbb{R}^3, \mathbb{C}^q)$*

$$\mathrm{Tr}[\sqrt{-\Delta} - U]_- \leq q L_D \int U(x)^4 dx \tag{4.45}$$

where the constant L_D is bound above by 0.0245. *Moreover, for any normalized, antisymmetric wave function of N particles with q spin states*

$$\langle \Psi, \sum_{j=1}^N \sqrt{-\Delta_j} \Psi \rangle \geq \frac{3}{4} \left(\frac{1}{4L_D} \right)^{1/3} q^{-1/3} \int_{\mathbb{R}^3} \rho_\Psi^{4/3}(x) dx \ , \tag{4.46}$$

where

$$\frac{3}{4} \left(\frac{1}{4L_D} \right)^{1/3} \geq 1.63 \ . \tag{4.47}$$

Proof of Theorem 4.2. With

$$\beta = \frac{\pi}{2} Z\alpha \tag{4.48}$$

and

$$\gamma = \frac{4}{3} \left[1.63 q^{-1/3} (1 - \frac{\pi}{2} Z\alpha) - 1.68\alpha \right] \tag{4.49}$$

we see that that for any antisymmetric wave function of N particles with q spin states

$$\langle \Psi, [\sum_{j=1}^{N} \sqrt{-\Delta_j} + \alpha V_c]\Psi \rangle \geq \mathcal{E}(\rho_\Psi) \tag{4.50}$$

where we have used, in addition, Theorem 3.5. By Theorem 4.4 we have stability provided that

$$\frac{\pi}{2} Z + 2.2159 q^{1/3} Z^{2/3} + 1.0307 q^{1/3} \leq \frac{1}{\alpha} \ . \tag{4.51}$$

□

Relativistic Stability of Matter is one of the main theorems in this whole field of research. Apart from the question of "good constants", Theorem 4.3 is a stronger theorem than the one concerning non-relativistic matter as the following shows.

First note that by Schwarz's inequality

$$(\Psi, \sqrt{-\Delta}\Psi) \leq \|\Psi\| (\Psi, -\Delta\Psi)^{1/2} \ . \tag{4.52}$$

Hence, since Ψ is normalized

$$N^{1/2} \left[\sum_{j=1}^{N} (\Psi, -\Delta\Psi) \right]^{1/2} \geq \sum_{j=1}^{N} (\Psi, -\Delta\Psi)^{1/2} \geq \sum_{j=1}^{N} (\Psi, \sqrt{-\Delta}\Psi) \tag{4.53}$$

From this we get that for any $a > 0$

$$\sum_{j=1}^{N} (\Psi, -\Delta\Psi) \geq \frac{2}{a} \sum_{j=1}^{N} (\Psi, \sqrt{-\Delta}\Psi) - N \frac{1}{a^2} \ . \tag{4.54}$$

Thus,

$$\sum_{j=1}^{N} (\Psi, -\Delta\Psi) + (\Psi, V_c\Psi) \geq \frac{2}{a} \sum_{j=1}^{N} (\Psi, \sqrt{-\Delta}\Psi) + (\Psi, V_c\Psi) - N \frac{1}{a^2} \tag{4.55}$$

$$\geq -\frac{N}{a^2} \tag{4.56}$$

provided that a is chosen such that

$$\frac{\pi}{2} Z + 2.2159 q^{1/3} Z^{2/3} + 1.0307 q^{1/3} \leq \frac{2}{a} \ , \tag{4.57}$$

obviously we do best choosing the equality sign. For $Z = 1$ and $q = 2$ we get

$$\frac{2}{a} = 5.6611 \tag{4.58}$$

and obtain the lower bound

$$-8.012N \ . \tag{4.59}$$

Note that the bound depends only on N and not on K.

5 Interaction with magnetic fields

So far the spin of the electron did not play any significant role in the stability problem. This situation changes drastically if one considers the interaction of an electron with a magnetic field $B(x)$. We consider in the following the magnetic field to be static and with finite field energy

$$\frac{1}{8\pi\alpha^2}\int_{R^3}|B(x)|^2dx\ . \tag{5.1}$$

We use again units explained at the beginning of Section 2 which leads to the appearance of α in the expression for the field energy.

The magnetic field $B(x)$ is divergence free on all of $\mathbb{R}^3$ and hence we can always find a 'Vector potential' $A(x)$ with the property that

$$\mathrm{curl}A(x) = B(x)\ . \tag{5.2}$$

Clearly A is only specified up to a **gauge transformation** $A \to A + \nabla\chi$ for some function χ.

The electron interacts with the magnetic field according to Lorenz's force law Force $= eB \times v$ where v is the velocity of the particle. In Quantum Mechanics one describes this interaction, analogous to classical mechanics by replacing the momentum $p = -i\nabla$ by $p - eA$. The kinetic energy is now given by

$$(p - eA)^2\ . \tag{5.3}$$

As usual we set the charge of the electron $e = -1$.

The first result for matter interacting with radiation is due to Avron - Herbst - Simon [1] and, independently, Combes - Schrader - Seiler [5], who consider the Hamiltonian (3.4) but with

$$\tau = (-i\nabla + A(x))^2\ . \tag{5.4}$$

Their result is that the system is stable of the second kind, independent of the magnetic field and with the same constant as the one furnished by the Lieb - Thirring proof without the magnetic field. The result hinges on the diamagnetic inequality which says that the Greens function

$$|[(-i\nabla + A(x)) + \lambda^2]^{-2}(x,y)| \leq (-\Delta + \lambda^2)^{-2}(x,y) = \frac{1}{4\pi}\frac{e^{-\lambda|x-y|}}{|x-y|}\ . \tag{5.5}$$

where $\lambda \geq 0$ is a constant. From this one deduces the Lieb-Thirring inequality (3.26) with the same constants independent of the magnetic field.

The spin of the electron introduces an additional complication. The electron behaves to some extent like a magnetized needle, that is, it has a magnetic moment μ. The coupling between this moment and the magnetic field is given by $-\mu \cdot B$. Recall that the electron wave function is given by a 'spinor', i.e., a two component wave function. It is best to think of it as a vector

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{5.6}$$

The magnetic moment is now given by $(\psi, \sigma\psi)(x)$, where σ are the three Pauli matrices

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\ \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\ , \tag{5.7}$$

and (,) denotes the inner product in $\mathbb{C}^2$.

The part of the energy describing the interaction of the electron with the magnetic field is then given by the **Pauli operator**

$$\tau = (p + A)^2 + \sigma \cdot B = [\sigma \cdot (p + A)]^2 \ , \tag{5.8}$$

which is a positive matrix valued operator. The second equality is an easy calculation. The model under consideration is the Hamiltonian (3.4) with τ as above. The ground state energy $E(B)$ of this system (the Pauli principle is always included) will depend on the particle numbers N, K, the position of the nuclei, R, and their charges, Z, and it will be a complicated function of the magnetic field. It is known [2] that when the magnetic field gets large, then $E(B)$ gets (negative) large too. This is the effect of the $\sigma \cdot B$–term. Producing large magnetic fields, however, requires energy and the energy stored in a magnetic field is proportional to

$$\frac{1}{8\pi} \int_{\mathbb{R}^3} |B(x)|^2 dx \ . \tag{5.9}$$

Thus, one possibility is to include the magnetic field as a dynamical variable and define stability of the second kind by requiring

$$E(B) + \frac{1}{8\pi\alpha^2} \int_{\mathbb{R}^3} |B(x)|^2 dx \geq -C(Z)(N + K) \ . \tag{5.10}$$

Let us emphasize the requirement that the constant $C(Z)$ must not depend on the number of particles, the position of the nuclei and *not on the magnetic field either.*

Despite being non-relativistic this problem has a behavior similar to the relativistic one in that the fine structure constant and the nuclear charges have to be small enough in order for stability to hold. This may seem surprising, since the kinetic energy scales like the square of an inverse length whereas the potential energy scales only like an inverse length. From those facts one would expect stability irrespective of the values of α and Z.

The 'hydrogenic atom' offers a first clue [22], [36]. In that case the problem is to find a lower bound to the functional

$$\mathcal{E}(\psi, A) = \langle \sigma \cdot (p - A)\psi, \sigma \cdot (p - A)\psi \rangle - Z\langle \psi | \frac{1}{|x|} | \psi \rangle + \varepsilon \int_{R^3} |B(x)|^2 dx \ . \tag{5.11}$$

Here $\varepsilon = 1/(8\pi\alpha^2)$. Note that $B = \mathrm{curl} A$ and that we minimize over A as well as the spinor ψ. Also recall that we have used (5.8) to display the interaction of the electron with the magnetic field as a complete square. In this form the kinetic energy and the $\sigma \cdot B$ term are not separate anymore.

It was shown in [49] (see also [14] for further examples) that there exist **zero modes** of the three dimensional Dirac equation

$$\sigma \cdot (-i\nabla - A(x))\psi = 0 \ , \tag{5.12}$$

that is to say there exists a spinor $\psi(x)$ and $A(x)$ such that $\psi(x)$ is square summable and the field $B = \mathrm{curl} A(x)$ has finite energy. A simple exercise in scaling shows that if we set

$$\psi_\lambda(x) = \lambda^{\frac{3}{2}} \psi(\lambda x) \ , \tag{5.13}$$

and

$$A_\lambda(x) = \lambda A(\lambda x) \ , \tag{5.14}$$

then ψ_λ and A_λ still satisfy the three dimensional Dirac equation (5.12). If we choose ψ_λ and A_λ as trial functions in (58) then we obtain

$$\mathcal{E}(\psi_\lambda, A_\lambda) = \lambda\big[-Z\langle\psi|\frac{1}{|x|}|\psi\rangle + \varepsilon\int_{R^3}|B(x)|^2dx\big] \ , \tag{5.15}$$

which becomes arbitrarily negative provided

$$Z\alpha^2 > \frac{1}{8\pi}\frac{\int_{\mathbb{R}^3}|B(x)|^2dx}{\langle\psi, \frac{1}{|x|}\psi\rangle} \ . \tag{5.16}$$

In a certain sense this is a breakdown of the uncertainty principle. If we squeeze the wave function and the magnetic field simultaneously, the kinetic energy does not necessarily become large.

The collapse for a large fine structure constant can be understood in a manner similar to the relativistic case and we shall not repeat the argument here.

As far as the full problem (5.10) is concerned, Fefferman [16, 17, 18] was the first to provide a proof, however, he required that the fine structure constant had to be sufficently small. This was improved in [38] where a simple proof of (5.10) was given requiring that $\alpha \leq 0.041$. In particular it was found that for $\alpha = 1/137$ stability holds up to $Z = 1050$.

We shall follow this argument, but using the improved constants in the eigenvalue bounds of Theorem 3.4 as well as Theorem 4.3.

THEOREM 5.1. *The ground state energy of N electrons interacting with K fixed nuclei each having charge Z, i.e., the ground state energy $E_0(Z, B, N, K)$ of the Hamiltonian*

$$H = \sum_{j=1}^{N}[\sigma\cdot(-i\nabla_j + A(x_j))]^2 + V_c \tag{5.17}$$

acting on antisymmetric functions in $\wedge_{j=1}^{N}L^2(\mathbb{R}^3;\mathbb{C}^2)$, satisfies the estimate

$$E_0(Z, B, N, K) + \frac{1}{8\pi\alpha^2}\int B(x)^2dx \geq -C(N+K) \tag{5.18}$$

provided that

$$\frac{3\sqrt{3}}{32\alpha^2} > \frac{\pi}{2}Z + 2.2159q^{1/3}Z^{2/3} + 1.0307q^{1/3} \ . \tag{5.19}$$

In particular for $\alpha = 1/137$ and $q = 2$ there is stability for $Z < 1728$. for Z close to zero there is stability provided that $\alpha < 0.396q^{-1/6}$ which yields for $q = 2$ the bound $\alpha < 0.352$.

We see that in this theorem the parameter ranges contain the physical values.

What is the implication of such a result? We usually think of a magnetic field as external, that is, the magnetic field is produced by some external currents and not by the currents that are generated inside the piece of matter itself. The above result, however, holds irrespective of whether the magnetic field is external or not. This means that we allow the electrons to interact with the magnetic field that they create themselves. Thus, one must view the above model as a semiclassical caricature of non-relativistic quantum electrodynamics, that is as a 'theory' that incorporates the fields as dynamical quantities that fully interact with matter.

Proof of Theorem 5.1. We follow closely [38]

Using Theorem 4.3 we eliminate the Coulomb potential, i.e.,

$$H \geq \sum_{j=1}^{N} \left\{ [\sigma_j \cdot (p_j + A(x_j))]^2 - D|p_j + A(x_j)| \right\} \tag{5.20}$$

where D satisfies the estimate

$$D \geq \frac{\pi}{2} Z + 2.2159 q^{1/3} Z^{2/3} + 1.0307 q^{1/3} \ . \tag{5.21}$$

Thus we have reduced the problem to one of a single particle operator. Denote by $-e_1 \leq -e_2 \leq$ the negative eigenvalues of

$$h = [\sigma \cdot (p + A(x))]^2 - D|p + A(x)| \ , \tag{5.22}$$

then we have that

$$H \geq -q \sum_{i=1}^{[\frac{N}{q}]} e_i \tag{5.23}$$

where $[\frac{N}{q}]$ denote the smallest integer greater or equal to $\frac{N}{q}$. Obviously

$$a = \int_0^\infty \chi_{a>e}(e) de \tag{5.24}$$

and hence

$$q \sum_{i=1}^{[\frac{N}{q}]} e_i = q \int_0^\infty \sum_{i=1}^{[\frac{N}{q}]} \chi_{e_i>e}(e) de \tag{5.25}$$

$$= q \int_0^\infty \min \left(N_e(h), [\frac{N}{q}] \right) de \ , \tag{5.26}$$

where $N_e(h)$ is the number of eigenvalues of h below $-e$.

Now, recalling that the system must have a certain natural energy scale if it is stable at all, we denote this unknown energy scale by μ. Thus, we rewrite (5.26) as

$$q \int_0^\mu \min \left(N_e(h), [\frac{N}{q}] \right) de + q \int_\mu^\infty \min \left(N_e(h), [\frac{N}{q}] \right) de \ , \tag{5.27}$$

and bound it from above by

$$q \quad [\frac{N}{q}]\mu + q \int_\mu^\infty N_e(h) de \tag{5.28}$$

$$\leq \quad (N+q)\mu + q \int_\mu^\infty N_e(h) de \tag{5.29}$$

The first term should estimate the energy of the system, while the second term should be compensated by the field energy. Thus, we have to relate this term to something that has the same dimensionality as the field energy. This is achieved by a sliding energy scale; for all $e \geq \mu$ we have

$$[\sigma \cdot (p + A(x))]^2 \geq \frac{\mu}{e} [\sigma \cdot (p + A(x))]^2 \geq \frac{\mu}{e} (p + A(x))^2 - \frac{\mu}{e} \sigma \cdot B \ . \tag{5.30}$$

Further

$$|p + A(x)| \leq \frac{D}{2e} |p + A(x)|^2 + \frac{e}{2D} \ . \tag{5.31}$$

Hence,

$$h \geq \left(\frac{\mu}{e} - \frac{D^2}{2e}\right) |p + A(x)|^2 - \frac{\mu}{e}\sigma \cdot B - \frac{e}{2} =: h'_e =: \frac{1}{e}k - \frac{e}{2} \ , \tag{5.32}$$

where

$$k = \left(\mu - \frac{D^2}{2}\right) |p + A(x)|^2 - \mu\sigma \cdot B \tag{5.33}$$

and

$$N_e(h) \leq N_e(h'_e) \leq N_{e^2/2}(k) \ . \tag{5.34}$$

Now,

$$\begin{aligned} \int_\mu^\infty N_e(h)de \leq \int_0^\infty N_{e^2/2}(k)de &= \sqrt{2}\int_0^\infty N_{e^2}(k)de & (5.35)\\ &= \sqrt{2}\sum_j \lambda_j^{1/2} \ , & (5.36) \end{aligned}$$

where $-\lambda_j$ are the negative eigenvalues of the operator k.

Using the Lieb-Thirring inequality (3.29) for the square root of the eigenvalues leads to the estimate

$$\sum_j \lambda_j^{1/2} \leq 2\mu^2 \left(\mu - \frac{D^2}{2}\right)^{-3/2} \frac{1}{8\pi} \int B(x)^2 dx \ . \tag{5.37}$$

This naive application of the Lieb-Thirring inequality leads to a constant that is by a factor of two larger. This is due to the spin which forces us to count each eigenvalue twice. It is not known whether this factor 2 can be omitted if the bounds (3.29) are applied to our problem. (See [48] for another method where the factor of 2 can indeed be removed.)

Optimizing the above expression over μ yields

$$\mu = 2D^2$$

which yields the bound

$$H \geq -(N+q)2D^2 - \frac{32D}{3\sqrt{3}} \frac{1}{8\pi} \int B(x)^2 d^3x \ . \tag{5.38}$$

Thus, we get the stability condition $1/\alpha^2 > 32D/3\sqrt{3}$, i.e.,

$$\frac{3\sqrt{3}}{32\alpha^2} > \frac{\pi}{2}Z + 2.2159q^{1/3}Z^{2/3} + 1.0307q^{1/3} \ . \tag{5.39}$$

□

6 Interaction with the radiation field

As indicated in the last chapter, stability of matter interacting with magnetic fields has implications for matter interacting with the radiation field. We discuss first the results for non-relativistic quantum electrodynamics (QED), where the charges are treated as non - relativistic particles and the (relativistic) radiation field is quantized. The Hamiltonian, including spin in suitable units, is

$$H = \sum_{j=1}^N [\sigma \cdot (-i\nabla + \sqrt{\alpha}A(x)]^2 + \alpha V_c + H_f \ . \tag{6.1}$$

The field energy H_f is

$$H_f = \sum_{\lambda=1}^{2} \int_{\mathbb{R}^3} |k| a_\lambda^*(k) a_\lambda(k) dk \tag{6.2}$$

and the A-field is

$$A(x) = \frac{1}{2\pi} \sum_{\lambda=1}^{2} \int_{\mathbb{R}^3} \frac{dk}{\sqrt{|k|}} \chi_\Lambda(k) \varepsilon_\lambda(k) \left[e^{ik\cdot x} a_\lambda(k) + e^{-ik\cdot x} a_\lambda^*(k) \right] \ . \tag{6.3}$$

The coefficients a and a^* are now operators, the destruction and creation operators. They satisfy the commutation relations

$$[a_\mu(p), a_\nu^*(q)] = \delta_{\mu,\nu} \delta(p-q) \tag{6.4}$$

and the a's commute among each other as well as the a^*'s. The vectors $\varepsilon_\lambda(k), \lambda = 1, 2$ are the polarization vectors. Together with $k/|k|$ they form an orthonormal triple for every $k \neq 0$. The function χ_λ is a smooth function whose support is in a ball of radius Λ. This function imposes an *ultraviolet cutoff* on the vector potential $A(x)$. This is essential; without a cutoff the Hamiltonian (6.1) does not make any sense.

The Hilbert space of the photons is the Fock - space $\mathcal{F}$. This space is defined as

$$\mathcal{F} = \oplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^{n} L^2(\mathbb{R}^3; \mathbb{C}^2) \ , \tag{6.5}$$

where $\otimes_{\mathrm{s}}^{n}$ denotes the n-fold symmetric tensor product. Hence, the Hilbert space for the operator (6.1) is

$$\mathcal{H} = \wedge^N L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathcal{F} \tag{6.6}$$

The Hamiltonian (6.1) describes matter interacting with radiation at least at low energies. Unfortunately one has to impose an ultraviolet cutoff Λ to make the theory finite. It turns out that the physical quantities one calculates in practical applications depend weakly on Λ. So far, however, one does not know how to remove this cutoff.

Stability of the second kind for the Hamiltonian (6.1) was proved by Bugliaro, Fröhlich and Graf [3]. We state here another result due to Fefferman, Fröhlich and Graf [19, 20].

THEOREM 6.1. *For any value of α, for any value of Z and any value of the cutoff, there exists a constant $C(Z, \alpha)$ so that on $\mathcal{H}$*

$$H \geq -C(Z, \alpha)(\Lambda + 1)M \ . \tag{6.7}$$

The fact that no condition on the fine structure is required stems from the ultraviolet cutoff. This cutoff does not allow for the scaling of the magnetic field which was crucial for understanding the instabilities.

While the proof of this theorem is involved, it is not hard to describe one of the difficulties. The field energy H_f cannot be compared easily with the classical field energy $\int B(x)^2 dx$ which is in our context also an operator. The reason is that in passing from the classical field energy to the quantized field energy one has to normal order the operators, i.e., bring all the creation operators to the left and all the destruction operators to the right. Thus if we write $B(x) = D(x) + D^*(x)$ where D contains only destruction operators and D^* only creation operators then

$$\int_{\mathbb{R}^3} B(x)^2 dx = \int_{\mathbb{R}^3} \left[D^2(x) + D^{*2}(x) + 2D^*(x)D(x) + [D(x), D^*(x)] \right] dx \ . \tag{6.8}$$

However,

$$[D(x), D^*(x)] = -\frac{1}{4\pi^2} \int_{\mathbb{R}^3} \chi_\Lambda^2(k) |k| dk \tag{6.9}$$

which is not integrable with respect to x. The way out of this apparent dilemma is to consider a stronger version of the stability of matter problem (5.10) where the field energy is replaced by

$$\int_C B(x)^2 dx \tag{6.10}$$

where C is a set that is a union of balls whose center are at the position of the nuclei and whose radius is of the order of Λ^{-1}, the ultraviolet cutoff. Clearly, this expression does not have the problems mentioned above, i.e., the commutator when integrated over this set C yields a finite contribution.

Relativistic quantum electrodynamics is, in the perturbative realm, a very successful theory for describing certain interactions of charges particles with photons. One has not succeeded, however, to go beyond perturbation theory, in particular there is no Hamiltonian that describes the interaction of matter and radiation. There have been attempts at creating certain caricatures. The true equation governing the motion of a single electron is the Dirac equation. For a hydrogenic atom interacting with a meagnetic field the Dirac equation reads

$$D(A) := \boldsymbol{\alpha} \cdot (-i\nabla + \sqrt{\alpha} A(x)) + m\beta - \frac{Z\alpha}{|x|} \tag{6.11}$$

with

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \ \beta = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix},$$

Note that the Hilbert space is $L^2(\mathbb{R}^3; \mathbb{C}^4)$. The spectrum of the Dirac operator extends all the way to $-\infty$ and hence it makes little sense to try to define stability. While the positive energy states are electrons states, the negative energy states are associated with positrons and to turn this into a feasible model requires second quantization of the electron positron field. As mentioned before there is so far no consistent theory of this kind.

Another possibility taken by physicists is to find an approximate theory for the case where pair creation is neglected. It is natural to interpret the positive energy states of the Dirac operator as electrons and the negative energy states as positrons. Clearly, this depends strongly on what kind of Dirac operator one considers. One way, which is standard in the physics literature is to define the electron space as the positive spectral subspace of the the free Dirac operator. Denote by P_j^+ the projection onto the positive spectral subspace of the *free* Dirac operator acting on the coordinates of the j-th particle. Next we define the N-particle Hilbert space

$$\mathcal{H} = P^+ \wedge_{j=1}^N L^2(\mathbb{R}^3; \mathbb{C}^4) \tag{6.12}$$

where $P^+ = \prod_{j=1}^N P_j^+$. It is easy to see that the this space is anti-symmetric under exchange of particle labels. For classical magnetic fields one defines the Hamiltonian

$$H(A(x)) = P^+ \left[D_j(A) - \alpha V_c \right] P^+ + \frac{1}{8\pi} \int |B(x)|^2 dx \ . \tag{6.13}$$

Note that this Hamiltonian is no longer gauge invariant and hence one expects some problems with it. This is indeed the case as was shown in [41]. We quote their result in the version of [23]. They show that for any fixed $\alpha > 0$ there exists a field $A(x)$ and a constant $C(A)$ so that

$$\inf_{\Psi, \|\Psi\|=1, \gamma, \delta > 0} \langle \Psi, H(\gamma A(\delta x)) \Psi \rangle = -\infty \ . \tag{6.14}$$

Thus there is no stability of any kind.

If one decides to keep gauge invariance one has to consider the spectral projection $P^+_{j,A}$ of the Dirac operator including the magnetic field. This leads to a Hilbert space that depends on the vector potential A

$$\mathcal{H}_A = P^+_A \wedge_{j=1}^N L^2(\mathbb{R}^3;\mathbb{C}^4) \tag{6.15}$$

where as before $P^+_A = \prod_{j=1}^N P^+_{j,A}$. Likewise the Hamiltonian for a classical field is now given by

$$H(A(x)) = P^+_A \left[D_j(A) - \alpha V_c\right] P^+_A + \frac{1}{8\pi}\int |B(x)|^2 dx \ . \tag{6.16}$$

It was shown in [41] that for classical fields and sufficiently small values of α and $Z\alpha$ this Hamiltonian is stable on the Hilbert space $\mathcal{H}_A$.

Next let us consider the same model, except that the magnetic field is quantized, i.e., we consider the Hamiltonian

$$H_N^{\rm phys} = P^+ H'_N P^+ \ . \tag{6.17}$$

where

$$H'_N = \sum_{i=1}^N D_i(A) + \alpha V_c + H_f \ . \tag{6.18}$$

The vector potential $A(x)$ is now given by (6.3). Again, we have to decide which Hilbert space to take. One possibility is

$$\mathcal{H} = P^+ \wedge_{j=1}^N L^2(\mathbb{R}^3;\mathbb{C}^4) P^+ \otimes \mathcal{F} \tag{6.19}$$

where P^+ is again defined using the positive spectral subspace of the free Dirac operator. It is shown in [23] that for this model there is no stability of the second kind.

Defining the electrons as the states in the positive spectral subspace of the Dirac operator including the *quantized radiation field* leads to the Hilbert space

$$\mathcal{H} = P^+_A \wedge_{j=1}^N L^2(\mathbb{R}^3;\mathbb{C}^4) \otimes \mathcal{F} P^+_A \tag{6.20}$$

where one should take note that since the A field acts nontrivially on the Fock space, so does the projection P^+_A. *The separation of electrons and photons is now lost.*

The following theorem is proved in [37].

THEOREM 6.2 (Relativistic Quantum electrodynamic Stability)**.** *Assume that Z and α are such that there is a solution κ and $\varepsilon \geq 0$ to the three inequalities*

$$\kappa \ \geq \ \max\{64.5,\, \pi Z\} \tag{6.21}$$

$$(\kappa\alpha)^2 \ < \ 1 - \varepsilon \leq 1 \tag{6.22}$$

$$\frac{(1-\varepsilon)^2\alpha}{(1-\varepsilon-\kappa^2\alpha^2)^{3/2}} \ \leq \ \frac{1}{8\pi(0.060)} \tag{6.23}$$

Then H_N^{phys} in (6.17) with P^+ replaced by P^+_A, is bounded below by

$$H_N^{phys} \geq +\sqrt{\varepsilon}\ m\, N - \frac{18\Lambda}{\pi} K C_2^3 \ , \tag{6.24}$$

where

$$C_2^4 = \frac{N}{K}\frac{6\sqrt{1-\varepsilon} + (\alpha/2)(\sqrt{2Z}+2.3)^2}{27/2\pi} \ . \tag{6.25}$$

In particular, $Z \leq 42$ is allowed when $\alpha = 1/137$.

Note that κ is an auxiliary quantity that does not show up in the actual estimates. The proof of this theorem although somewhat involved follows closely the one given in [41]. The main difference is that in order to deal with the commutator (6.9) the problem is reduced to a relativistic problem where the kinetic energy is localized in the vicinity of the nuclei.

Thus, depending on the various parameters there are various possibilites of stability and instability, stability of the second kind. These are all listed in the table below taken from [37].

Electrons defined by projection onto the positive subspace of $D(0)$, the free Dirac operator

	Classical or quantized field without cutoff Λ $\alpha > 0$ but arbitrarily small.	Classical or quantized field with cutoff Λ $\alpha > 0$ but arbitrarily small.
Without Coulomb potential αV_c	Instability of the first kind	Instability of the second kind
With Coulomb potential αV_c	Instability of the first kind	Instability of the second kind

Electrons defined by projection onto the positive subspace of $D(A)$, the Dirac operator with field

	Classical field with or without cutoff Λ or quantized field with cutoff Λ
Without Coulomb potential αV_c	The Hamiltonian is positive
With Coulomb potential αV_c	Instability of the first kind when either α or $Z\alpha$ is too large
	Stability of the second kind when both α and $Z\alpha$ are small enough

One can see from these considerations that the requirement that matter be stable poses restrictions for some of the models. Some require that certain coupling constants be small while others require that the theory be modified.

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