Appendix B

Nonparametric Riemannian geodesics

I'm going to reproduce here some work of Ty Bondurant on Exercise 1.41 and attempt to extend that work. Bondurant points out that starting with

$$\text{length}_{\mathcal{B}}[\alpha] = \int_{(a,b)} \frac{4}{4 + |\alpha|^2} \, |\alpha'|$$

one can take the special case $\alpha(x) = (x, h(x))$ to obtain a functional $\ell : C^1[a,b] \to [0,\infty)$ given by

$$\ell[h] = \int_{(a,b)} \frac{4}{4 + x^2 + h^2} \, \sqrt{1 + h'^2}.$$

Assuming h minimizes ℓ one should have $F'(0) = 0$ for

$$F(t) = \ell[h + t\phi]$$

whenever $\phi \in C_0^1[a,b]$. That is,

$$F(t) = \int_{(a,b)} \frac{4}{4 + x^2 + (h + t\phi)^2} \, \sqrt{1 + (h' + t\phi')^2}.$$

so

$$
\begin{aligned}
F'(t) &= \frac{d}{dt}\int_{(a,b)} \frac{4}{4+x^2+(h+t\phi)^2}\ \sqrt{1+(h'+t\phi')^2} \\
&= \int_{(a,b)} \frac{\partial}{\partial t}\left(\frac{4}{4+x^2+(h+t\phi)^2}\ \sqrt{1+(h'+t\phi')^2}\right) \\
&= 4\int_{(a,b)} \left(-\frac{2(h+t\phi)\phi}{[4+x^2+(h+t\phi)^2]^2}\ \sqrt{1+(h'+t\phi')^2}\right. \\
&\qquad\qquad\qquad \left. + \frac{1}{4+x^2+(h+t\phi)^2}\ \frac{(h'+t\phi')\phi'}{\sqrt{1+(h'+t\phi')^2}}\right) \\
&= 4\int_{(a,b)} \left(-\frac{2(h+t\phi)}{[4+x^2+(h+t\phi)^2]^2}\ \sqrt{1+(h'+t\phi')^2}\ \phi\right. \\
&\qquad\qquad\qquad \left. + \frac{1}{4+x^2+(h+t\phi)^2}\ \frac{(h'+t\phi')}{\sqrt{1+(h'+t\phi')^2}}\ \phi'\right).
\end{aligned}
$$

Evaluating at $t=0$, this gives

$$
\begin{aligned}
F'(0) &= 4\int_{(a,b)} \left(-\frac{2h}{(4+x^2+h^2)^2}\ \sqrt{1+h'^2}\ \phi\right. \\
&\qquad\qquad \left. + \frac{1}{4+x^2+h^2}\ \frac{h'}{\sqrt{1+h'^2}}\ \phi'\right) \\
&= -8\int_{(a,b)} \frac{h}{(4+x^2+h^2)^2}\ \sqrt{1+h'^2}\ \phi \\
&\qquad\qquad + 4\int_{(a,b)} \frac{1}{4+x^2+h^2}\ \frac{h'}{\sqrt{1+h'^2}}\ \phi'.
\end{aligned}
$$

Assuming the factor

$$
\frac{1}{4+x^2+h^2}\ \frac{h'}{\sqrt{1+h'^2}}
$$

in the integrand of the second integral is continuously differentiable and integrating by parts we find

$$\begin{aligned}\int_{(a,b)} \frac{1}{4+x^2+h^2}\ \frac{h'}{\sqrt{1+h'^2}}\ \phi' &= \left(\frac{1}{4+x^2+h^2}\ \frac{h'}{\sqrt{1+h'^2}}\ \phi\right)\Big|_{x=a}^{b} \\ &\qquad - \int_{(a,b)} \left(\frac{1}{4+x^2+h^2}\ \frac{h'}{\sqrt{1+h'^2}}\right)'\ \phi \\ &= -\int_{(a,b)} \left(\frac{1}{4+x^2+h^2}\ \frac{h'}{\sqrt{1+h'^2}}\right)'\ \phi\end{aligned}$$

since $\phi(a) = \phi(b) = 0$. Recombining the integrals gives

$$\begin{aligned}F'(0) = -4\int_{(a,b)} \Bigg[&\frac{2h}{(4+x^2+h^2)^2}\ \sqrt{1+h'^2} \\ &+ \left(\frac{1}{4+x^2+h^2}\ \frac{h'}{\sqrt{1+h'^2}}\right)'\ \Bigg]\ \phi.\end{aligned}$$

By the fundamental lemma of vanishing integrals we obtain the ordinary differential equation

$$\left(\frac{h'}{(4+x^2+h^2)\sqrt{1+h'^2}}\right)' + \frac{2h\sqrt{1+h'^2}}{(4+x^2+h^2)^2} = 0.$$

There is nothing particularly singular looking about this ordinary differential equation. Expanding the derivative and multiplying by a factor

$$(4+x^2+h^2)^2\sqrt{1+h'^2}$$

we obtain a quasilinear form that may be more suitable for numerical consideration, though some mathematical software may handle the form above directly. Specifically, for $h \in C^2[a,b]$ we can write

$$(4+x^2+h^2)h'' - \left(2x + 2hh' + (4+x^2+h^2)\frac{h'h''}{1+h'^2}\right) h' + 2h(1+h'^2) = 0$$

or

$$(4+x^2+h^2)\left(1 - \frac{h'^2}{1+h'^2}\right) h'' + 2\,(h - xh') = 0.$$

That is,

$$\frac{4+x^2+h^2}{1+h'^2}\,h'' + 2\,(h - xh') = 0. \tag{B.1}$$

I'm going to pair with this first an initial condition $h(a) = y_a$ for some a with $-1 < a < 0$ and some y_a with $0 \le y_a < \sqrt{1-a^2}$. In Figure B.1 I've solved numerically for solutions with $h'(a) = 0$, 0.1, 0.2, 0.3, and 0.4. One of

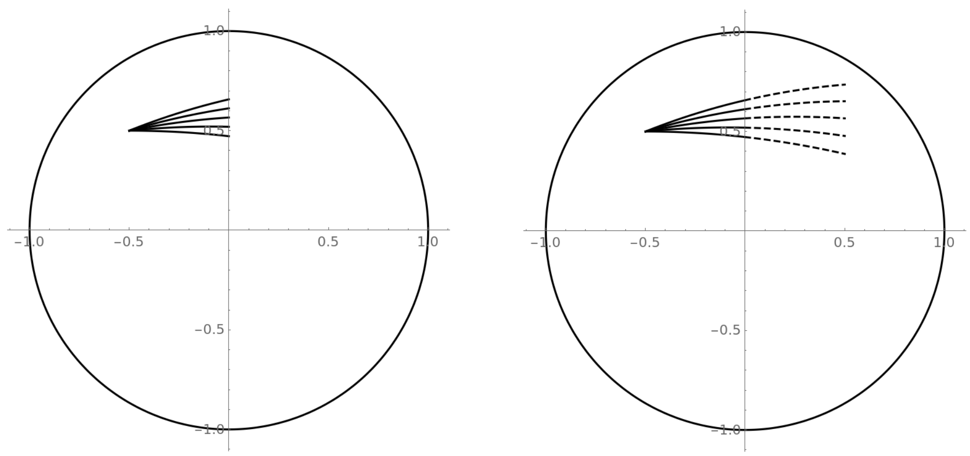

Figure B.1: Numerical approximation of solutions of equation (B.1).

the first things that strikes me is the (vague) resemblance of these solutions to the circular paths illustrated in Figure 3.2 accompanying Exercise 3.3 in Chapter 3. Of course, those "circles" are illustrated in the manifold $\mathcal{B}$, and here I've produced an illustration with evident coordinates as in $B_1(\mathbf{0})$. Also, there is a 90 degree rotation and the paths in Exercise 3.3 are explicit circular paths while these paths are some (presumably complicated) solutions of an ODE. Nevertheless, my inclination is to take a close look at Exercise 3.3.

Let me rotate by 90 degrees in order to match more closely Ty's calculation. After the rotation, I'm considering circular arcs parameterized by

$$\alpha(t) = (0, y) + r(-\sin t, \cos t) \qquad \text{for } 0 \le t \le \theta$$

with $y < 0$,

$$r = \sqrt{a^2 + (y_a - y)^2}, \qquad \text{and} \qquad \theta = \tan^{-1}\left(\frac{a}{y - y_a}\right).$$

In this case, $\alpha' = -r(\cos t, \sin t)$ and

$$|\alpha|^2 = r^2 \sin^2 t + (y + r\cos t)^2 = r^2 + y^2 + 2ry\cos t.$$

Thus,

$$\begin{aligned}
\text{length}_{\mathcal{B}}[\alpha] &= \int_0^\theta \frac{4r}{4 + r^2 + y^2 + 2ry\cos t}\,dt \\
&= \frac{4r}{4 + r^2 + y^2} \int_0^\theta \frac{1}{1 - c\cos t}\,dt
\end{aligned}$$

where

$$c = -\frac{2ry}{4 + r^2 + y^2} > 0.$$

Substituting $1 = \cos^2(t/2) + \sin^2(t/2)$ and $\cos t = \cos^2(t/2) - \sin^2(t/2)$, and noting also that $c < 1$, we can write

$$\begin{aligned}
\text{length}_{\mathcal{B}}[\alpha] &= -\frac{2c}{y} \int_0^\theta \frac{1}{(1-c)\cos^2(t/2) + (1+c)\sin^2(t/2)}\,dt \\
&= -\frac{2c}{y(1-c)} \int_0^\theta \frac{\sec^2(t/2)}{1 + \left(\sqrt{\frac{1+c}{1-c}}\tan(t/2)\right)^2}\,dt \\
&= -\frac{4c}{y\sqrt{1-c^2}} \int_0^{\sqrt{\frac{1+c}{1-c}}\tan(\theta/2)} \frac{1}{1+u^2}\,du \\
&= -\frac{4c}{y\sqrt{1-c^2}} \tan^{-1}\left(\sqrt{\frac{1+c}{1-c}}\tan(\theta/2)\right).
\end{aligned}$$

Notice that c and θ are functions of y, we have thus reduced $\text{length}_{\mathcal{B}}[\alpha]$ to a function of y. This is a little bit of a complicated function of y.

In Exercise 3.3 we are also asked to compute the Euclidean length, and in this case, the Euclidean length is given by

$$\text{length}[\alpha] = r\theta = \sqrt{a^2 + (y_a - y)^2}\ \tan^{-1}\left(\frac{a}{y - y_a}\right).$$

There are perhaps a couple interesting things to do at this point. One is that we expect the Euclidean length is a monotone function of the center height

y. In fact, length$[\alpha]$ should be monotonically increasing in y. This may not be entirely obvious from the formula. The radius

$$\sqrt{a^2 + (y_a - y)^2}$$

is of course a decreasing function in y while the angle

$$\tan^{-1}\left(\frac{a}{y - y_a}\right)$$

is clearly increasing. These two assertions are easy to verify. The product, however, is a little more tricky. We first observe that there is a limit as y tends to $-\infty$, and that limit should be the euclidean lenghth of the segment connecting (a, y_a) to $(0, y_a)$, namely $-a$. In fact,

$$\lim_{y \searrow -\infty} \sqrt{a^2 + (y_a - y)^2} = +\infty$$

and

$$\lim_{y \searrow -\infty} \tan^{-1}\left(\frac{a}{y - y_a}\right) = 0,$$

so

$$\frac{\tan^{-1}\left(\frac{a}{y - y_a}\right)}{\left(a^2 + (y_a - y)^2\right)^{-1/2}}$$

is indeterminate. Thus, we consider

$$\frac{d\theta}{dy} = -\frac{a}{(y - y_a)^2} \; \frac{(y - y_a)^2}{(y - y_a)^2 + a^2} = -\frac{a}{a^2 + (y - y_a)^2} > 0$$

as expected and

$$\frac{d}{dy}\left(\frac{1}{r}\right) = (y_a - y)\left(a^2 + (y_a - y)^2\right)^{-3/2} > 0$$

as expected. Most importantly,

$$\frac{\frac{d\theta}{dy}}{\frac{d}{dy}\left(\frac{1}{r}\right)} = -\frac{a}{y_a - y} \; \sqrt{a^2 + (y - y_a)^2} \tag{B.2}$$

which tends to $-a$ as expected. As for the monotonicity, the expression for

$$\frac{d}{dy} \, \text{length}[\alpha]$$

is not so obviously positive (at least not obviously to me). Sometimes in cases like this, the second derivative can clarify what is going on:

$$\frac{d^2}{dy^2}\,\text{length}[\alpha] = \frac{a^2}{(a^2 + (y_a - y)^2)^{3/2}}\ \tan^{-1}\left(\frac{a}{y - y_a}\right) > 0. \tag{B.3}$$

In view of the limiting value

$$\lim_{y \searrow -\infty} \text{length}[\alpha] = -a$$

obtained via (B.2) and the convexity expressed in (B.3) we conclude that indeed $\text{length}[\alpha]$ is increasing as a function of y as expected.

We may have much less intuition concring the behavior of $\text{length}_{\mathcal{B}}[\alpha]$ as a function of y, but the monotonicity of the Euclidean length $\text{length}[\alpha]$ also suggests the possibility of considering $\text{length}_{\mathcal{B}}[\alpha]$ as a function of $\text{length}[\alpha]$. We know there should hold $\text{length}_{\mathcal{B}}[\alpha] < \text{length}[\alpha]$ and also, there should be a limit

$$\begin{aligned}
\lim_{y \searrow -\infty} \text{length}_{\beta}[\alpha] &= \int_a^0 \frac{4}{4 + y_a^2 + t^2}\, dt \\
&= \frac{4}{4 + y_a^2} \int_a^0 \frac{1}{1 + \left(\frac{t}{\sqrt{4+y_a^2}}\right)}\, dt \\
&= \frac{4}{\sqrt{4 + y_a^2}} \int_{a/\sqrt{4+y_a^2}}^0 \frac{1}{1 + u^2}\, du \\
&= \frac{4}{\sqrt{4 + y_a^2}}\ \tan^{-1}\left(\frac{-a}{\sqrt{4 + y_a^2}}\right).
\end{aligned} \tag{B.4}$$

An initial numerical plot of $\ell(y) = \text{length}_{\mathcal{B}}[\alpha]$ for the values

$$(a, y_a) = 0.5(\cos(3\pi/4), \sin(3\pi/4))$$

gives some confirmation that the limiting value is correctly calculated, though the monotonicity (or lack thereof) is not clearly visible. See Figure B.2 (upper left). A more exaggerated plot over $-8 < y < -3$ shows an apparent local minimum, but the limiting value is lost at this scale as illustrated in Figure B.2 (upper right). Adjusting parameters we can see plausible evidence (lower left and right) that $\text{length}_{\mathcal{B}}[\alpha]$ has a unique minimum value among the

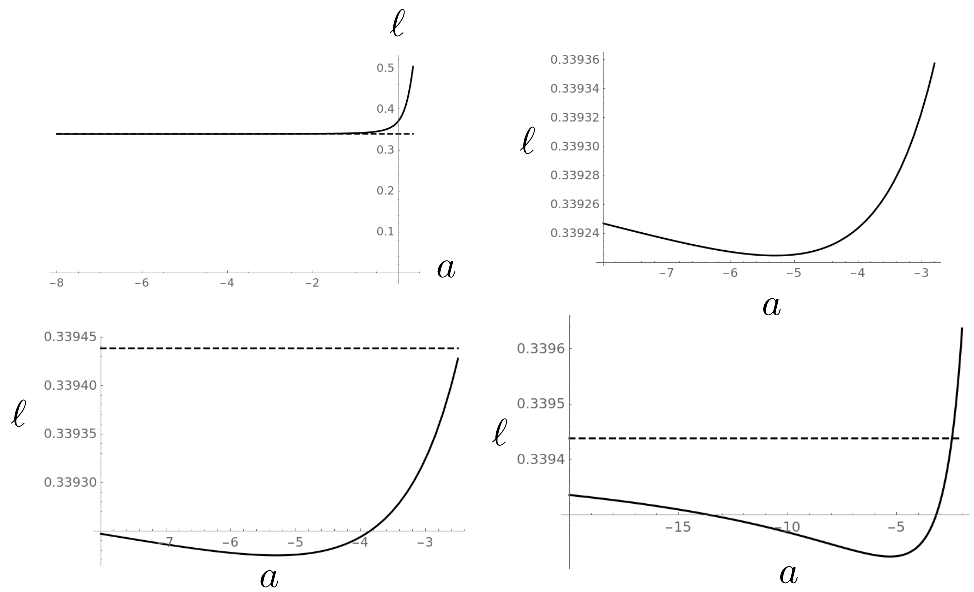

Figure B.2: Numerical plot(s) of $\text{length}_{\mathcal{B}}[\alpha]$ as a function of y for $(a, y_a) = 0.5(\cos(3\pi/4), \sin(3\pi/4))$. For $-8 < y < 0.5\sin(3\pi/4)$ (upper left) the limiting value looks correct, but the monotonicity is unclear. The upper right and lower left plots are approximately on the interval $-8 < y < 3$ with the latter including enough y values to show the relation with the limiting value. The last plot is for the interval $-20 < y < -2$ and suggests clearly a unique local minimum Riemannian length for a circular arc (not a straight line) at least among the circular arcs considered.

circular arcs under consideration corresponding to a parameter value around $y = -5.3$. As a further indication that values of $\text{length}_{\mathcal{B}}[\alpha]$ tend monotonically to the calculated limiting value given in (B.4) given approximately by 0.339438 in this case one can calculate values like those given in Table B.1.

Returning to consideration of the minimum value of $\text{length}_{\mathcal{B}}[\alpha]$ when considered as a function of y, one can also plot $\text{length}_{\mathcal{B}}[\alpha]$ as a function of the Euclidean length $\text{length}[\alpha]$ of these circular arcs which is better understood. Figure B.3 gives plots for the special case $(a, y_a) = 0.5(\cos(3\pi/4), \sin(3\pi/4))$ considered above. Again, these plots suggest a unique value $y_{\min} < 0$ corresponding to a circular arc for which $\text{length}_{\mathcal{B}}[\alpha]$ takes a minimum among the

$y \doteq$	$\text{length}_{\mathcal{B}}[\alpha] \doteq$
-10	0.339269
-100	0.339415
-1000	0.339436
-10 000	0.339438

Table B.1: Approximate Riemannian lengths of “circular” arcs passing through (a, y_a) in $\mathcal{B}$ in the case $a = 0.5\cos(3\Pi/4) < 0$ and $y_b = 0.5\sin(3\pi/4) > 0$.

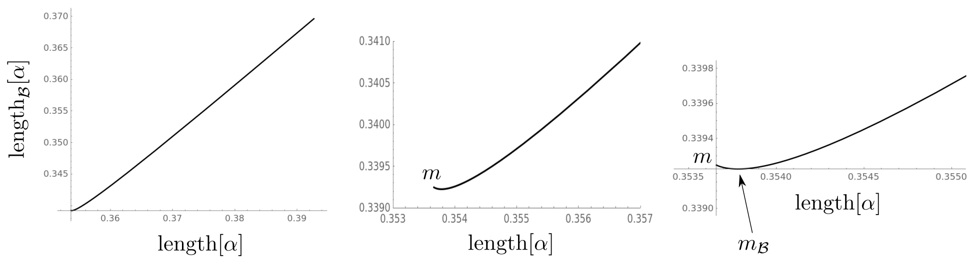

Figure B.3: Numerical plot(s) of $\text{length}_{\mathcal{B}}[\alpha]$ as a function of $\text{length}[\alpha]$ for $(a, y_a) = 0.5(\cos(3\pi/4), \sin(3\pi/4))$. $-8 < y < 0.5\sin(3\pi/4)$ (left). The point m indicates the limiting (Euclidean) straight line segment with minimum Euclidean length. On the right the point $m_{\mathcal{B}}$ suggests a circular arc with minimum Riemannian length.

circular arcs under consideration. The expression for

$$\frac{d}{dy}\text{length}_{\mathcal{B}}[\alpha]$$

is somewhat complicated, but it can be calculated, and a numerical solution for the equation

$$\frac{d}{dy}\text{length}_{\mathcal{B}}[\alpha] = 0$$

when a particular point (a, y_a) is specified is easy to find. One obtains a value

$$y_{\min} \doteq -5.5033 \tag{B.5}$$

for $(a, y_a) = 0.5(\cos(3\pi/4), \sin(3\pi/4))$.

At this point, let me pause to summarize. We have a nonsingular second order ordinary differential equation

$$N[h] = \left(4 + x^2 + h^2\right) h'' + 2\,(h - xh')\left(1 + h'^2\right) = 0. \tag{B.6}$$

given in (B.1), and we've plotted solutions of this equation in a special case in Figure B.1. I have not plotted large enough portions of these solutions to suggest they are circular arcs, but were I to plot these solutions over longer intervals, what I would find woudl be indeed consistent with the "guess" that they are (or might be) circular arcs. On the other hand, we've considered a certain family of circular arcs passing through the point (a, y_a) and found that in this family continuously depending on a parameter y there appears to be one of them with $\text{length}_{\mathcal{B}}[\alpha]$ smaller than all the others. Hopefully, all this should motivate expressing these circular arcs as graphs of functions $h = h(x)$ depending on the center height parameter y and seeing what happens if we substitute these particular functions h in the differential equation (B.6). In any case, that is the calculation I am going to make. The circular graphs with center at $(0, y)$ and passing through (a, y_a) are given by

$$h(x) = y + \sqrt{a^2 + (y_a - y)^2 - x^2}.$$

Calculating I find

$$h' = -\frac{x}{\sqrt{a^2 + (y_a - y)^2 - x^2}},$$

$$\begin{aligned} h'' &= -\frac{1}{\sqrt{a^2 + (y_a - y)^2 - x^2}} - \frac{x^2}{[a^2 + (y_a - y)^2 - x^2]^{3/2}} \\ &= -\frac{a^2 + (y_a - y^2)}{[a^2 + (y_a - y)^2 - x^2]^{3/2}}, \end{aligned}$$

$$4 + x^2 + h^2 = 4 + y^2 + a^2 + (y_a - y)^2 + 2y\ \sqrt{a^2 + (y_a - y)^2 - x^2},$$

$$\begin{aligned} h - xh' &= y + \sqrt{a^2 + (y_a - y)^2 - x^2} + \frac{x^2}{\sqrt{a^2 + (y_a - y)^2 - x^2}} \\ &= y + \frac{a^2 + (y_a - y^2)}{\sqrt{a^2 + (y_a - y)^2 - x^2}}, \qquad \text{and} \end{aligned}$$

$$1 + h'^2 = \frac{a^2 + (y_a - y)^2}{a^2 + (y_a - y)^2 - x^2}.$$

Consequently,

$$\begin{aligned} N[h] &= -(4 + y^2)\frac{a^2 + (y_a - y)^2}{[a^2 + (y_a - y)^2 - x^2]^{3/2}} - \frac{[a^2 + (y_a - y^2)]^2}{[a^2 + (y_a - y)^2 - x^2]^{3/2}} \\ &\quad - 2y\ \frac{a^2 + (y_a - y^2)}{a^2 + (y_a - y)^2 - x^2} \\ &\qquad + 2y\frac{a^2 + (y_a - y^2)}{a^2 + (y_a - y)^2 - x^2} + 2\frac{[a^2 + (y_a - y^2)]^2}{[a^2 + (y_a - y)^2 - x^2]^{3/2}} \\ &= -(4 + y^2)\frac{a^2 + (y_a - y)^2}{[a^2 + (y_a - y)^2 - x^2]^{3/2}} + \frac{[a^2 + (y_a - y^2)]^2}{[a^2 + (y_a - y)^2 - x^2]^{3/2}} \\ &= [a^2 + (y_a - y)^2]\ \frac{a^2 + (y_a - y)^2 - 4 - y^2}{[a^2 + (y_a - y)^2 - x^2]^{3/2}} \\ &= [a^2 + (y_a - y)^2]\ \frac{-2y_a\ y - (4 - a^2 - y_a^2)}{[a^2 + (y_a - y)^2 - x^2]^{3/2}}. \end{aligned}$$

A first observation is that this quantity vanishes (and h is a circular arc solution of the ODE) exactly if y takes the unique negative value

$$y = -\frac{4 - a^2 - y_a^2}{2y_a}.$$

Taking $(a, y_a) = 0.5(\cos(3\pi/4), \sin(3\pi/4))$ we find this value to satisfy $y_{\min} \doteq -5.3033$. This, of course, might be viewed initially as something of a coincidence. In fact, however, if there is a Riemannian length minimizing path for some Weierstrass minimization problem resulting in the ODE (B.1)/(B.6), then we have pretty good reason to believe that an admissible solution of that problem will be a minimizer. At this point, we have in hand a solution of an ODE, but we need to go back and properly pose the minimization problem for which this solution gives the minimizer.

Before I proceed to consideration of the appropriate variational problem, I wish to point out something about the value of the ordinary differential operator $N[h]$ on the functions h corresponding to circular arcs which are **not** solutions of the ordinary differential equation. They have an interesting property, which may not seem interesting at first but, as we should see, does turn out to be interesting. Let us consider the nonsingular ordinary differential equation

$$M[h] = \frac{1}{(1+h'^2)^{3/2}}\ N[h] = \frac{4 + x^2 + h^2}{(1+h'^2)^{3/2}}\ h'' + 2\frac{h - xh'}{\sqrt{1+h'^2}} = 0 \qquad \text{(B.7)}$$

involving the ordinary differential operator $M : C^2[a,b] \to C^0[a,b]$. If we evaluate M on the functions $h(x) = y + \sqrt{a^2 + (y_a - y)^2 - x^2}$ we find

$$M[h] = -\frac{2y_a\ y + 4 - a^2 - y_a^2}{\sqrt{a^2 + (y_a - y)^2}}. \qquad \text{(B.8)}$$

Of course, the vanishing of M gives the same (probable) minimizer corresponding to $y = y_{\min}$, but notice the value of M on the other circular arcs. The ODE is **nonautonomous**, by virtue of the explicit dependence on the independent variable x, so we should expect $M[h]$ also, in most instances, to depend on x. However, the value $M[h]$ in (B.8) does not depend on x. The value is nonzero in general, but it is a nonzero **constant**, which is unexpected—and says something interesting about those other circular arcs.

Exercise B.1. If you think back to Weierstrass' original problem about minimizers for the Euclidean length of a path connecting two points in the plane, for which paths is the value of the associated ordinary differential operator (or some appropriate modification of it making it more geometric) a constant?

Though I did not write it down above, Ty started with a minimization problem based on the admissible class

$$\mathcal{A} = \{h \in C^1[a,b] : h(a) = y_a,\ h(b) = y_b\}.$$

If we consider the circular arcs which are graphs over the interval $[a,0]$ and their lengths as calculated above, then they do not fall into such an admissible class. Notice that if we take $b = 0$, then the values

$$h(0) = y + \sqrt{a^2 + (y_a - y)^2}$$

are not all the same (independent of the center paramter y). If we take, however, $b = -a > 0$ and consider

$$\mathcal{A}_{\text{sym}} = \{h \in C^1[a,-a] : h(a) = y_a,\ h(-a) = y_a\},$$

Then for each $h(x) = y + \sqrt{a^2 + (y_a - y)^2 - x^2}$ we have

$$h(-a) = y + \sqrt{(y_a - y)^2} = y_a.$$

Thus, all the circular graphs are admissible for this particular problem and the lengths of each such circular graph is exactly twice the value we have calculated for it. In particular, the same half-arc we have identified as minimizing among the (non-admissible) circular graphs over $[a,0]$ will be admissible and also minimizing over $[a,-a]$, that is in the particular admissible class $\mathcal{A}_{\text{sym}}$.

To summarize, there are two Weierstrass type minimization for Riemannian length we can pretty confidently solve in $\mathcal{B}$ using the ODE (B.1)/(B.6)/(B.7), then for $a < 0$ and $y_a > 0$ we can take

$$\mathcal{A}_0 = \{h \in C^1[a,0] : h(a) = y_a,\ h(0) = 0\} \tag{B.9}$$

and

$$\mathcal{A}_{\text{sym}} = \{h \in C^1[a,-a] : h(a) = y_a,\ h(-a) = y_a\}. \tag{B.10}$$

There are several observations allowing one to solve more general problems using just what we have observed above. The first important observation involves the theory of ordinary differential equations. Each **initial value problem**

$$\begin{cases} M[h] = 0, & a < x < b \\ h(a) = y_a, & h'(a) = m \end{cases} \tag{B.11}$$

we can solve has precisely one solution. Thus, on the one hand, if we can find a solution $h_1 \in C^2[a,b]$ of the **two point boundary value problem**

$$\begin{cases} M[h] = 0, & a < x < b \\ h(a) = y_a, & h(b) = y_b, \end{cases} \tag{B.12}$$

then the function we have in hand, is the unique solution of (B.11) with $m = h_1'(a)$. This has the following application: If we have a solution h_1, for example, a minimizing solution associated with (B.9) or (B.10), then the restriction (or extension) of that solution satisfying

$$\begin{cases} M[h] = 0, & a < x < b \\ h(a) = y_a, & h(b) = h_1(b), \end{cases}$$

is also a probable minimizer associated with the Weierstrass minimization of $\text{length}_{\mathcal{B}}[\alpha]$ in

$$\mathcal{A} = \{h \in C^1[a,b] : h(a) = y_a,\ h(b) = h_1(b)\}.$$

Note carefully that our application of the uniqueness theorem for solutions of initial value problems does not immediately rule out the possibility of some other function $h_2 \in \mathcal{A}$ satisfying (B.12) and having $h_2'(b) \neq h_1'(b) = m$. If that were to happen, then one would need to compare the lengths associated with the graphs h_1 and h_2 (and any other possible candiate minimizers). Also, in the general setting of such minimization problems, it can turn out that no function satisfies the ordinary differential equation resulting from the variational procedure (and that there is no minimizer). It turns out that none of these complicated things happen when it comes to minimizing $\text{length}_{\mathcal{B}}$ in $\mathcal{B}$. If you find a solution of the ODE, you are seeing an actual minimizer, but realize that I'm just telling you that. We have not proved anything like that. On the toher hand, if you believe $\text{length}_{\mathcal{B}}$ minimizing paths exist, then these paths will solve the ODE, so if you understand all solutions of the ODE, then you understand something about the structure of the collection of the actual length minimizing paths. This is certainly a good first goal.

Returning to the consideration of existence and uniqueness for ODEs, it is also the case that for every $m \in \mathbb{R}$, the initial value problem (B.11) has a unique **local solution** meaning that for some $\epsilon > 0$, there is a unique solution $h \in C^2[a, a+\epsilon]$ of (B.11). This also has consequences. Any one of these solutions $h_1 \in C^2[a, a+\epsilon]$ is also a probable minimizer for the Weierstrass type minimization problem associated with

$$\mathcal{A}_\epsilon = \{h \in C^1[a, a+\epsilon] : h(a) = y_a,\ h(a+\epsilon) = h_1(a+\epsilon)\}.$$

In particular, if we knew other solutions (with different values for $h'(a) = m$) were circular arcs/graphs, then we could pretty confidently conlude that all shortest paths in $\mathcal{B}$ were circular arcs (or straight lines).

To illustrate what this might mean consider the situation of minimizing Euclidean length in $B_1(\mathbf{0})$. Each point $\mathbf{x} \in B_1(\mathbf{0})$ (or in $\mathbb{R}^2$ for that matter) has a collection of length minimizing paths **emanating** from it. These paths are straight lines or (geodesic) rays. Also, given any other point $\mathbf{y} \in \mathbb{R}^2$ with $\mathbf{y} \neq \mathbf{x}$, there is precisely one Euclidean geodesic ray emanating from $\mathbf{x}$ and connecting to $\mathbf{y}$. Consider for a moment what it would mean to obtain an assertion like this for $\mathcal{B}$ or one of the supermanifolds containing $\mathcal{B}$. Before I state a "proposition" for you to consider as a possible outcome of such imagining, let me make two preliminary comments.

1. It would be natural and useful to consider this question for more general paths than those which can be expressed as graphs. This should be expected. Even in Euclidean space some adjustment should be considered to deal with length minimizing paths (straight lines) connecting two points (a, y_1) and (a, y_2) on a vertical line. These are graphs over the x_2 axis, but they cannot be expressed as graphs $\{(x, h(x)) : x \in [a, b]\}$. Thus, we should expect (at least eventually) to have a system of ordinary equations for the component functions $\alpha(t) = (\alpha_1(t), \alpha_2(t))$ of a parametric curve which supercedes the ODE (B.1)/(B.6)/(B.7).

2. In the special case of $(a, y_a) = \mathbf{0} = (0, 0)$, we know solutions of the ODE and (presumably) all minimizing paths. For each $P = (x_1, x_2) \in \mathcal{B}$, the unique minimizer of

$$\text{length}_{\mathcal{B}}[\alpha] = \int_{(a,b)} \frac{4}{4 + |\alpha|^2}\ |\alpha'|$$

in

$$\mathcal{A}_0 = \{\alpha \in C^1[a, b] : \alpha(a) = \mathbf{0},\ \alpha(b) = P\}$$

is the (parameterized) straight line segment connecting $\mathbf{0}$ to P. It is the picture associated with this observation which we now generalize.

Proposition. For each $P \in \mathcal{B} \backslash \{\mathbf{0}\}$, there is a collection $\mathcal{C}$ of circular arcs/rays (two of which are straight lines, i.e., circular arcs of infinite radius) emanating from P and having the following properties:

(i) For each $\mathbf{v} \in \mathbb{S}^1 = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| = 1\}$, there exists a unique circular arc in $\mathcal{C}$ which (when geometrically regularly parameterized for example by $\alpha \in C^1([a,b] \to \mathcal{B})$ on some interval $[a,b]$ so that $\alpha(a) = P$) there holds

$$\frac{\alpha'(a)}{|\alpha'(a)|} = \mathbf{v}.$$

(ii) For each $Q \in B_1(\mathbf{0}) \backslash \{P\}$, there is a unique circular arc Γ in $\mathcal{C}$ with $Q \in \Gamma$. That is to say the (open) arcs in $\mathcal{C}$ emanating from Γ foliate $B_1(\mathbf{0}) \backslash \{P\}$.

(iii) The portion of an arc Γ in $\mathcal{C}$ connecting P to $Q \in B_1(\mathbf{0})$ is the (unique) minimizer of $\text{length}_{\mathcal{B}}[\alpha]$ among paths α connecting P to Q.