Though you may have hopefully been able to complete the exercises above and start to get a feel for how Riemannian manifolds work, you may have been left with the feeling that the form of the matrix assignment (g_{ij}) is not so well motivated and the same applies to the definitions of Riemannian length, angle, and area. These are all very good things to be worried about, and we should give serious attention to providing some motivation for all of these things. For the moment we set aside what should be some good questions you are starting to formulate based on these observations about motivation and turn to other directions of inquiry peculiar to Riemannian manifolds as well as some more examples.

3.4 Linear structure(s) on $\mathcal{B}$

$\mathbb{R}^n$ is a **vector space** by which we mean a linear space, i.e., a commutative group under addition with a scaling $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ for which

$$\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v} \qquad \text{for } \alpha, \beta \in \mathbb{R} \text{ and } \mathbf{v} \in \mathbb{R}^n,$$

$$(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v} \qquad \text{for } \alpha, \beta \in \mathbb{R} \text{ and } \mathbf{v} \in \mathbb{R}^n,$$

$$\alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \beta\mathbf{w} \qquad \text{for } \alpha \in \mathbb{R} \text{ and } \mathbf{v}, \mathbf{w} \in \mathbb{R}^n,$$

$$1\mathbf{v} = \mathbf{v} \qquad \text{for } \mathbf{v} \in \mathbb{R}^n,$$

and

$$|\mathbf{v}| = \sqrt{\sum_{j=1}^{n} v_j^2}$$

where $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ defines a norm. That is, $\mathbb{R}^n$ is a normed linear space, i.e., a vector space. When we think of $\mathbb{R}^n$ as a Riemannian manifold, with the identity as a global chart function, we associate with each point $\mathbf{x} \in \mathbb{R}^n$ a copy of $\mathbb{R}^n$ of displacements, or tangent vectors, from $\mathbf{x}$. This is called the tangent space at $\mathbf{x}$ and is denoted by $T_{\mathbf{x}}\mathbb{R}^n$. It happens in this case that there is a natural association of a vector $\mathbf{v} \in T_{\mathbf{x}}\mathbb{R}^n$ with another point in the manifold $\mathbb{R}^n$ by addtion, namely

$$\mathbf{x} + \mathbf{v}.$$

More generally, given a vector $\mathbf{v} \in T_{\mathbf{x}}\mathbb{R}^n$ and some $t > 0$, we can find another vector

$$\mathbf{x} + t\mathbf{v} \in \mathbb{R}^n.$$

We wish to extract certain aspects of this construction for application in a general Riemannian manifold M. Let me briefly repeat this discussion of the manifold $M = \mathbb{R}^n$ with some additional detail and clearly indicating the roles of the two distinct copies of $\mathbb{R}^n$. One copy is the manifold itself $M = \mathbb{R}^n$ containing the point $\mathbf{x}$ and other points like $\mathbf{x}$. The other copy $T_{\mathbf{x}}\mathbb{R}^n = \mathbb{R}^n$ is the vector space associated with a particular point $\mathbf{x}$, and $T_{\mathbf{x}}\mathbb{R}^n$ contains vectors $\mathbf{v}$ and other vectors like $\mathbf{v}$. For each $\mathbf{v} \in T_{\mathbf{x}}\mathbb{R}^n \backslash \{\mathbf{0}\}$, there exists a unique geodesic ray, which in this case is a straight line. Every point $\mathbf{x} + t\mathbf{v} \in M = \mathbb{R}^n$ with $t > 0$ lies on this geodesic ray.

In a general Riemannian manifold M, some aspects of this description may not be possible, but many aspects of the description can be recreated. We will now begin the task of recreating the aspects which can be obtained for the Riemannian manifold $\mathcal{B}$.

We first associate a **linear space** (without a norm or means to measure magnitudes) with each point $P \in \mathcal{B}$. Let us call this linear space $\mathcal{L}_P\mathcal{B}$.

This space is constructed as follows: For each function $\alpha : [a, b] \to \mathcal{B}$ with $a, b \in \mathbb{R}$ and $a < b$ for which

(i) $\alpha(t_0) = P$ for some $t_0 \in (a, b)$ and

(ii) $\xi \circ \alpha \in C^1[a, b]$

we say $\alpha \in c\mathfrak{P}^1(\mathcal{B})$, that is α is a **chart C^1 path in $\mathcal{B}$**, with $\alpha(t_0) = P$. If $\alpha \in c\mathfrak{P}^1(\mathcal{B})$ satisfies $\alpha(t_1) \neq \alpha(t_2)$ for $a \leq t_1 < t_2 \leq b$, then we write $\alpha \in c\mathfrak{I}^1(\mathcal{B})$. That is,

$$c\mathfrak{I}^1(\mathcal{B}) = \{\alpha \in c\mathfrak{P}^1(\mathcal{B}) : \alpha(t_1) \neq \alpha(t_2),\ t_1 < t_2\}.$$

Now consider the collection of embedded paths

$$A_P = \{\alpha \in c\mathfrak{I}^1(\mathcal{B}) : \alpha(t_0) = P\}. \tag{3.10}$$

Exercise 3.22. Show there exists a path in the set A_p defined in (3.10). That is, show this collection of paths is nonempty.

On the set A_P we define an equivalence relation as follows: Two paths $\alpha_1, \alpha_2 \in A_P$ are equivalent, and we write $\alpha_1 \sim \alpha_2$ if

$$(\xi \circ \alpha_1)'(t_1) = (\xi \circ \alpha_2)'(t_2) \tag{3.11}$$

where $\alpha_j(t_j) = P$ for $j = 1, 2$.

Exercise 3.23. Show the relation on A_P defined by (3.11) is an equivalence relation:

(a) $\alpha \sim \alpha$ for every $\alpha \in A_P$.

(b) $\alpha_1 \sim \alpha_2$ implies $\alpha_2 \sim \alpha_1$.

(c) If $\alpha_1 \sim \alpha_2$ and $\alpha_2 \sim \alpha_3$, then $\alpha_1 \sim \alpha_3$.

Recall that whenever one has an equivalence relation on a set, then the set is naturally partitioned into equivalence classes. Specifically, if $\alpha_1, \alpha_2 \in A_P$, then there are precisely two possibilites for the sets

$$[\alpha_1] = \{\alpha \in A_P : \alpha \sim \alpha_1\} \qquad \text{and} \qquad [\alpha_2] = \{\alpha \in A_P : \alpha \sim \alpha_2\}, \tag{3.12}$$

namely either

$$[\alpha_1] \cap [\alpha_2] = \phi \qquad \text{or} \qquad [\alpha_1] = [\alpha_2]. \tag{3.13}$$

Exercise 3.24. Show the property (3.13) for the equivalence classes holds when (3.12) holds.

We designate the collection of equivalence classes

$$\mathcal{L}_P\mathcal{B} = \{[\alpha] : \alpha \in A_P\}.$$

As a set, this will be the linear space assigned to the point $P \in \mathcal{B}$. In order to have a linear structure on $\mathcal{L}_P\mathcal{B}$, we need operations of addition and scaling:

$$[\alpha_1] + [\alpha_2] = [\alpha_3] \tag{3.14}$$

where $(\xi \circ \alpha_3)'(t_3) = (\xi \circ \alpha_1)'(t_1) + (\xi \circ \alpha_2)'(t_2)$ and, as may be expected, $\alpha(t_j) = P$ for $j = 1, 2, 3$. Given $c \in \mathbb{R}$,

$$c[\alpha_1] = [\alpha_2] \tag{3.15}$$

where $(\xi \circ \alpha_2)'(t_2) = c(\xi \circ \alpha_1)'(t_1)$ and $\alpha_j(t_j) = P$ for $j = 1, 2$.

Exercise 3.25. Show the definitions of addition and scaling associated with (3.14) and (3.15) respectively are well-defined and make $\mathcal{L}_P\mathcal{B}$ a linear space.

The global chart function $\mathbf{p} : B_1(\mathbf{0}) \to \mathcal{B}$ and the global coordinate function $\xi = \mathbf{p}^{-1} : \mathcal{B} \to B_1(\mathbf{0})$ in particular played a prominent role in the introduction of the linear structure constituted by the assignment

$$P \mapsto \mathcal{L}_P\mathcal{B}.$$

The paths $\alpha \in c\mathfrak{P}^1(\mathcal{B})$ themselves, leading to the identification of the sets A_P and finally $\mathcal{L}_P\mathcal{B}$, were differentiated from other paths using the notion of regularity called "chart C^1" which seemingly relies on the chart function. Furthermore, each following step, from defining the equivalence classes to defining the operations, relied directly on the use of the global chart.

On the other hand, like the open sets in the topology on $\mathcal{B}$, the paths themselves can be considered as objects having an identity only with respect to the manifold $\mathcal{B}$, and the same can be said concerning the sets A_p, $\mathcal{L}_P\mathcal{B}$, and the resulting sums $[\alpha_1] + [\alpha_2]$ and scalings $c[\alpha]$. In order to better appreciate the extent to which $\mathcal{B}$ exerts its own identity in regard to the linear structure constituted by the assignment

$$P \mapsto \mathcal{L}_P\mathcal{B},$$

we consider in the next sections the proposition that while the particular global chart function $\mathbf{p} : B_1(\mathbf{0}) \to \mathcal{B}$ was used to define the linear structure, this particular chart function was not the only possibility. In particular while the use of some chart function or functions is generally required to define a linear structure, the particular chart functions used are, in a certain sense, peripheral to the structure created.

3.5 More than one chart—an atlas

I'm going to briefly start again in a somewhat more general setting. This will clean up some details that were glossed over above. We start with a Poincaré manifold M. Recall that this means

(i) M is a topological space, and

(ii) Associated with each point $P \in M$ there is at least one chart function $\mathbf{p} : B_2(\mathbf{0}) \to M$ defined on $B_2(\mathbf{0}) \subset \mathbb{R}^n$ for which the following hold:

(a) $P \in \mathbf{p}(B_1(\mathbf{0})) \subset \overline{\mathbf{p}(B_1(\mathbf{0}))} \subset \mathbf{p}(B_2(\mathbf{0}))$,

(b) $\mathbf{p} : B_2(\mathbf{0}) \to \mathbf{p}(B_2(\mathbf{0}))$ is a homeomorphism, and

(c) the restrictions

$$\mathbf{p}\big|_{B_1(\mathbf{0})} : B_1(\mathbf{0}) \to \mathbf{p}(B_1(\mathbf{0})) \qquad \text{and}$$
$$\mathbf{p}\big|_{\overline{\mathbf{p}(B_1(\mathbf{0}))}} : \overline{\mathbf{p}(B_1(\mathbf{0}))} \to \overline{(\mathbf{p}(B_1(\mathbf{0})))}$$

are also homeomorphisms.

3.5.1 Atlas theory

The collection of chart functions $\{\mathbf{p}\}_{\mathbf{p}\in\Gamma}$ required to exist in the definition have the property that the codomains $\mathbf{p}(B_2(\mathbf{0}))$ for which $\mathbf{p} : B_2(\mathbf{0}) \to U_{\mathbf{p}}$ is a homeomorphism satisfy

$$M = \bigcup_{\mathbf{p}\in\Gamma} \mathbf{p}(B_2(\mathbf{0})). \tag{3.16}$$

That is to say $\{\mathbf{p}(B_2(\mathbf{0}))\}_{\mathbf{p}\in\Gamma}$ is an open cover of M. Consequently, the pairs

$$\mathcal{A}^0 = \{(B_2(\mathbf{0}), \mathbf{p})\}_{\mathbf{p}\in\Gamma}$$

provide an example of what is called a covering atlas. Clearly this particular atlas and these particular chart functions are somewhat special.[12] Anytime, however, we have a collection of open domain sets $U_{\mathbf{p}} \subset \mathbb{R}^n$ along with homeomorphisms $\mathbf{p} : U_{\mathbf{p}} \to \mathbf{p}(U_{\mathbf{p}}) \subset M$ for $\mathbf{p}$ in some indexing set Γ satisfying

$$M = \bigcup_{\mathbf{p}\in\Gamma} \mathbf{p}(U_{\mathbf{p}}) \tag{3.17}$$

[12] This is not to say the atlas $\mathcal{A}^0$ is unique in any particular sense. We can call such an atlas in which each chart function $\mathbf{p} : B_2(\mathbf{0}) \to M$ illustrates the locally Euclidean structure of M a **topological base atlas**, but there could be many different such atlases. We always build upon some topological base atlas. It doesn't matter which one you start with, and (somewhat paradoxically) the chart functions in other atlases may not include any element in the topological base atlas. As a related note, one can start with a collection of chart functions which are not homeomorphisms but simply bijections $\mathbf{p} : B_2(\mathbf{0}) \to M$ into a (structureless) set M with images covering M as in (3.16). Then one can construct a topology on M using the bijections. We have done this in the case where there exists a global chart. If there is no global chart, construction of the topology is more complicated. This is one reason we started again with a topology in place.

we say

$$\mathcal{A} = \{(U_{\mathbf{p}}, \mathbf{p})\}_{\mathbf{p}\in\Gamma}$$

is a **covering atlas** or just an **atlas**. Of course, in principle, we could call any collection of (chart, chart function) pairs an atlas, but in practice, we almost always require the covering atlas property (3.17).

Exercise 3.26. Show that whenever $\mathbf{p} : U_{\mathbf{p}} \to \mathbf{p}(U_{\mathbf{p}})$ and $\mathbf{q} : U_{\mathbf{q}} \to \mathbf{q}(U_{\mathbf{q}})$ are chart functions (homeomorphisms) for a Poincaré manifold and

$$W = \mathbf{p}(U_{\mathbf{p}}) \cap \mathbf{q}(U_{\mathbf{q}}) \neq \phi$$

then

$$\begin{aligned} \psi &= \eta \circ \mathbf{p}\big|_{\xi(W)} : \xi(W) \to \eta(W) \qquad \text{and} \\ \phi &= \psi^{-1} = \xi \circ \mathbf{q}\big|_{\eta(W)} : \eta(W) \to \xi(W) \end{aligned}$$

are homeomorphisms where $\xi = \mathbf{p}^{-1}$ and $\eta = \mathbf{q}^{-1}$ are the associated coordinate functions.

The property described in Exercise 3.26 is called an **overlap property**, and with the starting point we have chosen this particular overlap property comes for "free," that is the property can be shown to hold as an exercise. Very often an overlap property is something that is assumed instead (as we shall soon see).

A covering atlas $\mathcal{A}_*$ is said to be a **maximal atlas** for the Poincaré manifold M if every homeomorphism $\mathbf{p} : U \to \mathbf{p}(U)$ with domain an open subset of $\mathbb{R}^n$ and $\mathbf{p}(U) \subset M$ is in $\mathcal{A}_*$. In fact, there is clearly a unique maximal atlas:

$$\mathcal{A}_* = \left\{ (U, \mathbf{p}) \in \mathcal{T} \times \left(\bigcup_{V\in\mathcal{T}} C^0(V \to M) \right) : \right. \\ \left. \mathbf{p} : U \to \mathbf{p}(U) \text{ is a homeomorphism} \right\}$$

where $\mathcal{T}$ denotes the (usual) topology on $\mathbb{R}^n$. Generally speaking $\mathcal{A}_*$ contains every chart function we will ever want to consider; it is a really big atlas. It will be important, however, to consider some specified subatlases of $\mathcal{A}_*$. The subatlases of interest are usually specified by some kind of overlap condition. For example, say we can find an atlas $\mathcal{A} \subset \mathcal{A}_*$ for which

$$\psi = \eta \circ \mathbf{p}\Big|_{\xi(W)} : \xi(W) \to \eta(W) \qquad \text{and} \tag{3.18}$$

$$\phi = \psi^{-1} = \xi \circ \mathbf{q}\Big|_{\eta(W)} : \eta(W) \to \xi(W) \tag{3.19}$$

are **Lipschitz homeomorphisms** whenever $\mathbf{p} : U_{\mathbf{p}} \to \mathbf{p}(U_{\mathbf{p}})$ and $\mathbf{q} : U_{\mathbf{q}} \to \mathbf{q}(U_{\mathbf{q}})$ are chart functions (homeomorphisms) with

$$(U_{\mathbf{p}}, \mathbf{p}), (U_{\mathbf{q}}, \mathbf{q}) \in \mathcal{A},$$

$W = \mathbf{p}(U_{\mathbf{p}}) \cap \mathbf{q}(U_{\mathbf{q}}) \neq \phi$, and $\xi = \mathbf{p}^{-1}$ and $\eta = \mathbf{q}^{-1}$ the associated coordinate functions as usual. Then we say $\mathcal{A}$ is a **Lipschitz atlas** and/or $\mathcal{A}$ gives M a **Lipschitz structure**. Here we assume $\mathcal{A}$ is again a covering atlas, so $\mathcal{A}$ in this case is a **covering Lipschitz atlas**. Let us denote one such atlas by $\mathcal{A}^{\mathrm{Lip}}$. Then we can consider the maximal atlas

$$\mathcal{A}_*^{\mathrm{Lip}} = \mathcal{A}^{\mathrm{Lip}} \bigcup \Big\{ (U_{\mathbf{q}}, \mathbf{q}) \in \mathcal{A}_* : \text{the maps in (3.18) and (3.19)} \\ \text{are Lipschitz homomeomorphisms whenever } (U_{\mathbf{p}}, \mathbf{p}) \in \mathcal{A}^{\mathrm{Lip}} \Big\}.$$

The reference to (3.18) and (3.19) in the definition of the maximal atlas $\mathcal{A}_*^{\mathrm{Lip}}$ assumes $W = \mathbf{p}(U_{\mathbf{p}}) \cap \mathbf{q}(U_{\mathbf{q}}) \neq \phi$ with $\xi = \mathbf{p}^{-1}$ and $\eta = \mathbf{q}^{-1}$ the associated coordinate functions as usual. Note the construction of $\mathcal{A}_*^{\mathrm{Lip}}$ is quite different from the construction of the unique maximal topological atlas $\mathcal{A}_*$. In particulr, the overlap condition appears as an important assumption rather than a necessary consequence.

Exercise 3.27. Show that if

(i) $(U_{\mathbf{p}}, \mathbf{p}), (U_{\mathbf{q}}, \mathbf{q}) \in \mathcal{A}_*^{\mathrm{Lip}}$, and

(ii) $W = \mathbf{p}(U_{\mathbf{p}}) \cap \mathbf{q}(U_{\mathbf{q}}) \neq \phi$,

then the maps in (3.18) and (3.19) are Lipschitz homomeomorphisms where $\xi = \mathbf{p}^{-1}$ and $\eta = \mathbf{q}^{-1}$ are the associated coordinate functions as usual.

Exercise 3.28. Show that if $\mathcal{A}$ is a Lipschitz atlas satisfying

(i) $\mathcal{A}^{\text{Lip}} \subset \mathcal{A}$,

(ii) If $(U_{\mathbf{p}}, \mathbf{p}) \in \mathcal{A}^{\text{Lip}}$, $(U_{\mathbf{q}}, \mathbf{q}) \in \mathcal{A}$ with

- **(a)** $W = \mathbf{p}(U_{\mathbf{p}}) \cap \mathbf{q}(U_{\mathbf{q}}) \neq \phi$, and
- **(b)** $\xi = \mathbf{p}^{-1}$ and $\eta = \mathbf{q}^{-1}$ are the associated coordinate functions as usual,

then the maps in (3.18) and (3.19) are Lipschitz homomeomorphisms,

then $\mathcal{A} \subset \mathcal{A}_*^{\text{Lip}}$.

Two interesting and important questions arise immediately:

1. Does there exist a Lipschitz (covering) atlas?
2. Assuming there exist two Lipschitz atlases (or atlantes) $\mathcal{A}^{\text{Lip}}$ and $\mathcal{A}$ in $\mathcal{A}_*$, is the "unique" maximal atlas $\mathcal{A}_*^{\text{Lip}}$ containing $\mathcal{A}^{\text{Lip}}$ and discussed in Exercises 3.27 and 3.28 the same as the "unique" maximal atlas obtained by applying the same discussion to $\mathcal{A}$? (Hint: The answer is "no" (!).)

I suggest we set these questions aside for a moment and continue our discussion of special subatlantes.

A C^1 atlas is defined in the same way a Lipschitz atlas is defined, except the overlap changes of variables are required to be C^1 instead of Lipschitz.

Exercise 3.29. Adapt the discussion above concerning Lipschitz atlantes to C^1 atlantes, including versions of Exercises 3.26, 3.27, and 3.28.

If we have a C^1 atlas $\mathcal{A} = \mathcal{A}^1$ and the corresponding maximal atlas $\mathcal{A}_*^1$, then we are in a position to discuss **chart C^1 regularity** for paths into and real valued functions on M. This allows the construction of a linear space

$$\mathcal{L}_P M = \{[\alpha] : \alpha \in c\mathfrak{I}^1(M),\ \alpha(t_0) = P\}$$

with elements equivalence classes of chart C^1 paths as discussed for $\mathcal{B}$ above. I suggest we call these particular elements of this linear space **filaments**. This name stands in contrast to vectors, which would be the elements of a vector space. A little reflection suggests the utility of having a general name distinguishing elements of a general linear space from the elements of a vector space in the special case when the linear space has a norm (or inner product). It is customary to call the elements of a normed linear space (i.e., vector space) vectors. Perhaps the term filament can be used similarly: An element of a linear space is a **filament**, and a linear space may also be referred to as a **filament space** if we wish to emphasize the absence of a norm.

Notice there was nothing particularly special about the use of the global chart we used for $\mathcal{B}$. The basic situation is illustrated in Figure 3.4. If we

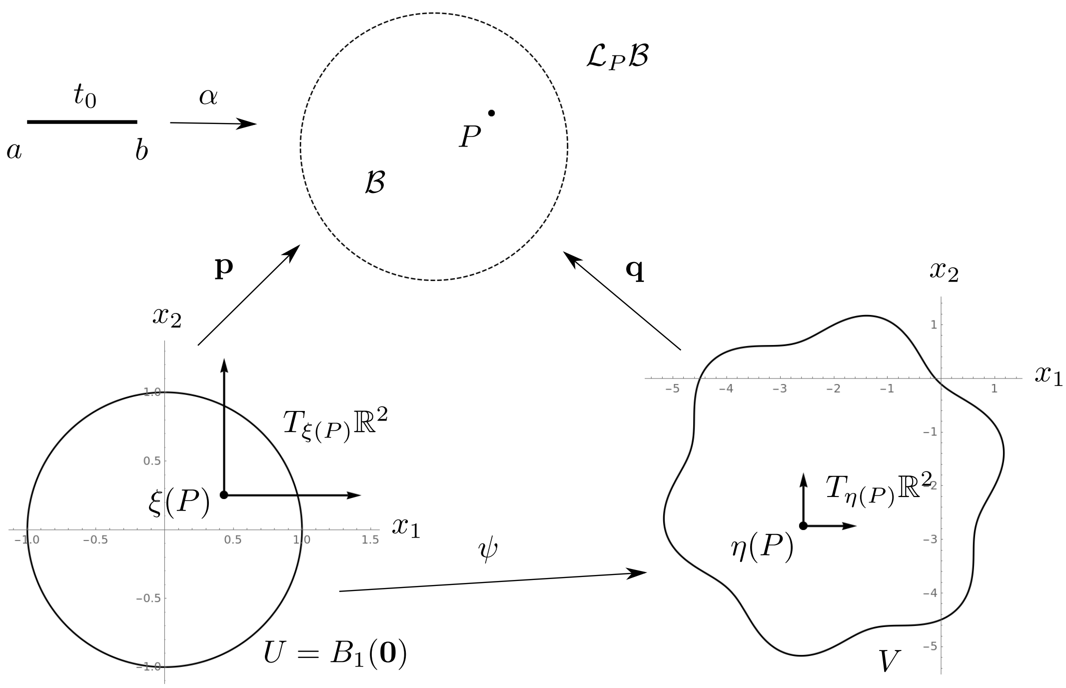

Figure 3.4: Another chart for $\mathcal{B}$, tangent spaces in the charts, and a linear space at a point $P \in \mathcal{B}$

have a global chart, say $\mathbf{p} : U \to M$ (for example $\mathbf{p} : U = B_1(\mathbf{0}) \to \mathcal{B}$), then every other chart $\mathbf{q} : V \to M$ defined on an open set $V \subset \mathbb{R}^n$ will have $W = \mathbf{p}(V) \subset M = \mathbf{p}(U)$. In partiucular, $W = \mathbf{p}(U) \cap \mathbf{q}(V)$ is an open subset of M and the restrictions

$$\mathbf{p}\big|_{\xi(W)} \in C^0(\xi(W) \to M), \qquad \text{and} \qquad \mathbf{q}\big|_{\eta(W)} \in C^0(\eta(W) \to M)$$

are well defined as well as the **change of coordinate functions**

$$\psi = \eta \circ \mathbf{p}\big|_{\xi(W)} : \xi(W) \to \eta(W) \qquad \text{and}$$
$$\phi = \psi^{-1} = \xi \circ \mathbf{q}\big|_{\eta(W)} : \eta(W) \to \xi(W).$$

All **overlap conditions** apply to changes of coordinates.

Even in this case when one has a global chart, it is natural to consider other charts in an atlas, and if other charts and the associated changes of coordinates are considered, then the overlap conditions are natural to consider as well and, in cases of higher regularity, necessary to consider.

Exercise 3.30. Say you have a global chart U and a global chart function $\mathbf{p} : U \to M$. You then have the unique maximal atlas

$$\mathcal{A}_* = \Bigg\{ (V, \mathbf{q}) \in \mathcal{T} \times \left(\bigcup_{A \in \mathcal{T}} C^0(A \to M) \right) : \\ \mathbf{q} : V \to \mathbf{p}(V) \text{ is a homeomorphism} \Bigg\}$$

where $\mathcal{T}$ denotes the (usual) topology on $\mathbb{R}^n$.

(a) Find (i.e., show the existence of by finding) a Lipschitz subatlas, i.e., a Lipschitz covering atlas,
$$\mathcal{A}^{\text{Lip}} \subset \mathcal{A}_*.$$

(b) A (covering) subatlas $\mathcal{A} \subset \mathcal{A}^*$ is said to be a C^1 **atlas** if

$$\psi = \eta \circ \mathbf{p}\big|_{\xi(W)} : \xi(W) \to \eta(W) \qquad \text{and} \tag{3.20}$$
$$\phi = \psi^{-1} = \xi \circ \mathbf{q}\big|_{\eta(W)} : \eta(W) \to \xi(W) \tag{3.21}$$

satisfy

$$\psi \in C^1(\xi(W) \to \eta(W)) \qquad \text{and} \qquad \phi \in C^1(\eta(W) \to \xi(W))$$

for every $(V, \mathbf{q}) \in \mathcal{A}$ where $\xi = \mathbf{p}^{-1}$, $\eta = \mathbf{q}^{-1}$, and $W = \mathbf{p}(U) \cap \mathbf{p}(V)$ as usual. Find a C^1 atlas.

(c) Define and find for each $k \in \mathbb{N} \cup \{\infty\}$ (in this special case where there exists a global chart function $\mathbf{p} : U \to M$) a C^k **atlas** for M.

Filaments determined by a global chart

Given the context of Exercise 3.30, we need a C^1 atlas[13] to consider $\mathcal{L}_P M$ for each $P \in M$. Specifically, given a C^1 atlas $\mathcal{A}^1$ with $(U, \mathbf{p}) \in \mathcal{A}$, we can construct the maximal C^1 subatlas

$$\mathcal{A}^1_* = \mathcal{A}^1 \bigcup \Big\{ (V, \mathbf{q}) \in \mathcal{A}_* : \text{the maps in (3.20) and (3.21) satisfy} \\ \psi \in C^1(\xi(W) \to \eta(W)) \text{ and } \phi \in C^1(\eta(W) \to \xi(W)) \Big\}.$$

Given $\alpha_1 \in c\mathfrak{I}^1(M)$ with $P \in \alpha_1(I)$, we can define the equivalence class $[\alpha_1] \in \mathcal{L}_P M$ by

$$[\alpha_1] = \left\{ \alpha \in c\mathfrak{I}^1(M) : (\xi \circ \alpha)'(\alpha^{-1}(P)) = (\xi \circ \alpha_1)'(\alpha_1^{-1}(P)) \right\},$$

but we would like to make sure that if we happen to use a different chart $(V, \mathbf{q}) \in \mathcal{A}^1_*$ with $P \in \mathbf{q}(V)$, then

$$\left\{ \alpha \in c\mathfrak{I}^1(M) : (\eta \circ \alpha)'(\alpha^{-1}(P)) = (\eta \circ \alpha_1)'(\alpha_1^{-1}(P)) \right\}$$

defines the same equivalence class. The key to this verification is a formula for $(\eta \circ \alpha)'(\alpha^{-1}(P))$ in terms of $(\xi \circ \alpha)'(\alpha^{-1}(P))$. Such a formula can be obtained by the chain rule applied to

$$\eta \circ \alpha = \eta \circ \mathbf{p} \circ \xi \circ \alpha$$

as long as the change of variables $\psi = \eta \circ \mathbf{p}\big|_{\xi(W)}$ satisfies $\psi \in C^1$. This condition should have been part of the definition of a C^1 atlas discussed in the solution of Exercise 3.29. Thus, by the chain rule

$$(\eta \circ \alpha)' = D\psi(\xi \circ \alpha)\ (\xi \circ \alpha)' \tag{3.22}$$

and

$$(\eta \circ \alpha)'(\alpha^{-1}(P)) = D\psi(\xi \circ \alpha(\alpha^{-1}(P)))\ (\xi \circ \alpha)'(\alpha^{-1}(P))) \tag{3.23}$$

in particular where

$$D\psi = \left(\frac{\partial \psi^i}{\partial x_j} \right) \tag{3.24}$$

is the $n \times n$ matrix of partial derivatives of ψ.

[13] You should know one.

Notational abuse of matrix multiplication in calculus

This is perhaps a convenient place to pause and point out a small abuse of notation in which we have indulged before and perhaps went unnoticed. We have generally considered vectors in $\mathbb{R}^n$ as row vectors with

$$\begin{aligned}
\mathbf{e}_1 &= (1, 0, 0, \ldots, 0), \\
\mathbf{e}_2 &= (0, 1, 0, \ldots, 0), \\
&\vdots \\
\mathbf{e}_n &= (0, 0, 0, \ldots, 1) \\
\mathbf{v} &= (v_1, v_2, \ldots, v_n) \\
\alpha' &= (\alpha'_1, \alpha'_2, \ldots, \alpha'_n), \quad \text{and} \\
Df &= (D_{\mathbf{e}_1} f, D_{\mathbf{e}_2} f, \ldots, D_{\mathbf{e}_n} f), \text{etc.}
\end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathfrak{P}^1(\mathbb{R}^n)$ is a parameterized path in $\mathbb{R}^n$ and $f \in C^1(U)$ is a real valued function defined on an open set $U \subset \mathbb{R}^n$. Perhaps the primary reason for the consideration of row vectors is typographical so that repeated use of expansive symbolic expressions like

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

can be avoided, especially in normal paragraphs of text. Thus one avoids $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ or having to always write $\mathbf{v}^T$ in inline text. On the other hand, the convention according to which the gradient vector Df is the row vector of partial derivatives is entirely compatible with the expression for the total derivative in (3.24) in which each row is the gradient of the component function ψ^i. Perhaps the natural convention, were typographical matters of no concern, would be to have vectors in $\mathbb{R}^n$ be column vectors, but then Df for a real valued function $f \in C^1(U)$ would take values in the vector space

$$\mathbb{R}^n_r = \{\mathbf{v}^T : \mathbf{v} \in \mathbb{R}^n\}$$

consisting of row vectors in contrast to the space $\mathbb{R}^n = \mathbb{R}^n_c$ of column vectors with $\mathbb{R}^n_c$ and $\mathbb{R}^n_r$ being vector space isomorphic by the transpose, and of course with each having its own inner product.

The notational abuse comes when a professed row vector is multiplied by a matrix on the left as in (3.22), (3.23), or the formula

$$\langle (g_{ij})\mathbf{v}, \mathbf{w}\rangle_{\mathbb{R}^n}$$

which has been used above in the context of a matrix assignment $(g_{ij}) = (g_{ij})$, for example in the matrix assignment for the example Riemannian manifold $\mathcal{B}$. Technically, one should use the transpose operation according to which

$$\mathbf{v}^T = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

is a column vector and $(g_{ij})^T = (g_{ji})$. Thus, more or less proper ways to write some of the matrix multiplications mentioned are

$$[(\eta \circ \alpha)']^T = D\psi(\xi \circ \alpha)\ [(\xi \circ \alpha)']^T,$$

$$(\eta \circ \alpha)' = [\ D\psi(\xi \circ \alpha)\ [(\xi \circ \alpha)']^T\ ]^T,$$

$$[(\eta \circ \alpha)'(\alpha^{-1}(P))]^T = D\psi(\xi \circ \alpha(\alpha^{-1}(P)))\ [(\xi \circ \alpha)'(\alpha^{-1}(P)))]^T,$$

$$\langle [(g_{ij})\mathbf{v}^T]^T, \mathbf{v}\rangle_{\mathbb{R}^n},$$

and

$$\mathbf{v}(g_{ij})\mathbf{v}^T.$$

In most instances, it is just too much trouble to sort all these questions out carefully and include transposes in the appropriate places, but it is perhaps good to be able to do so in certain instances.

Differential map in calculus

Referring back to Figure 3.4, the value on the right in (3.23) is the value of the **differential**

$$d\psi_{\xi(P)} : T_{\xi(P)}\mathbb{R}^2 \to T_{\eta(P)}\mathbb{R}^2$$

on the tangent vector $(\xi \circ \alpha)'(\alpha^{-1}(P))$. This is a good time to note some fundamental differences between (3.22) and (3.23). The formula (3.22) is an expression of the chain rule which holds at all points along a particular path α. The evaluation (3.23) plays a special role and has a special name: This is called a **coordinate transformation rule at a point** which tells how the filament $[\alpha]$ transforms under a change of coordinates. Specifically, we can say $(\xi \circ \alpha)'(\alpha^{-1}(P))$ is the (or an) "expression" for $[\alpha]$ in local coordinates. In a certain sense $(\xi \circ \alpha)'(\alpha^{-1}(P))$ is a "representative" of $[\alpha]$.

Exercise 3.31. Show
$$[\alpha] \mapsto (\xi \circ \alpha)'(\alpha^{-1}(P))$$
defines a linear space isomorphism from $\mathcal{L}_P M$ to $T_{\xi(P)}\mathbb{R}^n$.

Notice that if α_1 is a different path in $[\alpha]$ then $\xi \circ \alpha_1$ will be a different path in U. Consequently, the formulas

$$(\eta \circ \alpha)' = D\psi(\xi \circ \alpha)\ (\xi \circ \alpha)' \qquad \text{and} \qquad (\eta \circ \alpha_1)' = D\psi(\xi \circ \alpha_1)\ (\xi \circ \alpha_1)'$$

are both correct espressions of the chain rule, but they are different. The evaulations associated with (3.23)

$$(\eta \circ \alpha)'(\alpha^{-1}(P)) = D\psi(\xi \circ \alpha(\alpha^{-1}(P)))\ (\xi \circ \alpha)'(\alpha^{-1}(P)))$$

and
$$(\eta \circ \alpha_1)'(\alpha_1^{-1}(P)) = D\psi(\xi \circ \alpha_1(\alpha_1^{-1}(P)))\ (\xi \circ \alpha_1)'(\alpha_1^{-1}(P)))$$
however are identical. Also, the expressions

$$(\eta \circ \alpha)'(\alpha^{-1}(P)) \qquad \text{and} \qquad (\eta \circ \alpha_1)'(\alpha_1^{-1}(P))$$

are identical; they are the same expressions of $[\alpha]$ in the coordinate η though the paths $\eta \circ \alpha$ and $\eta \circ \alpha_1$ may be different. This is what the definition of the equivalence relation used to define $\mathcal{L}_P M$ tells us.

Finally, the transformation rule (3.23) expresses how the expression for $[\alpha]$ in the coordinate ξ "transforms" when $[\alpha]$ is expressed in a different coordinate η. This concept of a **coordinate transformation rule at a point** is very central to the realization of the matrix assignment (g_{ij}) as an independent entity (called the Riemannian metric tensor) on $\mathcal{B}$ as discussed below.

To finish this section on atlases let me attempt a summary/review of some of the linear structures we have discussed in the general setting. If you have a C^1 atlas $\mathcal{A}^1$ on a Poincaré manifold M, then you are assuming changes of coordinates are C^1 diffeomorphisms. Within the maximal topological base atlas $\mathcal{A}_*$ there is a unique maximal C^1 atlas containing $\mathcal{A}^1$. This atlas is denoted by $\mathcal{A}^1_*$. In this case, we can discuss the linear space of filaments $\mathcal{L}_P M$ consisting of equivalence classes $[\alpha]$ of paths $\alpha \in c\mathfrak{P}^1(M)$. The filaments are added and scaled as follows:

$$[\alpha] + [\beta] = [\gamma]$$

where

$$(\xi \circ \gamma)'(\gamma^{-1}(P)) = (\xi \circ \alpha)'(\alpha^{-1}(P)) + (\xi \circ \beta)'(\beta^{-1}(P))$$

and

$$c[\alpha] = [\sigma]$$

where

$$(\xi \circ \sigma)'(\sigma^{-1}(P)) = c(\xi \circ \alpha)'(\alpha^{-1}(P)).$$

It was not emphasized above, but when one has a C^1 atlas $\mathcal{A}^1$, the linear space $cC^1(M)$ consisting of real valued functions $f : M \to \mathbb{R}$ with coordinate expressions $f \circ \mathbf{p} : U \to \mathbb{R}$ satisfying $f \circ \mathbf{p} \in C^1(U)$ is also part of the linear structure on M.

Exercise 3.32. Let $f : M \to \mathbb{R}$ where M is a Poincaré manifold considered with respect to a C^1 structure determined by a C^1 atlas $\mathcal{A}^1 \subset \mathcal{A}_*$. Show the following: If for each $P \in M$, there exists some $(U, \mathbf{p}) \in \mathcal{A}^1_*$ with $P \in \mathbf{p}(U)$ and for which $f \circ \mathbf{p} \in C^1(U)$, then $f \circ \mathbf{q} \in C^1(V)$ for each $(V, \mathbf{q}) \in \mathcal{A}^1_*$.

In a Cr atlas $\mathcal{A}^2$, the changes of coordinates are required to be C^2 diffeomorphisms. Associated with each C^2 atlas $\mathcal{A}^2$, there is a unique maximal C^2 atlas $\mathcal{A}^2_*$. If the manifold M has a C^2 structure (induced by a C^2 atlas $\mathcal{A}^2$), then it also has a C^1 structure induced by $\mathcal{A}^2$.

Exercise 3.33. Let $\mathcal{H}^1_*$ denote the unique maximal C^1 atlas containing a given C^2 atlas $\mathcal{A}^2$. Let $\mathcal{A}^1 \subset \mathcal{H}^1_*$ be a C^1 atlas (for the same manifold). Then $\mathcal{H}^1_* = \mathcal{A}^1_*$. (True of false?)

For any $k \in \mathbb{N}$ a C^k atlas $\mathcal{A}^k$ is required to have changes of coordinates which are C^k diffeomorphisms. Every such atlas $\mathcal{A}^k$ determines a unique maximal C^k atlas $\mathcal{A}^k_*$ with $\mathcal{A}^k \subset \mathcal{A}^k_*$. Each such manifold has a well-defined linear structure.

Exercise 3.34. Let $\mathcal{A}^k$ be C^k atlas on a Poincaré manifold M for some $k \in \mathbb{N}$. Also, let $\mathcal{A}^1$ be a C^1 atlas on the same Poincaré manifold. Consider the following collections of embedded paths:

$$\left\{ \alpha \in \mathfrak{I}^0(M) : \xi \circ \alpha\Big|_{\alpha^{-1}(\mathbf{p}(U))} \in C^1(\alpha^{-1}(\mathbf{p}(U)) \to U) \text{ for some } (U, \mathbf{p}) \in \mathcal{A}^k \right\}$$

and

$$\left\{ \alpha \in \mathfrak{I}^0(M) : \xi \circ \alpha\Big|_{\alpha^{-1}(\mathbf{p}(U))} \in C^1(\alpha^{-1}(\mathbf{p}(U)) \to U) \text{ for some } (U, \mathbf{p}) \in \mathcal{A}^1 \right\}.$$

(a) Are these sets the same, or can they be different?

(b) What does this tell you about the definition of $c\mathfrak{P}^1(M)$?

In a C^∞ atlas $\mathcal{A}^\infty$, changes of coordinates are required to be C^∞ diffeomorphisms. Every C^∞ atlas $\mathcal{A}^\infty$ determines a unique maximal C^∞ atlas $\mathcal{A}^\infty_*$.

Exercise 3.35. Let $\mathcal{A}^\infty$ be C^∞ atlas on a Poincaré manifold M, and let P be a point in M. Identify the following sets:

$$A = \Big\{ \alpha \in \mathfrak{I}^0(M) :$$
$$\xi \circ \alpha\Big|_{\alpha^{-1}(\mathbf{p}(U))} \in C^\infty(\alpha^{-1}(\mathbf{p}(U)) \to U) \text{ for some } (U, \mathbf{p}) \in \mathcal{A}^\infty \Big\},$$
$$B = \left\{ \alpha \in A : \alpha^{-1}(\{P\}) \neq \phi \right\}, \qquad \text{and}$$
$$C = \{[\alpha] : \alpha \in B\}.$$

Notice that if M has a C^k atlas $\mathcal{A}^k$ with corresponding maximal atlas $\mathcal{A}^k_*$ for some $k \in \mathbb{N} \cup \{\infty\}$, then for every $m \in \mathbb{N}$ with $m < k$, there is a unique maximal C^m atlas $\mathcal{A}^m_* \subset \mathcal{A}_*$ satisfying $\mathcal{A}^k \subset \mathcal{A}^m_*$. Like the question mentioned above concerning the existence of a Lipschitz atlas in $\mathcal{A}_*$, there is a sequence of interesting questions having the form(s)

1. Given a Poincaré manifold M and some $k \in \mathbb{N} \cup \{\infty\}$, does there exist a C^k atlas $\mathcal{A}^k \subset \mathcal{A}_*$?

2. Given a C^k atlas $\mathcal{A}^k_* \subset \mathcal{A}_*$ on a Poincaré manifold M for some $k \in \mathbb{N}$ and given some $m \in \mathbb{N} \cup \{\infty\}$ with $m > k$, does there exist a C^m atlas $\mathcal{A}^m \subset \mathcal{A}^k_*$?

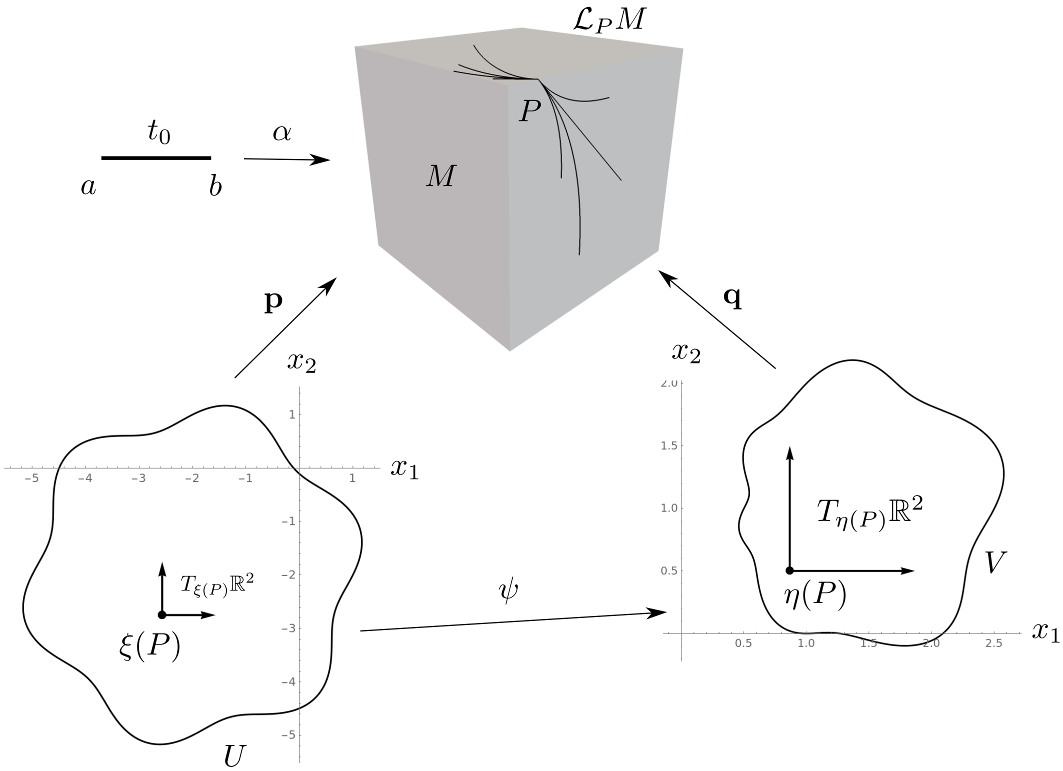

Figure 3.5: Filaments at a point P in a manifold M with C^1 (linear) structure.

Largely to avoid these interesting questions of existence, we assume for the moment (until further notice) that we have a C^∞ atlas $\mathcal{A}_*^\infty \subset \mathcal{A}_*$ on a Poincaré manifold M as illustrated in Figure 3.5.

3.5.2 Linear spaces of functions

Collections of real valued functions on M with specified chart regularity, when considered as linear spaces, may also be considered to be part of the linear structure on a manifold M. These spaces typically also have the structure of a ring. Examples include

$$c\,\mathrm{Lip}(M),\ cC^1(M),\ cC^2(M),\ldots,\ cC^k(M),\ldots,\ cC^\infty(M).$$

In many texts the last ring is often denoted $\mathscr{F}(M)$ and is called the ring of smooth functions on M. We will stick to the unified notation $cC^\infty(M)$ for this space, and the others, but we note that we consider these spaces not only as linear spaces with the usual operations determined by

$$(f+g)(P) = f(P) + g(P) \qquad \text{and} \qquad (cf)(P) = cf(P)$$

where $f, g \in cC^\infty(M)$ and $c \in \mathbb{R}$ but they are also algebraic rings with

$$(fg)(P) = f(P)g(P).$$

3.5.3 Filament fields

A function $\ell : M \to \bigcup_{P\in M} \mathcal{L}_P M$ satisfying $\ell(P) \in \mathcal{L}_P M$ is called a **filament field**. For a manifold with a C^∞ atlas $\mathcal{A}_*^\infty$, we can consider the chart regularity of a filament field in the "usual way." Here are the details: For $k \in \mathbb{N} \cup \{\infty\}$ it is convenient to denote the C^k vector fields on an open subset U in $\mathbb{R}^n$ by $\mathscr{V}^k(U)$. That is,

$$\mathscr{V}^k(U) = \Bigg\{ \mathbf{v} = (v^1, v^2, \ldots, v^n) \in \left(\bigcup_{\mathbf{x}\in U} T_{\mathbf{x}}\mathbb{R}^n \right)^U : \\ \mathbf{v}(\mathbf{x}) \in T_{\mathbf{x}}\mathbb{R}^n \text{ and } v^j \in C^k(U),\ j = 1, 2, \ldots, n \Bigg\}.$$

In this way, we know what it means for a vector field $\mathbf{v} : U \to \bigcup_{\mathbf{x}\in U} T_{\mathbf{x}}\mathbb{R}^n$, which assigns to each $\mathbf{x} \in U \subset \mathbb{R}^n$ an element of the tangent space at $\mathbf{x}$, to be smooth.

We can say a filament field is **chart** C^∞ if the field induced on a coordinate chart is actually C^∞. Specifically, if

$$\ell \in \left(\bigcup_{P\in M} \mathcal{L}_P M \right)^M$$

is a filament field, then

$$\ell \in cC^\infty \left(M \to \ell(M) \right)$$

if for each $(U, \mathbf{p}) \in \mathcal{A}_*^\infty$ there holds $\mathbf{v} \in \mathscr{V}^\infty(U)$ where

$$\mathbf{v}(\mathbf{x}) = (\xi \circ \alpha)'(\alpha^{-1}(\mathbf{p}(\mathbf{x})))$$

where $\alpha \in \ell(\mathbf{p}(\mathbf{x}))$.

Exercise 3.36. Given a filament field

$$\ell \in \left(\bigcup_{P \in M} \mathcal{L}_P M \right)^M$$

and $\alpha_1, \alpha_2 \in \ell(\mathbf{p}(\mathbf{x}))$ for some $\mathbf{x} \in U$ and $(U, \mathbf{p}) \in \mathcal{A}_*^\infty$, show

$$(\xi \circ \alpha_1)'(\alpha_1^{-1}(\mathbf{p}(\mathbf{x}))) = (\xi \circ \alpha_2)'(\alpha_2^{-1}(\mathbf{p}(\mathbf{x})))$$

where $\xi = \mathbf{p}^{-1}$ is the coordinate function associated with $(U, \mathbf{p})$ as usual.

The collection of all cC^∞ filament fields on a given manifold M with smooth structure $\mathcal{A}_*^\infty$ is denoted by

$$\mathscr{X}^\infty(M)$$

or just $\mathscr{X}(M)$. That is,

$$\mathscr{X}^\infty(M) = \Bigg\{ \ell \in \left(\bigcup_{P \in M} \mathcal{L}_P M \right)^M : \ell(P) \in \mathcal{L}_P M,$$
$$\mathbf{v} \in \mathcal{V}(U), \qquad (U, \mathbf{p}) \in \mathcal{A}_*^\infty;$$
$$\mathbf{v}(\mathbf{x}) = (\xi \circ \alpha)'(\alpha^{-1}(\mathbf{p}(\mathbf{x}))), \qquad \alpha \in \ell(\mathbf{p}(\mathbf{x})) \Bigg\}$$

Exercise 3.37. If $\ell : M \to \bigcup_{P \in M} \mathcal{L}_P M$ is a filament field and $\mathbf{v} \in \mathscr{V}^\infty(U)$ where

$$\mathbf{v}(\mathbf{x}) = (\xi \circ \alpha)'(\alpha^{-1}(\mathbf{p}(\mathbf{x}))), \qquad \alpha \in \ell(\mathbf{p}(\mathbf{x}))$$

for $(U, \mathbf{p})$ in some covering atlas $\mathcal{A} \subset \mathcal{A}_*^\infty$, then $\ell \in \mathscr{X}(M)$.

Exercise 3.38. Consider $A : T_{\xi(P)}\mathbb{R}^n \to \mathcal{L}_P M$ by

$$A(\mathbf{v}) = [\alpha],$$

where $\alpha : (-\epsilon, \epsilon) \to M$ by $\alpha(t) = \mathbf{p}(\xi(P) + t\mathbf{v})$, $(U, \mathbf{p}) \in \mathcal{A}_*^\infty$, and $\epsilon > 0$ is some positive number for which $\xi(P) + t\mathbf{v} \in U$ for $|t| < \epsilon$. Show A is a linear space isomorphism. In particular, $\mathcal{L}_P M$ is a linear space of dimension n (linear space) isomorphic to $\mathbb{R}^n$.

Exercise 3.39. If $\ell : M \to \bigcup_{P\in M} \mathcal{L}_P M$ is a filament field, can you find a natural topology on $\ell(M)$ so $C^0(M \to \ell(M))$ makes sense directly?

3.5.4 The space $\mathscr{X}(M)$

The collection $\mathscr{X}(M)$ of chart C^∞ filament fields introduced above admits the structure of a linear space over $\mathbb{R}$ with operations

$$(\ell_1 + \ell_2)(P) = \ell_1(P) + \ell_2(P)$$

for $\ell_1, \ell_2 \in \mathscr{X}(M)$ and

$$(c\ell)(P) = c\ \ell(P)$$

for $\ell \in \mathscr{X}(M)$ and $c \in \mathbb{R}$. Notice the addition and scaling on the right take place in $\mathcal{L}_P M$ which is a linear space.

There is another scaling $cC^\infty(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ by

$$(f\ell)(P) = f(P)\, \ell(P)$$

according to which

$$(fg)\ell = f(g\ell),$$

$$(f + g)\ell = f\ell + g\ell,$$

$$f(\ell_1 + \ell_2) = f\ell_1 + f\ell_2,$$

and

$$1\ell = \ell \tag{3.25}$$

for $f, g \in cC^\infty(M)$ and $\ell, \ell_1, \ell_2 \in \mathscr{X}(M)$ where 1 in (3.25) is the multiplicative identity in the ring $cC^\infty(M)$. This scaling makes the additive abelian group of filament fields $\mathscr{X}(M)$ a **module** over the ring $cC^\infty(M)$.

In summary, the collection $\mathscr{X}(M)$ of filament fields

$$\ell \in cC^\infty \left(M \to \ell(M) \subset \bigcup_{P\in M} \mathcal{L}_P M \right)$$

is a linear space over the field $\mathbb{R}$ and a module over the ring $cC^\infty(M)$.

3.5.5 Linear Leibnizian functionals

An alternative assignment of a linear space to each point P in a Poincaré manifold with a specified C^∞ atlas $\mathcal{A}_*^\infty \subset \mathcal{A}_*$ may be obtained as follows: Recall the linear space $cC^\infty(M)$ which is also an algebraic ring and is sometimes denoted $\mathscr{F}(M)$. Denote by $\mathcal{L}(cC^\infty(M))$ the collection of all **linear functionals** $v : cC^\infty(M) \to \mathbb{R}$. More generally, we can denote by $\mathcal{L}(V)$ the collection of all linear functionals defined on a linear space V and by $\mathcal{L}(V \to W)$ the collection of all linear operators $L : V \to W$ from a linear space V to any other linear space W over the same field. Generally, an element $L \in \mathcal{L}(V \to W)$ is required to satisfy

$$L(av + bw) = aLv + bLw$$

for $v, w \in V$ and a, b scalars in the field. The space $\mathcal{L}(V \to W)$ coincides with $\daleth(V \to W)$ when V and W have topologies and V is finite dimensional. In this application we have no topology on $cC^\infty(M)$ at this point, so we use $\mathcal{L}(cC^\infty(M))$. The space $\mathcal{L}(V) = \mathcal{L}(V \to \mathbb{R})$ is also sometimes called the **algebraic dual** of the linear space V while the collection of continuous linear functionals $\daleth(V) = \daleth(V \to \mathbb{R})$ is called the **analytic dual** or simply the **dual space** when V has a topology and is also denoted simply by V^*.

We assign to each $P \in M$ the linear space

$$\mathcal{L}\mathcal{L}_P M = \{v \in \mathcal{L}(cC^\infty(M)) : v[fg] = v[f]g(p) + f(p)v[g],\ f, g \in cC^\infty(M)\}.$$

A functional $v : cC^\infty(M) \to \mathbb{R}$ satisfying

$$v[fg] = v[f]g(p) + f(p)v[g] \qquad \text{for} \qquad f, g \in cC^\infty(M) \tag{3.26}$$

is said to be **Leibnizian at** $P \in M$. Thus, $\mathcal{L}\mathcal{L}_P M$ is the collection of linear Leibnizian functionals on $cC^\infty(M)$. These functions/functionals are often referred to as "differential operators" or "directional derivative operators," but they are not really differential operators. The Leibnizian property gives these functionals one of the formal properties of a directional derivative, but no derivatives are involved (or harmed) in the construction of $\mathcal{L}\mathcal{L}_P M$.

What $\mathcal{L}\mathcal{L}_P M$ does provide is a linear space which can be associated to each point $P \in M$. Moreover, this linear space is finite dimensional and isomorphic as a linear space to $\mathcal{L}_P M$ (and hence to $\mathbb{R}^n$).

Exercise 3.40. Consider $V : T_{\xi(P)}\mathbb{R}^n \to \mathcal{LL}_P M$ by

$$V(\mathbf{v}) = v,$$

where $v : cC^\infty(M) \to \mathbb{R}$ by $v[f] = D_{\mathbf{v}}(f \circ \mathbf{p})(\xi(P))$ where $(U, \mathbf{p}) \in \mathcal{A}_*^\infty$, and

$$D_{\mathbf{v}}(f \circ \mathbf{p})(\xi(P)) = \lim_{t \searrow 0} \frac{f \circ \mathbf{p}(\xi(P) + t\mathbf{v}) - f \circ \mathbf{p}(\xi(P))}{t|\mathbf{v}|}.$$

Show V is a linear space isomorphism. In particular, $\mathcal{LL}_P M$ is a linear space of dimension n (linear space) isomorphic to $\mathbb{R}^n$.

Like $\mathcal{L}_P M$, the linear space $\mathcal{LL}_P M$ has an identity depending on, but essentially independent from, coordinate expressions.

Exercise 3.41. Describe $\mathcal{LL}_{\mathbf{x}}\mathbb{R}^n$.

3.5.6 Linear Leibnizian operators

It is also possible to consider a "field" of linear Leibnizian functionals. Generally, such a field can be expressed as a function

$$w : M \to \bigcup_{P \in M} \mathcal{LL}_P M \qquad \text{with} \qquad w(P) = w_P \in \mathcal{LL}_P M.$$

In this case $w(P)$ is usually written $w_P : cC^\infty(M) \to \mathbb{R}$. Again, the chart regularity of a linear Leiznizian functional field w can be expressed in terms of the smooth vector fields $\mathcal{V}^\infty(U)$ where $(U, \mathbf{p}) \in \mathcal{A}_*^\infty$.

Exercise 3.42. Given a field of linear Leibnizian functionals

$$w : M \to \bigcup_{P \in M} \mathcal{LL}_P M$$

and a local chart U with $(U, \mathbf{p}) \in \mathcal{A}_*^\infty$, describe the vector field $\mathbf{v}$ induced on U and define the condition

$$w \in cC^\infty(M \to w(M)).$$

Hint: Review the definition of $\ell \in cC^\infty(M \to \ell(M))$ given above for a filament field ℓ.

Since $w_P[f] \in \mathbb{R}$ for each $f \in cC^\infty(M)$ and each $P \in M$, there is an alternative formulation for linear Leibnizian functional fields in terms of linear Leibnizian operators.

The notation is slightly abused in the following description relative to the description of elements of $v \in \mathcal{LL}_P M$ given above, so one should consider it carefully. In particular, I am going to denote a linear Leibnizian operator

$$v : cC^\infty(M) \to \mathbb{R}^M$$

using the same symbol v. Given this usage, the element $v \in \mathcal{LL}_P M$ considered in the previous section should/may be denoted v_P. Here are the details: A **linear Leibnizian operator**[14] is a function

$$v : cC^\infty(M) \to \mathbb{R}^M$$

satisfying

(i) $v[af + bg] = av[f] + bv[g]$ for $a, b \in \mathbb{R}$ and $f, g \in cC^\infty(M)$, and

(ii) $v[fg] = v[f]g + fv[g]$ for $f, g \in cC^\infty(M)$.

Note that $\mathbb{R}^M$ is a linear space, so the linearity condition **(i)** makes sense, that is, **(i)** requires $v \in \mathcal{L}(cC^\infty(M) \to \mathbb{R}^M)$, that is, v is a linear operator. Also, $cC^\infty(M)$ is a ring, so the generalized Leibnizian property **(ii)** makes good sense. What this definition does not include is a condition of chart regularity.

Exercise 3.43. Show a linear Leibnizian operator $v : cC^\infty(M) \to \mathbb{R}^M$ defines a unique linear Leibnizian functional field

$$w : M \to \bigcup_{P \in M} \mathcal{LL}_P M$$

by $w(P)[f] = v[f](P)$. Show furthermore that the condition $w \in cC^\infty(M \to w(M))$ defined in Exercise 3.42 above is equivalent to the condition

$$v(cC^\infty(M)) \subset cC^\infty(M).$$

Thus, a smooth (chart C^∞) linear Leibnizian operator is simply an operator $v : cC^\infty(M) \to cC^\infty(M)$ satisfying conditions **(i)** (linearity) and **(ii)** (Leibnizian) given above.

[14] which is equivalent to a field of linear Leibnizian functionals.

3.5.7 Riemannian metric tensor

Having reached the point where we have the linear/filament space $\mathcal{L}_P M$ associated with each point P in a manifold with a maximal C^∞ atlas $\mathcal{A}_*^\infty \subset \mathcal{A}_*$ (or even a maximal C^1 atlas $\mathcal{A}_*^1 \subset \mathcal{A}_*$), it is not too much trouble to explain how the Riemannian metric tensor is realized on M. The objective is to define an inner product

$$\langle \,\cdot\, , \,\cdot\, \rangle_P : \mathcal{L}_P \times \mathcal{L}_P \to \mathbb{R} \tag{3.27}$$

on each linear space $\mathcal{L}_P M$. There are basically two difficulties involved. First we need to explain clearly the relation of such an inner product with the functions/matrix assignment (g_{ij}) in local coordinates, and in particular explain how the inner product is defined in a way that is essentially independent of the particular matrix assignment arising in a chart. If we understand well that the elements $[\alpha]$ of $\mathcal{L}_P M$ have an independent identity, then this is relatively straightforward if the values of the inner product as suggested in (3.27) are well-defined. The second difficulty is to explain clearly what kind of regularity should be required of the inner product with respect to the variable/point $P \in M$ and the consequences for regularity implied by the assignment. This is somewhat similar to the discussion of the (chart) regularity of filament fields and fields of linear Leibnizian functionals (or linear Leibnizian operators) but is, on the one hand, somewhat more complicated and, on the other hand, somewhat more significant because the assignment, if it is done correctly, actually allows a reasonable notion of actual derivatives and, hence, calculus directly on the resulting Riemannian manifold M.

There is an obvious formula. At least this formula should be obvious at this point, so perhaps I will start by writing it down:

$$\langle [\alpha], [\beta] \rangle_P = \langle (g_{ij})(\xi \circ \alpha)'(\alpha^{-1}(P)), (\xi \circ \beta)'(\beta^{-1}(P)) \rangle_{\mathbb{R}^n} \tag{3.28}$$

where $[\alpha], [\beta] \in \mathcal{L}_P M$ and $(U, \mathbf{p}) \in \mathcal{A}_*^1$ as usual. Of course this is a formula in local coordinates, and the natural question might be:

> Where do the (g_{ij}) come from?

Hopefully we will give a satisfying answer to this question.

It turns out that it may be natural to take a somewhat more general starting point as well. A **two form at** P is a bilinear function

$$b : \mathcal{L}_P M \times \mathcal{L}_P M \to \mathbb{R},$$

and the innner product suggested by the formula (3.28) above is an example of such a bilinear function. That is,

$$b(c_1[\alpha] + c_1[\beta], [\gamma]) = c_1 b([\alpha], [\gamma]) + c_2 b([\beta], [\gamma])$$

and

$$b([\gamma], c_1[\alpha] + c_1[\beta]) = c_1 b([\gamma], [\alpha]) + c_2 b([\gamma], [\beta])$$

for $c_1, c_2 \in \mathbb{R}$ and $[\alpha], [\beta], [\gamma] \in \mathcal{L}_P M$. We call the collection of all bilinear functions (or two forms) at a point $P \in M$ the **two tensors at a point** P (or sometimes the zero-two tensors at P for reasons that will be explained later) and denote this collection by

$$\mathscr{T}_P^2(\mathcal{L}_P M).$$

Exercise 3.44. What is the relation between $\mathscr{T}_P^2(\mathcal{L}_P M)$ and the linear space $\mathcal{L}(\mathcal{L}_P M \times \mathcal{L}_P M)$ of linear functionals on $\mathcal{L}_P M \times \mathcal{L}_P M$?

Exercise 3.45. Show $\mathscr{T}_P^2(\mathcal{L}_P M)$ is a linear space with $(b_1 + b_2)([\alpha], [\beta]) = b_1([\alpha], [\beta]) + b_2([\alpha], [\beta])$ and $(cb)([\alpha], [\beta]) = cb([\alpha], [\beta])$ for $b_1, b_2, b \in \mathscr{T}_P^2(\mathcal{L}_P M)$ and $c \in \mathbb{R}$ as usual.

Symmetric and positive definite two forms

Recall that an $n \times n$ matrix $(g_{ij}) = (g_{ij}(\mathbf{x}))$ defined at a point $\mathbf{x} \in \mathbb{R}^n$ is **symmetric** if $g_{ij} = g_{ji}$ for $i, j = 1, 2, \ldots, n$. Also, (g_{ij}) is **positive definite** if

$$\langle (g_{ij})\mathbf{v}, \mathbf{v} \rangle_{\mathbb{R}^n} \geq 0 \qquad \text{for} \qquad \mathbf{v} \in \mathbb{R}^n$$

with equality only if $\mathbf{v} = \mathbf{0}$.

For a bilinear form, i.e., a two form, at $P \in M$,

1. $b : \mathcal{L}_P M \times \mathcal{L}_P M \to \mathbb{R}$ is **symmetric** if

$$b([\alpha], [\beta]) = b([\beta], [\alpha]) \qquad \text{for} \qquad [\alpha], [\beta] \in \mathcal{L}_P M,$$

and

2. $b : \mathcal{L}_P M \times \mathcal{L}_P M \to \mathbb{R}$ is **positive definite** if

$$b([\alpha], [\alpha]) \geq 0 \qquad \text{for} \qquad [\alpha] \in \mathcal{L}_P M$$

with equality if and only if $[\alpha] = \mathbf{0} \in \mathcal{L}_P M$.

It will be noted immediately that these are the two properties required to make b an inner product on $\mathcal{L}_P M$. See section 13.6 in Chapter 13.

At this point we are going to abuse notation a little bit in the way we did with linear Leibnizian functional fields. A function

$$b : M \to \bigcup_{P \in M} \mathscr{T}^2_P(\mathcal{L}_P(M)) \qquad \text{with} \qquad b(P) = b_P \in \mathscr{T}^2_P(\mathcal{L}_P M)$$

is called a **two tensor** (or two form field or two tensor field or zero-two tensor (field)) on M. These words all mean the same thing. Note that for each two form field there is, for each $P \in M$, a two form $b(P)$ at the point P denoted by

$$b_P : \mathcal{L}_P M \times \mathcal{L}_P M \to \mathbb{R}.$$

Notice the difference between a two tensor and a two tensor at a point.

The collection of all two tensors may be denoted by $\mathscr{T}^2(M)$. Notice the implicit reference to the filament spaces $\mathcal{L}_P M$ here: An element b of $\mathscr{T}^2(M)$ is a function

$$b : M \to \bigcup_{P \in M} \mathscr{T}^2_P(\mathcal{L}_P M) \qquad \text{with} \qquad b(P) = b_P \in \mathscr{T}^2_P(\mathcal{L}_P M).$$

Partial/Preliminary Definition 1. A **Riemannian metric tensor** μ on M is a two form field satisfying certain properties:

(i) For each $P \in M$, the two form

$$\mu_P = \langle \,\cdot\, , \,\cdot\, \rangle_P : \mathcal{L}_P M \times \mathcal{L}_P M \to \mathbb{R}$$

at P is symmetric and positive definite, and

(ii) μ satisfies some regularity condition.

There are various ways to desribe various regularity conditions which can be used to complete this partial definition. They are all, as far as I know, fairly complicated. In order to understand some such regularity conditions and preferably some of the simpler ones, let us review some aspects of the special case of a manifold U which is an open subset of $\mathbb{R}^n$ and make a connection between $T_{\mathbf{x}}\mathbb{R}^n$ and $\mathcal{L}_{\mathbf{p}(\mathbf{x})}M$ when $\mathbf{x}$ is in a local chart U in particular. In the special case when U is an open subset of $\mathbb{R}^n$ and we have both

$$T_{\mathbf{x}}\mathbb{R}^n \qquad \text{and} \qquad \mathcal{L}_{\mathbf{x}} U$$

we consider $\mathcal{L}_{\mathbf{x}}U$ as an inner product space with the inner product

$$\langle [\alpha], [\beta] \rangle_{\mathbf{x}} = \langle \alpha'(\alpha^{-1}(\mathbf{x})), \beta'(\beta^{-1}(\mathbf{x})) \rangle_{\mathbb{R}^n}.$$

This makes the mapping $A : T_{\mathbf{x}}\mathbb{R}^n \to \mathcal{L}_{\mathbf{x}}U$ by

$$A(\mathbf{v}) = [\alpha] \qquad \text{where} \qquad \alpha(t) = \mathbf{x} + t\mathbf{v}$$

not only a linear space isomorphism but an inner product space isomorphism or what is called an (inner product space) **isometry**. What this observation is really saying is that in this special case, we really do not need to consider $\mathcal{L}_{\mathbf{x}}U$. We already have a much simpler linear space assigned to each point $\mathbf{x} \in U$, and that space is already an inner product space; that space is just the tangent space $T_{\mathbf{x}}\mathbb{R}^n = \mathbb{R}^n$.

As a consequence, in this special case while we can consider filament spaces $\mathcal{L}_{\mathbf{x}}U$, filament fields on V, tensors (of order two) and inner products on filament spaces, we do not. Instead we consider the versions of these directly on the tangent spaces $T_{\mathbf{x}}\mathbb{R}^n$ with natural and standard coordinates. Thus, instead of filaments $[\alpha] \in \mathcal{L}_{\mathbf{x}}U$ we have vectors $\mathbf{v} \in T_{\mathbf{x}}\mathbb{R}^n$; instead of filament fields

$$\ell : U \to \bigcup_{\mathbf{x} \in U} \mathcal{L}_{\mathbf{x}}U$$

we have vector fields $\mathbf{v} : U \to \mathbb{R}^n$ or more properly

$$\mathbf{v} = (v^1, v^2, \ldots, v^n) : U \to \bigcup_{\mathbf{x} \in U} T_{\mathbf{x}}\mathbb{R}^n.$$

Finally, we have the inner product

$$\langle\ \cdot\ ,\ \cdot\ \rangle_{\mathbb{R}^2} : T_{\mathbf{x}}\mathbb{R}^n \times T_{\mathbf{x}}\mathbb{R}^n \to \mathbb{R}$$

which is a symmetric, positive definite, bilinear form directly on the tangent space $T_{\mathbf{x}}\mathbb{R}^n = \mathbb{R}^n$ instead of (the equivalent) bilinear form

$$\langle\ \cdot\ ,\ \cdot\ \rangle_{\mathbf{x}} : \mathcal{L}_{\mathbf{x}}U \times \mathcal{L}_{\mathbf{x}}U \to \mathbb{R}$$

mentioned above. When we refer to tensors on U or write $\mathscr{T}^2(U)$ we mean elements of

$$\left(\bigcup_{\mathbf{x} \in U} \mathscr{T}^2_{\mathbf{x}}(\mathbb{R}^n) \right)^U = \left(\bigcup_{\mathbf{x} \in U} \mathscr{T}^2_{\mathbf{x}}(T_{\mathbf{x}}\mathbb{R}^n) \right)^U$$

instead of (the usual)

$$\left(\bigcup_{\mathbf{x}\in U} \mathscr{T}^2_{\mathbf{x}}(\mathcal{L}_{\mathbf{x}}U)\right)^U.$$

For the promised connection between $T_{\mathbf{x}}\mathbb{R}^n$ and $\mathcal{L}_{\mathbf{p}(\mathbf{x})}M$, we know these are linear space isomorphic, and we consider the linear isomorphism determined by

$$\mathbf{v} \to [\alpha]$$

where $\alpha(t) = \mathbf{p}(\mathbf{x} + t\mathbf{v})$. We denote this linear isomorphism by

$$d\mathbf{p}_{\mathbf{x}} : T_{\mathbf{x}}\mathbb{R}^n \to \mathcal{L}_{\mathbf{p}(\mathbf{x})}M$$

and may call it the **differential map of p at x** though there is no formula for this map (at least at this point) in terms of a/any derivative.

We can then apply the above described conventions to the special case $U \subset \mathbb{R}^n$ when U is a local chart with a given chart function $\mathbf{p} : U \to M$. The Riemannian metric tensor induces a two form field

$$b : U \to \bigcup_{\mathbf{x}\in U} \mathscr{T}^2_{\mathbf{x}}(T_{\mathbf{x}}\mathbb{R}^n)$$

by

$$b_{\mathbf{x}}(\mathbf{v}, \mathbf{w}) = \mu_{\mathbf{q}(\mathbf{x})}(d\mathbf{p}_{\mathbf{x}}(\mathbf{v}), d\mathbf{p}_{\mathbf{x}}(\mathbf{w})).$$

This two form field on U determines n^2 functions $g_{ij} : U \to \mathbb{R}$ by

$$g_{ij} = g_{ij}(\mathbf{x}) = b_{\mathbf{x}}(\mathbf{e}_i, \mathbf{e}_j).$$

The resulting matrix $g = (g_{ij})$ is symmetric and positive definite. Thus, we can define functions

$$\begin{aligned} g_{ij} &: U \to \mathbb{R} \qquad \text{for } i, j = 1, 2, \ldots, n, \\ g &: U \to \mathrm{Sym}^+_n(\mathbb{R}) \subset GL_n(\mathbb{R}), \text{ and} \\ \rho \circ g &: U \to \mathbb{R}^{n^2} \end{aligned}$$

where ρ is the row injection of $GL_n(\mathbb{R})$ into $\mathbb{R}^{n^2}$. Equivalent minimal (chart) regularity conditions required to complete the definition of a Riemannian metric tensor given above are

$$g_{ij} \in C^0(U), \ i, j = 1, 2, \ldots, n, \tag{3.29}$$

and

$$\rho \circ g \in C^0(U \to \mathbb{R}^{n^2}) \tag{3.30}$$

for all $(U, \mathbf{p}) \in \mathcal{A}^1_*$.

Exercise 3.46. Show that if (3.29) holds for all $(U, \mathbf{p}) \in \mathcal{A}$ where $\mathcal{A}$ is some covering atlas in $\mathcal{A}^1$, then (3.29) also holds for every $(U, \mathbf{p}) \in \mathcal{A}^1_*$.

The usual assumption is

$$g_{ij} \in C^\infty(U), \ i, j = 1, 2, \ldots, n, \qquad (U, \mathbf{p}) \in \mathcal{A}^\infty_*. \tag{3.31}$$