Chapter 3

Starting with examples

I am now going to introduce several examples of Riemannian manifolds and attempt to give (or at least discuss) most of the details of a formal definition. I will assume here that the the real numbers $\mathbb{R}$ and the Euclidean spaces $\mathbb{R}^n$ for $n \in \mathbb{N} = \{1, 2, \ldots\}$ consisting of points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ with each entry x_j a real number for $j = 1, 2, \ldots, n$ are familiar. If you are perceptive enough to know these are actually very mysterious spaces in some ways, do not worry about that.

3.1 Preliminary calculations

My first example(s) involve the case $n = 2$ and the particular set

$$B_1(\mathbf{0}) = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$$

which is the **unit ball** in $\mathbb{R}^2$. So first of all, this set has its usual identity[1] as a (coordinatized Euclidean) subset of $\mathbb{R}^2$ and we're assuming here you know everything (or at least many things) about that set. We will feel free to use any aspects of the Euclidean structure on this set $B_1(\mathbf{0})$ up to and including the calculus of functions with domain the unit ball, though this topic in particular is addressed in some detail elsewhere in these notes. I will simply pause to say that this Euclidean structure we are essentially taking for granted is very crucial to the concept of a Riemannian manifold.

[1] I'm using the term "identity" here in an informal sense, like Superman has his identity as Clark Kent, rather than in any technical sense from algebra.

The idea of our first real example of a Riemannian manifold is that it is, as a set, the same as $B_1(\mathbf{0})$. I will, however, call that set by a different name, and my suggestion is that the new set or structure I introduce bears an identity quite distinct from that of the Euclidean ball. The new name is $\mathcal{B}$. Perhaps I could pick a more distinctive name like M or M^2, but I think $\mathcal{B}$ will do.

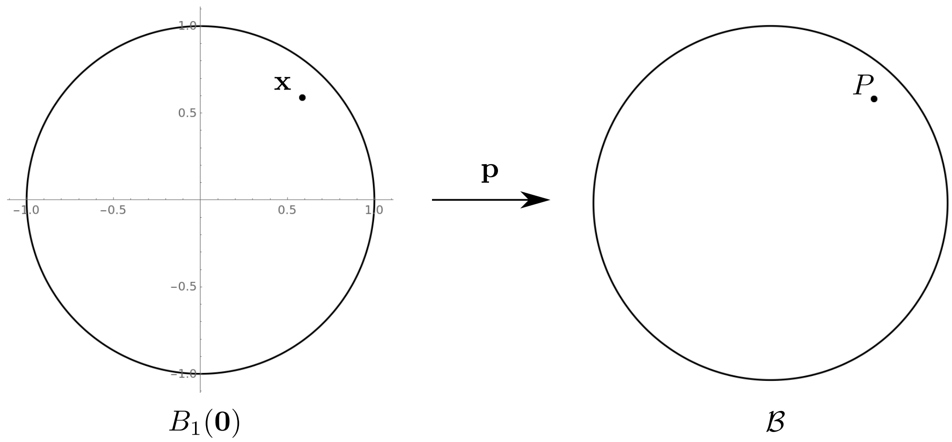

Figure 3.1: The Euclidean unit disk (left) and a Riemann surface $\mathcal{B}$ (right)

In what follows, making the distinction between $\mathcal{B}$ and the Euclidean ball $B_1(\mathbf{0})$ is both crucial and difficult. Before I attempt that crucial and difficult task, I am simply going to suggest some calculations without any notational or conceptual distinction. Then I will try to tease out the identity of $\mathcal{B}$ in stages from there. Each of the calculations involves the assignment of a certain 2×2 matrix to each $\mathbf{x} \in B_1(\mathbf{0})$, namely

$$\begin{pmatrix} \frac{16}{(4+|\mathbf{x}|^2)^2} & 0 \\ 0 & \frac{16}{(4+|\mathbf{x}|^2)^2} \end{pmatrix}. \tag{3.1}$$

I will first describe the suggested calculations in general terms and then give some exercises suggesting specific instances. For the general description, let $a, b \in \mathbb{R}$ be given with $a < b$. Given a path[2] $\alpha \in C^1([a,b] \to B_1(\mathbf{0}))$, recall

[2]Paths are discussed in more detail elsewhere in these notes, but we are assuming some familiarity here. In particular, a path $\alpha \in C^1([a,b] \to B_1(\mathbf{0}))$ as considered here has associated with it two real valued functions $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_j \in C^1[a,b]$, $j = 1, 2$

the (Euclidean) length is given by

$$\begin{aligned}\text{length}[\alpha] &= \int_a^b |\alpha'(t)|\,dt \qquad (3.2)\\ &= \int_a^b \sqrt{\langle \alpha'(t), \alpha'(t)\rangle_{\mathbb{R}^2}}\,dt\\ &= \int_a^b \sqrt{[\alpha_1'(t)]^2 + [\alpha_2'(t)]^2}\,dt.\end{aligned}$$

Both here and in (3.1) the Euclidean norm and innner product (or dot product) are used so that

$$|\mathbf{x}|^2 = \langle \mathbf{x}, \mathbf{x}\rangle_{\mathbb{R}^2} = x_1^2 + x_2^2.$$

3.1.1 Calculation of length in $\mathcal{B}$

For the same path $\alpha \in C^1([a,b] \to B_1(\mathbf{0}))$ discussed above, or more properly for the corresponding path[3] in $\mathcal{B}$, one can calculate the **Riemannian length**. As mentioned above the calculation involves the matrix assignment given in (3.1). For convenience, denote the entries in the matrix by $g_{ij} = g_{ij}(\mathbf{x})$ for $i, j = 1, 2$ so we can also write the matrix as

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix} = \frac{16}{(4 + |\mathbf{x}|^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (3.3)$$

Then the Riemannian length is given by

$$\text{length}_{\mathcal{B}}[\alpha] = \int_a^b \sqrt{\langle\ (g_{ij}(\alpha(t)))\ \alpha'(t),\ \alpha'(t)\ \rangle_{\mathbb{R}^2}}\,dt.$$

[3] We will not be careful about the details of the notation at this point because we have not carefully defined the identity of $\mathcal{B}$. Technically, we should consider the corresponding path $\tilde{\alpha} : [a,b] \to \mathcal{B}$ given by $\tilde{\alpha}(t) = \mathbf{p} \circ \alpha(t)$ where $\mathbf{p}$ is the "structure erasing map", i.e., the identity function on $B_1(\mathbf{0})$ which erases all the structure on $B_1(\mathbf{0})$ and gives us the structureless point set $\mathcal{B}$ on which we will build a/the Riemannian manifold (named) $\mathcal{B}$. Here, for these preliminary calcuations I will just write $\text{length}_{\mathcal{B}}[\alpha]$ instead of $\text{length}_{\mathcal{B}}[\tilde{\alpha}]$.

Here the usual conventions for matrix multiplication and the Euclidean inner product in $\mathbb{R}^2$ are used so that

$$\begin{aligned}(g_{ij}(\alpha(t)))\ \alpha'(t) &= \begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} \alpha_1' \\ \alpha_2' \end{pmatrix} \\ &= \begin{pmatrix} g_{11}\alpha_1' + g_{21}\alpha_2' \\ g_{12}\alpha_1' + g_{22}\alpha_2' \end{pmatrix}\end{aligned}$$

and

$$\langle \mathbf{y}, \mathbf{x} \rangle_{\mathbb{R}^2} = y_1 x_1 + y_2 x_2.$$

Exercise 3.1. Find the Euclidean length length$[\alpha]$ and Riemannian length length$_{\mathcal{B}}[\alpha]$ of the path $\alpha : [0, a] \to B_1(\mathbf{0})$ by $\alpha(t) = t(\cos\theta, \sin\theta)$ where $\theta \in \mathbb{R}$ and $a > 0$.

Exercise 3.2. Find the Euclidean length length$[\alpha]$ and Riemannian length length$_{\mathcal{B}}[\alpha]$ of the path $\alpha : [0, \theta] \to B_1(\mathbf{0})$ by $\alpha(t) = a(\cos t, \sin t)$ where $\theta \in \mathbb{R}$ and $a > 0$.

Exercise 3.3. For each x, a_0 and θ_0 with $x < 0 < a_0 < 1$ and $0 < \theta_0 \le \pi/2$ find the Euclidean length length$[\alpha]$ and Riemannian length length$_{\mathcal{B}}[\alpha]$ of the path $\alpha : [0, \theta] \to B_1(\mathbf{0})$ by $\alpha(t) = (x, 0) + a(\cos t, \sin t)$ where

$$a = \sqrt{(a_0 \cos\theta_0 - x)^2 + a_0^2 \sin^2\theta_0}$$

and

$$\theta = \tan^{-1}\left(\frac{a_0 \sin\theta_0}{a_0\cos\theta_0 - x}\right)$$

as illustrated in Figure 3.2.

Exercise 3.4. It is not clear that the notion of the **radius** of a Riemannian manifold makes sense in general, but I'm pretty sure it makes sense to talk about the radius of the Riemannian manifold $\mathcal{B}$. What is the radius of $\mathcal{B}$?

3.1.2 Angles in $\mathcal{B}$

For the second calculation I'd like to discuss, one is given two paths

$$\alpha \in C^1([a_1, b_1] \to B_1(\mathbf{0})) \qquad \text{and} \qquad \beta \in C^1([a_2, b_2] \to B_1(\mathbf{0}))$$

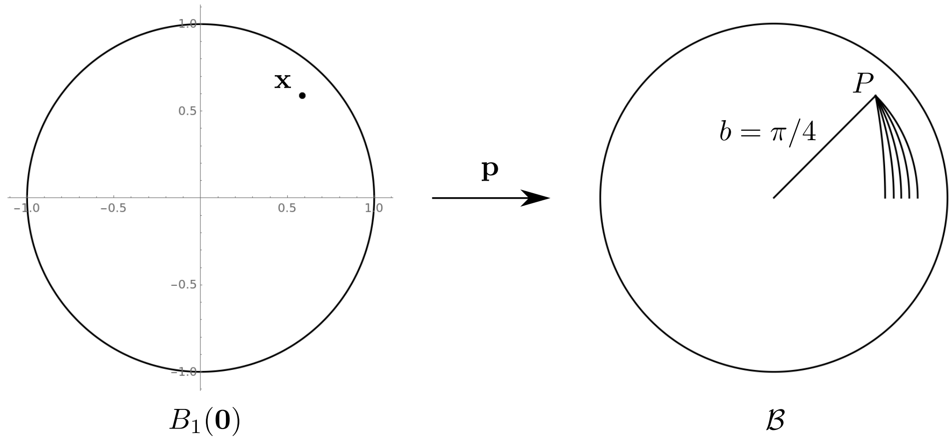

Figure 3.2: The Euclidean unit disk (left) and a Riemann surface $\mathcal{B}$ (right)

satisfying $\alpha(t_1) = \beta(t_2)$ for some t_j with $a_j < t_j < b_j$ for $j = 1, 2$. In this case, the paths are said to meet at the (Euclidean) angle $\theta \in [0, \pi]$ if

$$\cos\theta = \frac{\langle \alpha'(t_1), \beta'(t_2)\rangle_{\mathbb{R}^2}}{|\alpha'(t_1)|\ |\beta'(t_2)|}. \tag{3.4}$$

Of course, this does not quite always serve as a definition for the angle at which the paths meet because the value on the right in (3.4) may not be a well-defined real number. Specifically, if $\alpha'(t_1) = \mathbf{0}$ or $\beta'(t_2) = \mathbf{0}$, then there is a problem. If we rule out these possibilities by requiring $|\alpha'(t_1)|\ |\beta'(t_2)| > 0$, then there is no problem.

Exercise 3.5. Let $\cos^{-1} : [-1, 1] \to [0, \pi]$ denote the principal arccosine function. Plot $\cos^{-1}$ and explain why this function may be applied to both sides of (3.4).

Under the assumptions described above under which the Euclidean angle between paths α and β is well-defined, the **Riemannian angle** at which the paths α and β meet is given by

$$\theta_{\mathcal{B}} = \cos^{-1}\left(\frac{\langle (g_{ij})\ \alpha'(t_1),\ \beta'(t_2)\rangle_{\mathbb{R}^2}}{\sqrt{\langle (g_{ij})\ \alpha'(t_1),\ \alpha'(t_1)\rangle_{\mathbb{R}^2}\quad \langle (g_{ij})\ \beta'(t_2),\ \beta'(t_2)\rangle_{\mathbb{R}^2}}}\right). \tag{3.5}$$

Exercise 3.6. Assuming two paths

$$\alpha \in C^1([a_1, b_1] \to B_1(\mathbf{0})) \qquad \text{and} \qquad \beta \in C^1([a_2, b_2] \to B_1(\mathbf{0}))$$

meet at the Euclidean angle θ as described above, calculate the Riemannian angle $\theta_{\mathcal{B}}$ at which the two paths meet.

The exercise above should have been pretty anticlimactic. Comparison of (3.5) and (3.4) in general might suggest some interesting questions.

Other matrix assignments

Exercise 3.7. A **general inner product** on a real linear space[4] V is a function $\langle\ \cdot\ ,\ \cdot\ \rangle_V : V \times V \ \to \mathbb{R}$ satisfying the following conditions

(i) $\langle v, w\rangle_V = \langle v, w\rangle_V$ for $v, w \in V$.

(ii) $\langle v, v\rangle_V \geq 0$ for all $v \in V$ with equality if and only if $v = \mathbf{0} \in V$.

(iii) $\langle av + bw, z\rangle_V = a\langle v, z\rangle_V + b\langle w, z\rangle_V$ for all $a, b \in \mathbb{R}$ and $v, w, z \in V$.

If a function $\langle\ \cdot\ ,\ \cdot\ \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\langle \mathbf{v}, \mathbf{w}\rangle = \langle\ (g_{ij})\ \mathbf{v},\ \mathbf{w}\ \rangle_{\mathbb{R}^2},$$

where (g_{ij}) is a 2×2 matrix with real entries, then show $\langle\ \cdot\ ,\ \cdot\ \rangle$ defines an abstract inner product on $\mathbb{R}^2$ if and only if the following conditions are satisfied by the matrix (g_{ij}):

(i) $g_{ij} = g_{ji}$ for $i, j = 1, 2$,

(ii) $g_{11}, g_{22} > 0$, and

(iii) $g_{11}g_{22} - g_{12}^2 > 0$.

[4] I am going to try to make a distinction between a **linear space** and a **vector space** by reserving the term **vector space** to describe a **normed linear space**. All these concepts are reviewed in Chatpter 13 on different kinds of structures or spaces. Since we are discussing an inner product here, and as you can find in Chapter 13 (or probably already know) every **inner product space** is a **normed linear space** and thus a **vector space**, so the distinction is not so immediately crucial here. See the section below in which we introduce **linear structure** on $\mathcal{B}$. In that context the distinction becomes useful, and maybe even necessary and important.

Exercise 3.8. A **general bilinear form** on a real linear space V is a function $B : V \times V \to \mathbb{R}$ satisfying the following

(i) $B(av + bw, z) = aB(v, z) + bB(w, z)$ for all $a, b \in \mathbb{R}$ and $v, w, z \in V$.

(ii) $B(z, av + bw) = aB(z, v) + bB(z, w)$ for all $a, b \in \mathbb{R}$ and $v, w, z \in V$.

(a) Show every general bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ determines a unique matrix $(a_{ij}) \in \mathbb{R}^{n\times n}$ for which

$$B(\mathbf{v}, \mathbf{w}) = \langle (a_{ij})\ \mathbf{v},\ \mathbf{w}\rangle_{\mathbb{R}^n} \qquad \text{for all } \mathbf{v}, \mathbf{w} \in \mathbb{R}^n. \tag{3.6}$$

(b) Show conversely that given any real matrix $(a_{ij}) \in \mathbb{R}^{n\times n}$ the formula (3.6) determines a unique general bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

The matrix (a_{ij}) is called the **matrix of the bilinear form**.

A general bilinear form $B : V \times V \to \mathbb{R}$ is said to be **symmetric** if $B(v, w) = B(w, v)$ for all $v, w \in V$.

Exercise 3.9. A symmetric bilinear form $B : V \times V \to \mathbb{R}$ for which $B(v, w) = 0$ for all $w \in V$ implies $v = \mathbf{0}$ is said to be **nondegenerate**. Show the matrix of a nondegenerate (symmetric) bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is invertible.

3.1.3 Calculation of area in $\mathcal{B}$

The third calculation I would like to discuss is most easily described in terms of a **cyclic path**. To get started we can assume a cyclic path is parameterized for some $\epsilon > 0$ by a function $\alpha \in C^1([a - \epsilon, b + \epsilon] \to B_1(\mathbf{0}))$ satisfying the following:

$$\begin{aligned} \alpha(t_1) \neq \alpha(t_2) \qquad &\text{for} \qquad a \le t_1 < t_2 < b, \\ \alpha(t) = \alpha(b - a + t) \qquad &\text{for} \qquad a - \epsilon \le t \le a, \\ \alpha(t) = \alpha(a + t - b) \qquad &\text{for} \qquad b \le t \le b + \epsilon, \end{aligned}$$

and $|\alpha'(t)| \neq 0$. In such a case, the path itself is

$$\Gamma = \{\alpha(t) \in B_1(\mathbf{0}) : t \in [a, b]\}.$$

If $A \subset B_1(\mathbf{0})$ is a region with $\partial A = \Gamma$, then the (Euclidean) area of A is defined to be

$$\text{area}(A) = \int_A 1.$$

The meaning of this integral is assumed to be familiar here but is also discussed in some detail elsewhere in these notes. The **Riemannian area** of the region A enclosed by Γ is given by

$$\text{area}_{\mathcal{B}}(A) = \int_A \sqrt{g_{11}g_{22} - g_{12}^2}.$$

Exercise 3.10. For each $a > 0$ calculate the Euclidean area area(A) and the Riemannian area area$_{\mathcal{B}}(A)$ of the region A enclosed by $\alpha : [0, 2\pi] \to B_1(\mathbf{0})$ by $\alpha(t) = a(\cos t, \sin t)$.

Exercise 3.11. Notice I only gave a definition for Riemannian area for regions enclosed by C^1 (smooth) paths. Of course, it is not too much to ask to consider the Euclidean and Riemannian areas of the following regions:

(a) $R = \{\mathbf{x} = (x_1, x_2) : a_1 < x_1 < a_2, b_1 < x_2 < b_2\} \subset B_1(\mathbf{0})$.

(b) $S = \{\mathbf{x} = (x_1, x_2) : 0 < x_1 < a, x_2 < mx_1, x_1^2 + x_2^2 < r^2\} \subset B_1(\mathbf{0})$.

(c) $T = \{\mathbf{x} = (x_1, x_2) : \langle \mathbf{x} - \mathbf{x}_j \cdot \mathbf{n}_j\rangle_{\mathbb{R}^2} < 0, j = 1, 2, 3\} \subset \mathcal{B}$.

In these instances a_j and b_j, are appropriate real numbers for $j = 1, 2$, the numbers a, r and m are positive real numbers, and $\mathbf{x}_j$ and $\mathbf{n}_j$ are appropriate elements of $\mathbb{R}^2$ for $j = 1, 2, 3$. Give an appropriate definition of **piecewise C^1 cyclic paths** in $B_1(\mathbf{0})$ which allows the regions above to be considered as regions bounded by cyclic paths and regions of integration in particular.

Exercise 3.12. Find formulas for the Euclidean areas and Riemannian areas of some of the regions mentioned in **(a)**, **(b)**, and/or **(c)** of Exercise 3.11 above.

Exercise 3.13. If the Riemannian length of the radial segment indicated in Figure 3.2 is $b = \pi/4$, what is the Euclidean length a of the corresponding segment $\{t(\cos\theta, \sin\theta) \in B_1(\mathbf{0}) : 0 < t < a\}$?

Exercise 3.14. For some fixed a_0 and θ_0 in Problem 3.3, plot the value of length$_B[\alpha]$ as a function of x. Do you notice anything interesting?

3.2 Example $\mathcal{C}$

As a **point set** $\mathcal{B}$ is identical to $B_1(\mathbf{0}) \subset \mathbb{R}^2$. I've suggested that in order to think about and understand the identity of $\mathcal{B}$ as a Riemannian manifold, it is useful to "leave all the structure of $B_1(\mathbf{0}) \subset \mathbb{R}^2$ behind." After reflection on this suggestion, it strikes me that $\mathcal{B}$ is such a nice point set that it may be quite difficult to ignore the structure from $B_1(\mathbf{0})$ when contemplating $\mathcal{B}$. In anticipation of this problem, I've devised an alternative example designed to illustrate (to some extent) just how bad a Riemannian manifold's point set[5] can be. The second example is called $\mathcal{C}$, and $\mathcal{C}$ is a subset of $\mathbb{R}^3$. In order to describe $\mathcal{C}$, I'm going to use two familiar subsets of $\mathbb{R}$ and two familiar "quantities" associated with points in $\mathbb{R}^2$. The relevant subsets of $\mathbb{R}$ are the rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q} : q \in \mathbb{N} = \{1, 2, 3, \dots,\} \text{ and } p \in \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\} \right\}$$

and the irrational numbers $\mathbb{R} \backslash \mathbb{Q}$. The quantities associated with a point $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ are the polar radius

$$r = \sqrt{x_1^2 + x_2^2}$$

and the argument $\theta \in [0, 2\pi)$ for which $\mathbf{x} = r(\cos\theta, \sin\theta)$. There is one more (hopefully familiar) concept I need, namely that of a **characteristic function**: Given any set S and any subset $A \subset S$, the **characteristic function with support on** A is the function $\chi_A : S \to \mathbb{R}$ by

$$\chi_A(p) = \begin{cases} 1, & p \in A \\ 0, & p \in S \backslash A. \end{cases}$$

[5]Keep in mind that the point set by itself is not the Riemannian manifold in its entirety, but the Riemannian manifold is the point set along with the "structure" I am trying to describe and hopefully you are trying to understand. In principle, however, the point set has an obvious importance in the definition, and I view assertions like "... in the context of the preceeding definitions, one cannot distinguish between two homeomorphic manifolds nor between two diffeomorphic differentiable manifolds" which appears on page 3 in [4] as somewhat counterproductive, at least for those who are trying to understand those definitions. Hopefully, many differences between $\mathcal{B}$ and $\mathcal{C}$ are fairly obvious, though it is equally clear one does not discern those differences by looking at the charts alone.

In terms of these quantities, I'm going to define a function $h : \mathbb{R}^2 \to \mathbb{R}$ by

$$h(x_1, x_2) = \begin{cases} \chi_{\mathbb{R}\backslash\mathbb{Q}}(r) - \chi_{\mathbb{Q}}(r), & \theta/\pi \in \mathbb{Q} \\ \chi_{\mathbb{Q}}(r) - \chi_{\mathbb{R}\backslash\mathbb{Q}}(r), & \theta/\pi \in \mathbb{R}\backslash\mathbb{Q}. \end{cases} \tag{3.7}$$

Then

$$\mathcal{C} = \left\{ \left(x_1, x_2, h(x_1, x_2)\sqrt{x_1^2 + x_2^2} \right) : \mathbf{x} = (x_1, x_2) \in B_1(\mathbf{0}) \right\}. \tag{3.8}$$

The important points are the following:

1. $\mathcal{C}$ is in one-to-one correspondence with $B_1(\mathbf{0})$.
2. Riemannian lengths ($\text{length}_{\mathcal{C}}$), Riemannian angles ($\theta_{\mathcal{C}}$), and Riemannian areas ($\text{area}_{\mathcal{C}}$) can be computed using the same formulas used to find $\text{length}_{\mathcal{B}}$, $\theta_{\mathcal{B}}$, and $\text{area}_{\mathcal{B}}$.
3. $\mathcal{C}$ is an example of a Riemannian manifold just as much as $\mathcal{B}$.

Here are some exercises to walk you through some of the details of $\mathcal{C}$. The first question you might ask is: Is $\mathcal{C}$ well-defined?

Exercise 3.15. The quantity $r = \sqrt{x_1^2 + x_2^2}$, i.e., the polar radius, clearly corresponds to a well-defined function $r : \mathbb{R}^2 \to \mathbb{R}$. (I've written down the formula for this function.) The polar angle or argument θ is somewhat more complicated.

(a) Find/write down a formula for the argument $\theta : \mathbb{R}^2 \to \mathbb{R}$.

(b) Make an illustration of the graph

$$\left\{ \left(x_1, x_2, \sqrt{x_1^2 + x_2^2} \right) : \mathbf{x} = (x_1, x_2) \in B_1(\mathbf{0}) \right\}.$$

(c) Make an illustration of the graph

$$\{ (x_1, x_2, \theta(x_1, x_2)) : \mathbf{x} = (x_1, x_2) \in B_1(\mathbf{0}) \}.$$

Exercise 3.16. One consequence of the solution of Exercise 3.15 should be that the function h defined in (3.7) and the point set $\mathcal{C}$ defined in (3.8) are well-defined. In particular, there exists a well-defined **chart function**[6] $\mathbf{p} : B_1(\mathbf{0}) \to \mathcal{C}$ given by

$$\mathbf{p}(x_1, x_2) = (x_1, x_2, h(x_1, x_2)\ r(x_1, x_2)).$$

Find/write down a formula for the inverse $\xi : \mathcal{C} \to B_1(\mathbf{0})$ of $\mathbf{p}$. This function should have two coordinate functions $\xi = (\xi^1, \xi^2)$ with $\xi^j : \mathcal{C} \to \mathbb{R}$ for $j = 1, 2$. It's probably easiest to write down formulas in the form $\xi^j = \xi^j(x_1, x_2, x_3)$ for $j = 1, 2$ where $(x_1, x_2, x_3) \in \mathcal{C}$.

Exercise 3.17. Draw an illustration of the graph

$$\mathcal{C} = \left\{ \left(x_1, x_2, h(x_1, x_2)\sqrt{x_1^2 + x_2^2} \right) : \mathbf{x} = (x_1, x_2) \in B_1(\mathbf{0}) \right\}.$$

Terminology of charts

I think the idea Riemann (and Gauss) had in mind is best illustrated by imagining you are looking at a paper map. The next chapter, Chapter 4, is intended to help you put your mind in this mode of thinking. The entity corresponding to the paper map in the discussion of this chapter so far is the Euclidean disk $B_1(\mathbf{0})$. Generally, we can refer to the entity (or set) playing this role as the **chart**. So a **chart** for us is an open subset of (coordinatized) Euclidean space $\mathbb{R}^n$, in this case $B_1(\mathbf{0}) \subset \mathbb{R}^2$. As you look at the paper map/chart, you have in mind some kind of identification with some other "world," like the surface of the earth perhaps, and the functions

$$\mathbf{p} : B_1(\mathbf{0}) \to \mathcal{B} \qquad \text{and} \qquad \mathbf{p} : B_1(\mathbf{0}) \to \mathcal{C}$$

play this role. I wish to call functions like these **chart functions**.

Definition 3. (chart and chart function[7]) Given a Riemannian manifold M (whatever that is—some point set with a "structure") an open subset U of

[6] See below for further discussion of this terminology.

[7] More properly, this should probably be called an "informal definition" or a "preliminary definition," because it starts with the prerequisite assumption of a Riemannian manifold which we are in the process of trying to define. On the other hand, if one understands the definition of a Riemannian manifold, then this is a perfectly fine definition. It is perhaps just a little premature in a technical sense.

$\mathbb{R}^n$ in which geometric calculation can be carried out in reference to M, i.e., in which geometry in M can be "done," is called a **chart**. This set should be compared to a paper map which is used to, for example, determine distances in some "world" M in which the geometry may not be seen directly.

Associated with each chart, is a **chart function** $\mathbf{p} : U \to M$. If there is only one chart and only one chart function,[8] then for most practical purposes, the roles played by both the chart function and the point set M are secondary. The only property required of $\mathbf{p} : U \to M$ is that $\mathbf{p}$ is a bijection.

Of course, the matrix assignment (g_{ij}) on U is of central importance. You need to use (g_{ij}) in order to understand or "do" geometry in M. We will discuss that more later.

Generally a chart function is not required to be a bijection, but a chart function is always required to be a bijection onto the image

$$\mathbf{p}(U) = \{\mathbf{p}(\mathbf{x}) : \mathbf{x} \in U\} \subset M.$$

That is, $\mathbf{p} : U \to M$ is always required to be an injection. In cases where $\mathbf{p}$ is not surjective, i.e., when $\mathbf{p}(U)$ is a proper subset of M, then there must be other charts and chart functions around in order to navigate to all points in the world M. This almost goes without saying. The crucial consideration of situations in which there is more than one chart may be found in Section 3.6 below. For now, we can make a distinction among these two different kinds of charts and chart functions:

A chart U with a bijective chart function $\mathbf{p} : U \to M$ is called a **global chart/global chart function pair**. A chart U with a chart function $\mathbf{p} : U \to M$ for which $\mathbf{p}(U) \neq M$ is called a **local chart/local chart function pair**.

The set $\mathcal{B}$ or the set $\mathcal{C}$, which we wish to think of as the manifold M, is (when you are looking at the map/chart $B_1(\mathbf{0})$) **not seen directly**. The set M which "is" the manifold may be radically different as a point set from the way it is represented on the chart as illustrated by the Riemannian manifold $\mathcal{C}$. In order to understand the manifold, you have primary recourse to the chart itself, and both the manifold and the chart function play a secondary role with respect to the Riemannian geometry, though this state of affairs may be somewhat dependent on what you find when you actually "go out

[8] Note carefully that this is the case in both example $\mathcal{B}$ and example $\mathcal{C}$ where $U = B_1(\mathbf{0})$ and the single chart function $\mathbf{p} : B_1(\mathbf{0}) \to M$ is a bijection for $M = \mathcal{B}$ or $\mathcal{C}$.

into the world" and consider the manifold as a point set. It may be that you find M resembles the information contained in the chart so closely, it is difficult to tell M from U. This is the case with the Riemannian manifold $\mathcal{B}$ and the chart $B_1(\mathbf{0})$. It may also be the case that the point set M is easy to tell apart from the chart U as in the case of $\mathcal{C}$ and $B_1(\mathbf{0})$ or in the case of a paper map of Atlanta and a drive from Skiles to Stone Mountain.

In summary, Riemannian geometry is not a "visible" geometry. Though the angles are still "seen" in example $\mathcal{B}$ (and example $\mathcal{C}$) above, the notions of "length" (Riemannian length) and "area" (Riemannian area) are not directly seen, and this is the point. Very specifically, the observed Euclidean distance from the point $P_0 = \mathbf{p}(\mathbf{0}) \in \mathcal{B}$ corresponding to the origin $\mathbf{0} = (0,0) \in B_1(\mathbf{0})$ to the point $\mathbf{x} \in B_1(\mathbf{0})$ is $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2}$, and this is in general different from

$$\text{length}_{\mathcal{B}}[\alpha] = \int_0^{|\mathbf{x}|} \frac{4}{4+t^2}\,dt$$

which is the Riemannian distance[9] from $P_0 \in \mathcal{B}$ to $P = \mathbf{p}(\mathbf{x}) \in \mathcal{B}$. This is a length which is not "seen" and reminds me of something Spengler wrote about classical (western) mathematics:

> Every product of the waking consciousness of the Classical world, then, is elevated to the rank of actuality by way of sculptural definition. That which cannot be drawn is not "number."
>
> —Oswald Spengler

In contrast what one has in Riemannian geometry requires a strikingly different perspective:

> Numbers are the images of the perfectly desensualized understanding, of pure thought, and contain their abstract validity within themselves.
>
> —Oswald Spengler

We have already given three examples, though you may not have noticed the third: The Euclidean ball $B_1(\mathbf{0})$ is also a Riemannian manifold.

Exercise 3.18. What is the matrix assignment for each point in the Euclidean unit ball $B_1(\mathbf{0})$ when considered as a Riemannian manifold? (If you

[9] As Weierstrass might point out, it is also far from clear that this value is actually the Riemannian length of the path of **shortest** Riemannian length from P_0 to P.

do not know immediately, guess the simplest thing you can think of.) Go back and apply the definitions of Riemannian length, angle, and area to the Riemannian manifold $B_1(\mathbf{0})$. Explain carefully what you get in each case.

Exercise 3.19. Obviously regularity is not of primary interest in my application of the functions r and θ in constructing the example $\mathcal{C}$. However, these are interesting functions in general, and regularity is interesting in general.

(a) Show the polar radius satisfies $r \in C^0(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \backslash \{\mathbf{0}\})$.

(b) Show $r \notin C^1(\mathbb{R}^2)$.

(c) Show $r \in \text{Lip}(\mathbb{R}^2)$

(d) Show $\theta \in C^\infty(U)$ where

$$U = B_1(\mathbf{0}) \backslash \{(x, 0) : 0 \le x \le 1\}.$$

(e) Show θ has no continuous extension to

$$\overline{U} = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 1\}$$

where U is given in part **(d)** above.

(f) Let $\epsilon > 0$ and

$$V = \{\mathbf{x} = (x_1, x_2) \in B_1(\mathbf{0}) \backslash (\overline{B_\epsilon(\mathbf{0})} \cup \{s(\cos t, \sin t) : s \ge 0, \ |t| \le \epsilon\}).$$

Draw an illustration of $V \subset \mathbb{R}^2$.

(g) Let V be the open set defined in part **(f)** above. Show $\theta \in C^\infty(\overline{V})$ in the sense that for each partial derivative $D^\beta\theta : V \to \mathbb{R}$ there is some continuous function $g \in C^0(\overline{V})$ for which

$$g\big|_V = D^\beta\theta.$$

(h) With V as in part **(f)** above, show $\theta \in C^\infty(\overline{V})$ in the much stronger sense that $\theta \in C^\infty(V)$ and there is some open set $W \subset \mathbb{R}^2$ with $\overline{V} \subset W$ and a function $g \in C^\infty(W)$ such that

$$g\big|_V = \theta.$$

(i) Challenge: With V as in parts **(f)**, **(g)**, and **(h)** above, show there exists a function $g \in C^\infty(\mathbb{R}^2)$ for which

$$g\big|_V = \theta.$$

Here is another exercise which is a follow-up to the consideration of general bilinear forms as considered in Exercises 3.8 and 3.9. This is also a little outside the most direct narrative leading to the definition of a Riemannian manifold, but it may be interesting, and it will undoubtedly come up in the discussion at some point.

Exercise 3.20. Given a general bilinear form $B : V \times V \to \mathbb{R}$ as in Exercise 3.8, the **associated quadratic form** is the function $Q : V \to \mathbb{R}$ by $Q(v) = B(v, v)$.

(a) Show B is symmetric if and only if the **polarization identity**

$$B(v, w) = \frac{1}{2}[Q(v + w) - Q(v) - Q(w)]$$

holds.

(b) Show that if the matrix of a symmetric bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is invertible and $B(\mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{w} \in \mathbb{R}^n$, then $\mathbf{v} = \mathbf{0}$.

3.3 Structures on $\mathcal{B}$ and $\mathcal{C}$

I will now start in the direction of systematically introducing structure on the manifolds $\mathcal{B}$ and $\mathcal{C}$ which for all appearances have little or nothing to do with the geometrical calculations suggested above. Various functions will play a role in the development.[10]

[10] An outline of the first steps in the development are given in section 3.3.2 below when the manifold $\mathcal{B}$ is first allowed to start asserting its identity. What follows here is a somewhat rambling preliminary discussion, and you may want to skip down to section 3.3.2 at least before you start to try to wade through what follows/intervenes.

3.3.1 Preliminaries

Generally speaking there are functions with domain $\mathcal{B}$ and functions with codomain $\mathcal{B}$. The most important of the former are real valued functions $f : \mathcal{B} \to \mathbb{R}$. In order to emphasize the absence of any structure at all, that is to say consideration of $\mathcal{B}$ simply as a point set, I will denote the collection of real valued functions on $\mathcal{B}$ by $\mathbb{R}^{\mathcal{B}}$. Similarly, the initially important functions with codomain $\mathcal{B}$ have domain an interval in $\mathbb{R}$. Generally, I will denote an open interval $(a, b) \subset \mathbb{R}$ with $-\infty \le a < b \le \infty$ by I. Thus, two particular function families of interest are $\mathcal{B}^I$ and $\mathcal{B}^{\overline{I}}$ where $\overline{I} = [a, b]$ when $a, b \in \mathbb{R}$. In principle, the functions $\mathcal{B}^{\overline{I}}$ could include situations in which $a = -\infty$ and/or $b = \infty$, but I lay down now the convention that $\mathcal{B}^{\overline{I}}$ will only denote the functions defined on $\overline{I} = [a, b]$ when $a, b \in \mathbb{R}$.

Often discussions of such functions take place in the context of a fixed interval, very often $I = (0, 1)$ or $\overline{I} = [0, 1]$. We will attempt a somewhat more general approach in which, typically, $\mathcal{I}$ denotes the collection of all intervals under consideration. For example, we may specify

$$\mathcal{I} = \{I = (a, b) : a, b \in \mathbb{R},\ a < b\},$$

or

$$\mathcal{I} = \{I = (a, b) : -\infty \le a < b \le \infty\},$$

or

$$\mathcal{I} = \{I = [a, b] : a, b \in \mathbb{R},\ a < b\}.$$

Thus, the symbol $\mathcal{I}$ will generally denote some collection of intervals, but the particular collection under consideration should be specified in the particular context in which that collection is used.

Roughly speaking the sets of functions $\mathcal{B}^I$ and $\mathcal{B}^{\overline{I}}$ may be thought of as containing "paths," though usually this terminology implies at least some regularity and hence some structure on $\mathcal{B}$. We come to such structures shortly.

Paths in a topological space

If A is a topological space, then by paths in A, we usually mean **continuous** paths in A which we write as $C^0(I \to A)$ or $C^0(\overline{I} \to A)$. Let us denote by $\mathfrak{P}^0 = \mathfrak{P}^0(A)$ the collection of continuous **paths** in A, that is,

$$\mathfrak{P}^0 = \{\alpha \in C^0(I \to A) : I \in \mathcal{I}\}$$

where $\mathcal{I}$ is a collection of intervals specified in context as suggested above. Similarly, the collection of **embedded** paths is

$$\mathfrak{I}^0 = \{\alpha \in \mathfrak{P}^0 : \alpha(t_1) \neq \alpha(t_2) \text{ for } t_1 < t_2\}.$$

Note: It is standard to denote the continuous real valued functions $f : X \to \mathbb{R}$ with domain a topological space X by $C^0(X)$. Note carefully, that the codomain $\mathbb{R}$ is understood when this notation is used. This notation can be extended to indicate the codomain explicitly $C^1(X \to \mathbb{R})$ and to allow for other codomains $C^0(X \to A)$ where A is another topological space. We have done this above in the case $X = I$ an interval in $\mathbb{R}$ and A a topological space.

The notation $\mathfrak{P}^0 = \mathfrak{P}^0(A)$ for paths may be viewed as a kind of reversal of the notation $C^0(X)$ with an interval $I \subset \mathbb{R}$ taken as the domain. One significant difference is that any interval I in a specified collection $\mathcal{I}$ of intervals is allowed. The subcollection of paths with domain a specific interval I is adequately denoted by $C^0(I \to A)$ as it has been above. A similar comment applies to the subcollection of embedded or **injective paths** $\mathfrak{I}^0 = \mathfrak{I}^0(A)$.

You might be careful about the following: The terms **embedded** and **embedding** (or for some authors imbedding) are used to mean something somewhat different in a different context. In that different context differentiability is central to the discussion and, in fact, used to define the terms. We will get to that context later. The use is not entirely unrelated to my use of embeded here, but my embedded paths should more properly be called topologically embedded paths. In both contexts the terms embedded and embedding may be taken as complementary to the terms **immersed** and **immersion**. For us, the paths in $\mathfrak{P}^0 \backslash \mathfrak{I}^0$ might be called immersed (or strictly immersed) meaning there are some t_1 and t_2 with $t_1 \neq t_2$ but $\alpha(t_1) = \alpha(t_2)$. If you think about examples for a little while, you realize there exist (topologically, strictly) immersed paths in this context which are quite a mess. Nevertheless, the immersed paths (even very messy ones) are natural to consider when it comes to some topological concepts like connectedness, path connectedness, the fundamental group, and things like that. In fact, the formulation of one of the main problems I want to talk about later (the famous Poincaré conjecture) has as its main hypothesis an assumption about the fundamental group. On the other hand, it will be convenient for various other technical reasons below to have the (topologically) embedded paths $\mathfrak{I}^0$ (or the injective paths) available even with more regularity.

Paths in $B_1(\mathbf{0})$ and C^1 paths

Consider the special cases of $B_1(\mathbf{0})^I$ and $B_1(\mathbf{0})^{\overline{I}}$. Taking the latter case with

$$\mathcal{I} = \{I = [a, b] : a, b \in \mathbb{R},\ a < b\}$$

and the continuous injective paths

$$\mathfrak{I}^0(B_1(\mathbf{0})) = \{\alpha \in \mathfrak{P}^0 : \alpha(t_1) \neq \alpha(t_2) \text{ for } t_1 < t_2\}.$$

In order to assert $\alpha \in \mathfrak{P}^0(B_1(\mathbf{0}))$ is continuous, we need to use the topologies (in some way shape or form) from the domain $I = [a, b]$ of α and the codomain $B_1(\mathbf{0}) \subset \mathbb{R}^2$. In both cases a/the topology may be recognized in several different ways. One of those ways, and perhaps the usual way, is via a **basis** for the topology.

Definition 4. (basis of a topology) Given a topology $\mathfrak{T}$ on a set X, a **basis** for the topology is a set $\mathfrak{U} \subset \mathfrak{T}$ with the following properties:

(i)

$$\bigcup_{U \in \mathfrak{U}} U = X.$$

(ii) Given $U_1, U_2 \in \mathfrak{U}$ and some $x \in U_1 \cap U_2$, there exists some $U_3 \in \mathfrak{U}$ for which

$$x \in U_3 \subset U_1 \cap U_2.$$

An important basis for the topology on $\mathbb{R}$ consists of open intervals (a, b) with $a, b \in \mathbb{R}$ and $a < b$. That is,

$$\mathfrak{U}_1 = \{I = (a, b) : a, b \in \mathbb{R} \text{ and } a < b\}.$$

3.3.2 The identity of $\mathcal{B}$

You can perhaps think of what I am going to present below as happening in two steps. The first step is relatively easy, and the second step may be viewed as very difficult.

STEP 0 Easy distinctions and constructions

1. point sets in $\mathcal{B}$ and point sets in $B_1(\mathbf{0})$

2. functions on $\mathcal{B}$ and functions on $B_1(\mathbf{0})$
3. functions into $\mathcal{B}$ and functions into $B_1(\mathbf{0})$
4. the topology on $\mathcal{B}$ ($C^0(\mathcal{B})$ and $\mathfrak{P}^0(\mathcal{B})$)

STEP 1 Difficult distinctions and constructions

1. linear spaces associated with points in $\mathcal{B}$
2. vector spaces associated with points in $\mathcal{B}$
3. differentiation of functions on $\mathcal{B}$ ($C^1(\mathcal{B})$)
4. differentiation of functions into $\mathcal{B}$ ($\mathfrak{P}^1(\mathcal{B})$)
5. length in $\mathcal{B}$
6. shortest paths in $\mathcal{B}$ (geodesics)
7. distance in $\mathcal{B}$

Charts and coordinates

Recall $\mathcal{B}$ is a kind of copy of $B_1(\mathbf{0})$ in which all the rich (Euclidean) structure of $B_1(\mathbf{0})$ has been erased. We may take a function $\mathbf{p} : B_1(\mathbf{0}) \to \mathcal{B}$ to first represent this erasure of structure. Technically $\mathbf{p}$ is an identity function on the point sets, but psychologically it is actually doing something: It has done something (erased structure) and it is going to be used to introduce new structure(s). As I attempted to point out above, it is something of a technicality that $\mathbf{p}$ happens to be the identity on sets. If you want something without that psychological and technical artifact, you can use the function $\mathbf{p} : B_1(\mathbf{0}) \to \mathcal{C}$ considered in section 3.2.

In any case, there are **three important functions** to consider when illuminating the structure on a manifold. Those are chart functions, coordinate functions, and changes of coordinates. We will discuss the first two now and changes of coordinates shortly. Thus, we start with a main set $M = \mathcal{B}$ which is (or is going to be) our manifold and a secondary set called a chart $U \subset \mathbb{R}^n$. In our case, $U = B_1(\mathbf{0}) \subset \mathbb{R}^2$. A **chart function** is a bijection from a chart to the manifold. In our case, we have

$$\mathbf{p} : B_1(\mathbf{0}) \to \mathcal{B}$$

as a **global chart function** by which we mean simply that the image of $\mathbf{p}$ is the entire manifold. The inverse of a chart function is a **coordinate**

function $\xi : M \to \mathbb{R}^n$. For us we have $\xi = (\xi^1, \xi^2) : \mathcal{B} \to B_1(\mathbf{0})$, a function with two real valued "coordinate functions" $\xi^1, \xi^2 \in \mathbb{R}^{\mathcal{B}}$, though we refer to the entire inverse $\xi = \mathbf{p}^{-1}$ as the (global) coordinate function. Again, the fact that ξ is global refers to the fact that the domain of ξ is the entire manifold $\mathcal{B}$. This is really simple. (I said this part was going to be easy.) But it's really important, so I've drawn a picture and put in in Figure 3.3.

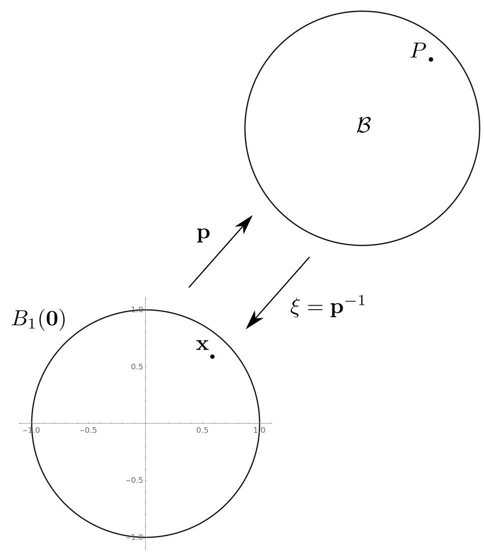

Figure 3.3: The Riemann surface $\mathcal{B}$ exerting its identity with a chart function $\mathbf{p}$ and a coordinate function $\xi = \mathbf{p}^{-1}$

By means of chart functions and coordinate functions a distinction may be made between functions, for example real valued functions on $\mathcal{B}$ and real valued functions on $B_1(\mathbf{0})$. Given a function $f : \mathcal{B} \to \mathbb{R}$, there is an associated function $f \circ \mathbf{p} : B_1(\mathbf{0}) \to \mathbb{R}$. And similarly, given a function $g : B_1(\mathbf{0}) \to \mathbb{R}$, there is a function $g \circ \xi : \mathcal{B} \to \mathbb{R}$. In this case, one can make sense of, and talk about, continuity and partial derivatives of the functions g and $f \circ \mathbf{p}$, but at the moment there is not enough structure to consider such concepts for the functions f and $g \circ \xi$.

Similarly, given a path $\beta \in \mathfrak{P}^0(B_1(\mathbf{0}))$, we can consider a function $\mathbf{p} \circ \beta : I \to \mathcal{B}$. Given a function $\alpha : I \to \mathcal{B}$ defined on an interval $I = [a, b] \subset \mathbb{R}$, there is a function $\xi \circ \alpha : I \to B_1(\mathbf{0})$. We can consider the question of whether or not the function $\xi \circ \alpha$ is a continuous path or a differentiable path. We can consider whether or not the path β is differentiable. We can not quite make sense of similar questions for the functions $\mathbf{p} \circ \beta$ and α.

As I mentioned, continuity is easy.

3.3.3 The topology on $\mathcal{B}$

In our example with $\mathcal{B} = \mathbf{p}(B_1(\mathbf{0}))$ it is easy to identify a topology on $\mathcal{B}$. We take the topology induced by the bijection $\mathbf{p} : B_1(\mathbf{0}) \to \mathcal{B}$. That is,

$$\mathfrak{T}_{\mathcal{B}} = \{\mathbf{p}(U) : U \text{ is open in } B_1(\mathbf{0})\}.$$

Exercise 3.21. Verify that $\mathfrak{T}_{\mathcal{B}}$ is a topology on $\mathcal{B}$.

I might point out that it is unusual to introduce the topology on a Riemannian manifold in this way, though not entirely unheard of. The usual (and sometimes simpler) approach is to assume apriori that the manifold M being defined is some kind of topological space. Then the chart functions are taken to be homeomorphisms with respect to the topology on M. We took this approach in our consideration of Poincaré manifolds and topological manifolds in Chapter 7 below which introduces Poincaré conjecture. In situations where there does not exist a global chart, the approach with an apriori topology makes things somewhat simpler. In the case under consideration, I have restricted attention to manifolds (and one manifold in particular) which admits a global chart. In that case using a topology induced by a bijection (global chart function) like $\mathfrak{T}_{\mathcal{B}}$ works just fine.

This allows us to consider continuous real valued functions on $\mathcal{B}$ and continuous paths in $\mathcal{B}$ directly. In particular, the notations $C^0(\mathcal{B})$ and $\mathfrak{P}^0(\mathcal{B})$ make good sense as well as the continuous embedded paths $\mathfrak{I}^0(\mathcal{B})$. In addition, the introduction of a topology on $\mathcal{B}$ allows us to apply topological concepts directly to $\mathcal{B}$. In particular, we know what it means to say $\mathcal{B}$ is connected (which it is) and compact (which it isn't). This is part of an answer to a question addressed in section C.7 of Appendix C which was asked when some additional structure(s) had been introduced.

In the ways mentioned above the introduction of a topology on $\mathcal{B}$ allows the manifold $\mathcal{B}$ to stand on its own or assert its own identity, though one must admit this is a rather timid start for $\mathcal{B}$. (The topology $\mathfrak{T}_{\mathcal{B}}$ is essentially identical to that of $B_1(\mathbf{0})$.) One has to start somewhere however.

Furthermore, I think we have pretty much completed STEP 0, and we are ready to move on to the difficult STEP 1.

More rambling

I have attempted to focus first on what is probably the most important structure making $\mathcal{B}$ different from $B_1(\mathbf{0})$. As a consequence I have introduced the assignment of a 2×2 matrix

$$\begin{pmatrix} \frac{16}{(4+|\mathbf{x}|^2)^2} & 0 \\ 0 & \frac{16}{(4+|\mathbf{x}|^2)^2} \end{pmatrix} \tag{3.9}$$

to each point $\mathbf{x} \in B_1(\mathbf{0})$. As we have seen, this has directed attention away from $\mathcal{B}$ and the points $P \in \mathcal{B}$ in particular. Nevertheless, the geometric quantities

1. The **Riemannian length** of a path,
2. The **Riemannian angle** between two paths, and
3. The **Riemannian area** enclosed by a path

are distinct from the usual quantities associated with the corresponding point sets in $B_1(\mathbf{0})$. While these "new quantities" may be formally interpreted as some kind of "geometric structure" on $\mathcal{B}$ distinct from the Euclidean structure on $B_1(\mathbf{0})$, because the matrices are assigned to points $\mathbf{x} \in B_1(\mathbf{0})$ rather than to $P \in \mathcal{B}$ it looks suspiciously like the structure is (1) not really on $\mathcal{B}$ and (2) not really constituting any actual "structure" on the point set $\mathcal{B}$ at all. If you are thinking this way, you may be comforted to know you are correct, and this turns out to be a pretty big problem. On the other hand, it is a problem I will attempt to address in detail below. If you like (and you want to be especially careful) you may refer to $\mathcal{B}$ as a "pre-Riemannian manifold" or something like that and perhaps think in the following informal way: There are "geometric objects" in $\mathcal{B}$, namely paths with measurable lengths, pairs of paths meeting at measurable angles, and regions with measurable areas.[11] As point sets these "geometric objects" correspond identically to familiar geometric objects in $B_1(\mathbf{0})$, but the measurments associated with these objects in $\mathcal{B}$ are different. In order to actually calculate the measurments, specific parameterizations are used and techniques from calculus. This is where the difficulty arises because reference back to $B_1(\mathbf{0})$ is, at least in this preliminary formulation, required.

[11] One might imagine that Euclid could consider these objects without any reference to Cartesian coordinates, parameterizations, or much of anything else.

Though you may have hopefully been able to complete the exercises above and start to get a feel for how Riemannian manifolds work, you may have been left with the feeling that the form of the matrix assignment (g_{ij}) is not so well motivated and the same applies to the definitions of Riemannian length, angle, and area. These are all very good things to be worried about, and we should give serious attention to providing some motivation for all of these things. For the moment we set aside what should be some good questions you are starting to formulate based on these observations about motivation and turn to other directions of inquiry peculiar to Riemannian manifolds as well as some more examples.

3.4 Linear structure(s) on $\mathcal{B}$

$\mathbb{R}^n$ is a **vector space** by which we mean a linear space, i.e., a commutative group under addition with a scaling $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ for which

$$\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v} \qquad \text{for } \alpha, \beta \in \mathbb{R} \text{ and } \mathbf{v} \in \mathbb{R}^n,$$

$$\begin{aligned} (\alpha+\beta)\mathbf{v} &= \alpha\mathbf{v} + \beta\mathbf{v} && \text{for } \alpha, \beta \in \mathbb{R} \text{ and } \mathbf{v} \in \mathbb{R}^n, \\ \alpha(\mathbf{v}+\mathbf{w}) &= \alpha\mathbf{v} + \beta\mathbf{w} && \text{for } \alpha \in \mathbb{R} \text{ and } \mathbf{v}, \mathbf{w} \in \mathbb{R}^n, \\ 1\mathbf{v} &= \mathbf{v} && \text{for } \mathbf{v} \in \mathbb{R}^n, \end{aligned}$$

and

$$|\mathbf{v}| = \sqrt{\sum_{j=1}^{n} v_j^2}$$

where $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ defines a norm. That is, $\mathbb{R}^n$ is a normed linear space, i.e., a vector space. When we think of $\mathbb{R}^n$ as a Riemannian manifold, with the identity as a global chart function, we associate with each point $\mathbf{x} \in \mathbb{R}^n$ a copy of $\mathbb{R}^n$ of displacements, or tangent vectors from $\mathbf{x}$. This is called the tangent space at $\mathbf{x}$ and is denoted by $T_{\mathbf{x}}\mathbb{R}^n$. It happens in this case that there is a natural association of a vector $\mathbf{v} \in T_{\mathbf{x}}\mathbb{R}^n$ with another point in the manifold $\mathbb{R}^n$ by addtion, namely

$$\mathbf{x} + \mathbf{v}.$$

More generally, given a vector $\mathbf{v} \in T_{\mathbf{x}}\mathbb{R}^n$ and some $t > 0$, we can find another vector

$$\mathbf{x} + t\mathbf{v} \in \mathbb{R}^n.$$

We wish to extract certain aspects of this construction for application in a general Riemannian manifold M. Let me briefly repeat this discussion of the manifold $M = \mathbb{R}^n$ with some additional detail and clearly indicating the roles of the two distinct copies of $\mathbb{R}^n$. One copy is the manifold itself $M = \mathbb{R}^n$ containing the point $\mathbf{x}$ and other points like $\mathbf{x}$. The other copy $T_{\mathbf{x}}\mathbb{R}^n = \mathbb{R}^n$ is the vector space associated with a particular point $\mathbf{x}$, and $T_{\mathbf{x}}\mathbb{R}^n$ contains vectors $\mathbf{v}$ and other vectors like $\mathbf{v}$. For each $\mathbf{v} \in T_{\mathbf{x}}\mathbb{R}^n \backslash \{\mathbf{0}\}$, there exists a unique geodesic ray, which in this case is a straight line. Every point $\mathbf{x} + t\mathbf{v} \in M = \mathbb{R}^n$ with $t > 0$ lies on this geodesic ray.

In a general Riemannian manifold M, some aspects of this description may not be possible, but many aspects of the description can be recreated. We will now begin the task of recreating the aspects which can be obtained for the Riemannian manifold $\mathcal{B}$.

We first associate a **linear space** (without a norm or means to measure magnitudes) with each point $P \in \mathcal{B}$. Let us call this linear space $\mathcal{L}_P\mathcal{B}$. This space is constructed as follows: For each function $\alpha : [a, b] \to \mathcal{B}$ with $a, b \in \mathbb{R}$ and $a < b$ for which

(i) $\alpha(t_0) = P$ for some $t_0 \in (a, b)$ and

(ii) $\xi \circ \alpha \in C^1[a, b]$

we say $\alpha \in c\mathfrak{P}^1(\mathcal{B})$, that is α is a **chart C^1 path in $\mathcal{B}$**, with $\alpha(t_0) = P$. If $\alpha \in c\mathfrak{P}^1(\mathcal{B})$ satisfies $\alpha(t_1) \neq \alpha(t_2)$ for $a \le t_1 < t_2 \le b$, then we write $\alpha \in c\mathfrak{I}^1(\mathcal{B})$. That is,

$$c\mathfrak{I}^1(\mathcal{B}) = \{\alpha \in c\mathfrak{P}^1(\mathcal{B}) : \alpha(t_1) \neq \alpha(t_2),\ t_1 < t_2\}.$$

Now consider the collection of embedded paths

$$A_P = \{\alpha \in c\mathfrak{I}^1(\mathcal{B}) : \alpha(t_0) = P\}. \tag{3.10}$$

Exercise 3.22. Show there exists a path in the set A_p defined in (3.10). That is, show this collection of paths is nonempty.

On the set A_P we define an equivalence relation as follows: Two paths $\alpha_1, \alpha_2 \in A_P$ are equivalent, and we write $\alpha_1 \sim \alpha_2$ if

$$(\xi \circ \alpha_1)'(t_1) = (\xi \circ \alpha_2)'(t_2) \tag{3.11}$$

where $\alpha_j(t_j) = P$ for $j = 1, 2$.

Exercise 3.23. Show the relation on A_P defined by (3.11) is an equivalence relation:

(a) $\alpha \sim \alpha$ for every $\alpha \in A_P$.

(b) $\alpha_1 \sim \alpha_2$ implies $\alpha_2 \sim \alpha_1$.

(c) If $\alpha_1 \sim \alpha_2$ and $\alpha_2 \sim \alpha_3$, then $\alpha_1 \sim \alpha_3$.

Recall that whenever one has an equivalence relation on a set, then the set is naturally partitioned into equivalence classes. Specifically, if $\alpha_1, \alpha_2 \in A_P$, then there are precisely two possibilites for the sets

$$[\alpha_1] = \{\alpha \in A_P : \alpha \sim \alpha_1\} \qquad \text{and} \qquad [\alpha_2] = \{\alpha \in A_P : \alpha \sim \alpha_2\}, \tag{3.12}$$

namely either

$$[\alpha_1] \cap [\alpha_2] = \phi \qquad \text{or} \qquad [\alpha_1] = [\alpha_2]. \tag{3.13}$$

Exercise 3.24. Show the property (3.13) for the equivalence classes holds when (3.12) holds.

We designate the collection of equivalence classes

$$\mathcal{L}_P\mathcal{B} = \{[\alpha] : \alpha \in A_P\}.$$

As a set, this will be the linear space assigned to the point $P \in \mathcal{B}$. In order to have a linear structure on $\mathcal{L}_P\mathcal{B}$, we need operations of addition and scaling:

$$[\alpha_1] + [\alpha_2] = [\alpha_3] \tag{3.14}$$

where $(\xi \circ \alpha_3)'(t_3) = (\xi \circ \alpha_1)'(t_1) + (\xi \circ \alpha_2)'(t_2)$ and, as may be expected, $\alpha(t_j) = P$ for $j = 1, 2, 3$. Given $c \in \mathbb{R}$,

$$c[\alpha_1] = [\alpha_2] \tag{3.15}$$

where $(\xi \circ \alpha_2)'(t_2) = c(\xi \circ \alpha_1)'(t_1)$ and $\alpha_j(t_j) = P$ for $j = 1, 2$.

Exercise 3.25. Show the definitions of addition and scaling associated with (3.14) and (3.15) respectively are well-defined and make $\mathcal{L}_P\mathcal{B}$ a linear space.

The global chart function $\mathbf{p} : B_1(\mathbf{0}) \to \mathcal{B}$ and the global coordinate function $\xi = \mathbf{p}^{-1} : \mathcal{B} \to B_1(\mathbf{0})$ in particular played a prominent role in the introduction of the linear structure constituted by the assignment

$$P \mapsto \mathcal{L}_P\mathcal{B}.$$

The paths $\alpha \in c\mathfrak{P}^1(\mathcal{B})$ themselves, leading to the identification of the sets A_P and finally $\mathcal{L}_P\mathcal{B}$, were differentiated from other paths using the notion of regularity called "chart C^1" which seemingly relies on the chart function. Furthermore, each following step, from defining the equivalence classes to defining the operations, relied directly on the use of the global chart.

On the other hand, like the open sets in the topology on $\mathcal{B}$, the paths themselves can be considered as objects having an identity only with respect to the manifold $\mathcal{B}$, and the same can be said concerning the sets A_p, $\mathcal{L}_P\mathcal{B}$, and the resulting sums $[\alpha_1] + [\alpha_2]$ and scalings $c[\alpha]$. In order to better appreciate the extent to which $\mathcal{B}$ exerts its own identity in regard to the linear structure constituted by the assignment

$$P \mapsto \mathcal{L}_P\mathcal{B},$$

we consider in the next sections the proposition that while the particular global chart function $\mathbf{p} : B_1(\mathbf{0}) \to \mathcal{B}$ was used to define the linear structure, this particular chart function was not the only possibility. In particular while the use of some chart function or functions is generally required to define a linear structure, the particular chart functions used are, in a certain sense, peripheral to the structure created.

3.5 More than one chart—an atlas

3.6 Changing coordinates and charts

Once it is realized the "geometry" (Riemannian legths, angles, and areas for example but other geometric quantities as well) are not visible within the particular set $\mathcal{B}$, then it is perhaps natural to discard the visible image of $\mathcal{B}$ altogether. Let us consider for example the set $U = B_{1/2}(\mathbf{0}) \subset \mathbb{R}^2$ and the function $\mathbf{q} : U \to \mathcal{B}$ by

$$\mathbf{q}(\mathbf{x}) = 2\mathbf{x}.$$