Chapter 17

Vector Fields

One notion of a filament/vector field on a (Riemannian) manifold M is simply an assignment

$$v : M \to \bigcup_{Q \in M} \mathcal{L}_Q M$$

for which $v(P) = v_P \in \mathcal{L}_P M = T_P M$. There is not much calculus one can hope to do with a vector field in this generality, and we are almost always interested in vector fields that are at least chart C^1. The notion of chart C^k for $k = 0, 1, 2, \ldots, \infty$ is relatively easy to define for filament fields as follows: Given any chart $(U, \mathbf{p}) \in \mathcal{A}_*^\infty$, there is a traditional vector field $\mathbf{v} : U \to \mathbb{R}^n$ given by $\mathbf{v}(\mathbf{x}) = d\xi_{\mathbf{p}(\mathbf{x})}(v_{\mathbf{p}(\mathbf{x})})$. The vector field $\mathbf{v}$ has coordinate functions $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ with each $v_j \in \mathbb{R}^U$ for $j = 1, 2, \ldots, n$. We say v is **chart** C^k if $v_j \in C^k(U)$ for each $j = 1, 2, \ldots, n$ (and each chart $(U, \mathbf{p}) \in \mathcal{A}_*^\infty$).

17.1 Traditional vector fields

17.1.1 Initial source of intuition

The total derivative of a traditional vector field $\mathbf{v} : U \to \mathbb{R}^n$ is given by a matrix (valued function)

$$D\mathbf{v} = \left(\frac{\partial v_i}{\partial x_j} \right) : U \to \mathbb{R}^{n \times n}$$

with rows $Dv_1, Dv_2, \ldots, Dv_n$ and columns

$$\frac{\partial \mathbf{v}}{\partial x_1}, \ \frac{\partial \mathbf{v}}{\partial x_2}, \ \ldots, \ \frac{\partial \mathbf{v}}{\partial x_n}$$

or technically

$$\left(\frac{\partial \mathbf{v}}{\partial x_1}\right)^T, \left(\frac{\partial \mathbf{v}}{\partial x_2}\right)^T, \ldots, \left(\frac{\partial \mathbf{v}}{\partial x_n}\right)^T.$$

We denote the chart C^k vector fields on M by $\mathcal{V}^k(M)$ and in the special case $k = \infty$, we write $\mathcal{V}^\infty(M) = \mathscr{X}(M)$. Each set $\mathcal{V}^k(M)$ is a module over the ring $cC^k(M)$.

17.1.2 The basic question

Aside from gaining some intuition for the basic calculus associated with traditional vector fields defined on open subsets of $\mathbb{R}^n$, the basic objective of this chapter is to understand/define a notion of **intrinsic derivative of a filament vector field**

$$v \in cC^1\left(M \to \bigcup_{Q \in M} \mathcal{L}_Q M\right).$$

This proves to be somewhat inconvenient in general since a loss of regularity is to be expected. Notice that for a traditional vector field $\mathbf{v} \in C^k(U \to \mathbb{R}^n)$, the total derivative satisfies $D\mathbf{v} \in C^{k-1}(U \to \mathbb{R}^{n \times n})$. This inconvenience goes away if we restrict to $\mathscr{X}(M)$, which at least for the moment is what I am going to do. Thus, I emphasize the basic and somewhat difficult question:

> What is the intrinsic derivative of $v \in \mathscr{X}(M)$?

This question parallels, in a certain sense, the question about the derivative of a real valued function $f : M \to \mathbb{R}$ considered in the last chapter, though that question was rather easier. Here too, a very natural place to start is groping for some kind of directional derivative

$$(D_u v)(P)$$

where $u \in \mathbb{S}_P^{n-1} = \{w \in T_P M : \mu_P(w, w) = 1\} \subset \mathcal{L}_P M = T_P M$.

There are a couple "obvious" sources of intuition and inspiration for such groping:

1. flat Euclidean space, and

2. submanifolds of Euclidean space (especially hypersurfaces).

Along with these special kinds of manifolds there is a particular class of vector fields which it is worth isolating for special attention:

Definition 24. We say a filament field $v_j \in \mathscr{X}(M)$ is **locally induced by $\mathbf{e}_j$ in the chart** $(U, \mathbf{p})$ **at** $P = \mathbf{p}(\mathbf{x})$ if there is some open set $V \subset\subset U$ for which

$$d\xi_Q((v_j)_Q) = \mathbf{e}_j = (\mathbf{e}_j)_{\xi(Q)} \qquad \text{for} \qquad Q \in \mathbf{p}(V).$$

Put another way, $v_j = d\mathbf{p}(\mathbf{e}_j) = [\mathbf{p} \circ \gamma_j]$ where $\gamma_j = \xi(Q) + t\mathbf{e}_j$, as discussed in the previous chapter.

17.1.3 Language: notation and terminology

A filament field having the property described in Definition 24 (or something rather like it) is usually said to mean v_j is a *coordinate field.* I do not really like this terminology because the field is up in the manifold M rather than down where the coordinates are, namely in the chart U. There aren't particularly meaningful coordinates "up in M," though one can see something about what is happening up in M by examining what is going on with coordinates "down in U." The field v_j in Definition 24 does, however, have something to do with the coordinates in U and the coordinate field $\mathbf{e}_j : U \to \mathbb{R}^n$ in particular. Let me suggest some possible (nonstandard) terminology. I have referred to "traditional" vector fields above and also I designated the filament field v_j in Definition 24 as "induced." Let me start by saying the traditional vector fields $\mathbf{e}_j$ for $j = 1, 2, \ldots, n$ on any open subset of $\mathbb{R}^n$ are **coordinate vector fields**. This can be the case with or without a specification of (nonstandard) metric tensor, but these coordinate vector fields will usually be considered at least implicitly with the standard dot product tensor $\langle \cdot , \cdot \rangle_{\mathbb{R}^n}$ in the background. I'm willing to say the filament field v_j satisfying the conditions of Definition 24 is **induced on M by the coordinate field $\mathbf{e}_j$ in the chart** U. For something shorter and adjectival rather than adverbial, I might suggest v_j is a/the **coordinate induced (filament) field**.

It is worth pointing out, with respect to the special cases which become important below, that the coordinate field $\mathbf{e}_j$ on U can also induce a traditional field when either

1. M is actually flat Euclidean space, or

2. M is a submanifold of some flat Euclidean space.

The latter special case would also include the situation in which the "ambient" Euclidean space has the same dimension as M, that is M is an open subset of $\mathbb{R}^n$ specifically with a nonstandard metric.

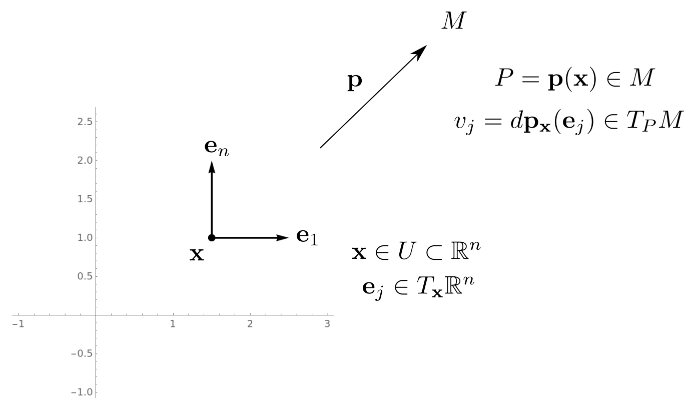

Figure 17.1: Filament field(s) v_j induced on a manifold from the coordiniate field $\mathbf{e}_j$ in a chart.

More generally, the term **induced** or **chart induced** may be used to deignate entities that have an intrinsic identity in a manifold M by somehow derive from entities in a chart. Thus, the **translator differential** $d\mathbf{p}_{\mathbf{x}} : T_{\mathbf{x}}U \to T_{\mathbf{p}(\mathbf{x})}M$ taking traditional vectors $\mathbf{w} \in T_{\mathbf{x}}U$ to filaments $w = d\mathbf{p}_{\mathbf{x}}(\mathbf{w})$ may be said to induce filaments or give induces filaments on M.

Somewhat more informally, we can say a quantity with an identity on M when expressed using elements from a chart is a **chart expression**. For example, given a function $f : U \to \mathbb{R}$ defined on a chart, we can say the chart expression for f is $f \circ \xi : \mathbf{p}(U) \to \mathbb{R}$. Simiarly, a path $\alpha \in c\mathfrak{P}(U)$ has chart expression $\mathbf{p} \circ \alpha : I \to M$.

I will try to a certain extent to reserve the term **coordinate**, e.g., coordinate field, coordinate expression, for quantities expressed in coordinates in a chart. The coordinate expression for a function $f : M \to \mathbb{R}$ is $f \circ \mathbf{p} : U \to \mathbb{R}$. The coordinate expression for a filament field

$$w : M \to \bigcup_{Q \in M} T_Q M$$

is a (traditional) vector field $\mathbf{w} = d\xi(w)$ where $\mathbf{w}_\mathbf{x} = d\xi_{\mathbf{p}(\mathbf{x})}(w_{\mathbf{p}(\mathbf{x})})$. Thus

$$\mathbf{w} : U \to \bigcup_{\mathbf{x}\in U} T_\mathbf{x}U = \mathbb{R}^n = \bigcup_{\mathbf{x}\in U} \mathbb{R}^n_\mathbf{x}.$$

(Sometimes the tangent space to $\mathbb{R}^n$ at a point $\mathbf{x} \in \mathbb{R}^n$ is expressed $\mathbb{R}^n_\mathbf{x}$ instead of $T_\mathbf{x}\mathbb{R}^n$.)

Finally, in view of the introduction of the semi-standard notation v_j for coordinate induced filament fields on a manifold (at a point $P \in M$), I will make the somewhat unfortunate accomodation of trying to avoid using the symbols v and $\mathbf{v}$ for general filament fields and general (traditional) vector fields respectively from now on. (Maybe there is a better notation for the coordinate induced filament fields $v_1, v_2, \ldots, v_n$.)

17.1.4 Derivatives of vector fields

The directional derivative $D_\mathbf{u}\mathbf{w}$ of a (traditional) vector field $\mathbf{w} : U \to \mathbb{R}^n$ with respect to a unit field $\mathbf{u} : U \to \mathbb{S}^{n-1}$ is, first of all, another vector field. In particular, at each point $\mathbf{x} \in U \subset \mathbb{R}^n$, the value

$$D_\mathbf{u}\mathbf{w}(\mathbf{x}) = D_{\mathbf{u}_\mathbf{x}}\mathbf{w}(\mathbf{x}) \in T_\mathbf{x}\mathbb{R}^n = \mathbb{R}^n,$$

and this value should depend only on $\mathbf{u}_\mathbf{x}$ instead of the entire unit field $\mathbf{u}$. In fact, if we recall the total derivative (matrix) $D\mathbf{w}$, then we can write

$$D_\mathbf{u}\mathbf{w}(\mathbf{x}) = D\mathbf{w}(\mathbf{x})\ \mathbf{u}_\mathbf{x} = D\mathbf{w}(\mathbf{x})\ (\mathbf{u}_\mathbf{x})^T$$

(where we often suppress the transpose as in the middle expression). This suggests that the generalized notation

$$D_\mathbf{z}\mathbf{w} = D\mathbf{w}(\mathbf{x})\ \mathbf{z}^T$$

for $\mathbf{z} \in \mathbb{R}^n$, and not just for $\mathbf{z} = \mathbf{u}$ a unit vector, makes sense in this case leading to a linear function of $\mathbf{z}$. As in the case of directional derivatives of a real valued function, if we are going to use this notation we should be careful to realize the value $D_\mathbf{z}\mathbf{w}$ is not properly the (intrinsic) directional derivative of the vector field $\mathbf{w}$ but rather a scaling

$$D_\mathbf{z}\mathbf{w} = |\mathbf{z}|\ D_\mathbf{u}\mathbf{w} \qquad \left(\mathbf{u} = \frac{\mathbf{z}}{|\mathbf{z}|}\right)$$

determined by the (Euclidean) norm of $\mathbf{z}$. If we wish to avoid this distinction, we can reserve the directional derivative notation $D_{\mathbf{u}}\mathbf{w}$ for situations in which $\mathbf{u} \in \mathbb{S}^{n-1}$ and denote the value $D\mathbf{w}(\mathbf{x})\ \mathbf{z}_{\mathbf{x}}$ as the (traditional) **differential of the vector field**:

$$d\mathbf{w}_{\mathbf{x}} = (d\mathbf{w})_{\mathbf{x}} : \mathbb{R}^n \to \mathbb{R}^n \qquad \text{by} \qquad d\mathbf{w}_{\mathbf{x}}(\mathbf{z}) = D\mathbf{w}(\mathbf{x})\ \mathbf{z}^T.$$

Either way, there is a nice linear function $d\mathbf{w}_{\mathbf{x}} : T_{\mathbf{x}}\mathbb{R}^n \to T_{\mathbf{x}}\mathbb{R}^n$ associated with the directional differentiation of a (traditional) vector field.

Exercise 17.1. Noting the parentheses $(d\mathbf{w})_{\mathbf{x}}$ inserted above, what can you say about $D(\mathbf{w}_{\mathbf{x}})$ and $d(\mathbf{w}_{\mathbf{x}})$? How about $D_{\mathbf{u}}(\mathbf{w}_{\mathbf{x}})$?

Exercise 17.2. Considering the differential of a vector field

$$\mathbf{w} : U \to \bigcup_{\mathbf{x}\in U} T_{\mathbf{x}}U$$

for U and open subset of $\mathbb{R}^n$ at the point $\mathbf{x}$ given by

$$(d\mathbf{w})_{\mathbf{x}} : T_{\mathbf{x}}U \to T_{\mathbf{x}}U$$

as introduced above, what would be the proper domain and codomain for the **differential field** $d\mathbf{w}$? What about for the **differential operator** d?

While $d\mathbf{w} : \mathscr{X}(U) \to \mathscr{X}(U)$ is linear when $\mathscr{X}(U)$ denotes the vector field of C^∞ traditional vector fields on U (over $\mathbb{R}$), this function also enjoys the additional property that

$$d\mathbf{w}(f\mathbf{z}) = f\,d\mathbf{w}(\mathbf{z}) \qquad \text{for} \qquad f \in C^\infty(U).$$

That is, $d\mathbf{w}$ is "linear" over $\mathscr{X}(U)$ also considered as a module over the ring $C^\infty(U)$. We can also consider a function with two vector field arguments. Let $L : \mathscr{X}(U) \times \mathscr{X}(U) \to \mathscr{X}(U)$ be given by

$$L(\mathbf{w}, \mathbf{z}) = d\mathbf{w}(\mathbf{z}) \qquad \text{where} \qquad L(\mathbf{w}, \mathbf{z})_{\mathbf{x}} = d\mathbf{w}_{\mathbf{x}}(\mathbf{z}_{\mathbf{x}}).$$

This function is bilinear over $\mathscr{X}(U)$ considered as a vector space over $\mathbb{R}$, but it does not enjoy the same scaling with respect to smooth functions when it comes to the $\mathbf{w}$ dependence (in the first argument of L and in the argument of the operator d). Specifically,

$$d(f\mathbf{w})(\mathbf{z}) = D_{\mathbf{z}}(f\mathbf{w}) = D_{\mathbf{z}}f\ \mathbf{w} + f\ D_{\mathbf{z}}\mathbf{w}.$$

This is, of course, a Liebnizian product rule, and it is natural (hopefully) to assume all these properties hold for the derivative(s) of a vector field on a manifold.

17.2 Preliminary calculations/reductions

17.2.1 Properties

Given $u = u_P \in \mathbb{S}_P^{n-1} \subset T_PM$ and $w \in \mathscr{X}(M) = c\mathcal{V}^\infty(M)$, I would like to define the (intrinsic) **directional derivative** $\nabla_u w(P)$, or more generally, if we take $u \in \mathscr{X}(M) = c\mathcal{V}^\infty(M)$ with $\|u_P\|_{T_PM} = 1$, a (chart smooth) **unit field**, I would like to define the vector field

$$\nabla_u w.$$

Hopefully, it's clear from the example of flat Euclidean space $\mathbb{R}^n$ that the directional derivative $\nabla_u w(P)$, which you can also call $D_u w(P)$ if you like, is (or should be) a vector in T_PM. Furthermore, it is natural to expect the following:

DVF1 point differential $\nabla_u w(P)$ taken with $u \in \mathbb{S}_P^{n-1}$ as an argument and $w \in \mathscr{X}(M)$ fixed determines a (nice) linear function $dw_P : T_PM \to T_PM$ with

$$dw_p(u) = \nabla_u w(P) \qquad \text{for} \qquad u \in \mathbb{S}_P^{n-1} \subset T_PM.$$

See Exercise 16.2 above and Exericse 17.4 below.

DVF2 global differential The linear functions

$$\left\{ dw_P \in \bigcup_{Q\in M} \beth(T_QM \to T_QM) : P \in M,\ w \in \mathscr{X}(M) \right\}$$

from **DVF1** determine a function $dw : \mathscr{X}(M) \to \mathscr{X}(M)$ by

$$dw(z)(P) = dw(z)_P = dw_P(z_P)$$

satisfying

(i) dw is linear when $\mathscr{X}(M)$ is considered as a vector field over $\mathbb{R}$ and

(ii) dw is module linear when $\mathscr{X}(M)$ is considered as a module over $cC^\infty(M)$.

Exercise 17.3. Given any $(0,1)$ tensor $\theta \in \mathscr{T}^1(M)$, that is a function $\theta : \mathscr{X}(M) \to \mathbb{R}$ for which

$$\theta(fw) = f\theta(w) \qquad for \qquad f \in cC^\infty(M)$$

also known as a **one form**, and a vector field $w \in \mathscr{X}(M)$, if we have an intrinsic derivative of vector fields satisfying **DVF1** and **DVF2**, then $\theta \circ dw$ is also a $(0,1)$ tensor. We can denote this particular tensor by $dw^*(\theta)$. What does this imply about the domain and co-domain of dw^*? (Think in terms of "dual spaces.")

Note that an important part/assertion of property **DVF2** is expressed in the specification of the codomain of the global differential dw. Specifically, for a smooth vector field $z \in \mathscr{X}(M)$, the vector field $dw(z)$ should be chart smooth.

DVF3 connection The global differential $dw : \mathscr{X}(M) \to \mathscr{X}(M)$ from **DVF2** determines a function $L : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ by

$$L(w,z)(P) = L(w,z)_P = dw(z)_P = dw_P(z_P)$$

which is

(a) bilinear when $\mathscr{X}(M)$ is considered as a linear space over $\mathbb{R}$,

(b) $cC^\infty(M)$ (modular) linear in the second argument

$$\begin{aligned} dw(fz_1 + gz_2) &= L(w, fz_1 + gz_2) \\ &= fL(w,z_1) + gL(w,z_2) \\ &= f\ dw(z_1) + g\ dw(z_2) \\ &(f, g \in cC^\infty(M),\ z_1, z_2 \in \mathscr{X}(M)) \end{aligned}$$

when $\mathscr{X}(M)$ is considered as a module over $cC^\infty(M)$, and

(c) Leibnizian

$$\begin{aligned} L(fw, z) &= d(fw)(z) \\ &= df(z)\ w + fdw(z) \\ &= df(z)\ w + fL(w,z) \\ &(f \in cC^\infty(M),\ z, w \in \mathscr{X}(M)) \end{aligned}$$

in the first argument.

Exercise 17.4. Let X be any real linear space. Consider in this exercise functions $G : T_PM \to X$ and $g : \mathbb{S}_P^{n-1} \to X$.

(i) Show that if G is linear, and

$$g = G\Big|_{\mathbb{S}_P^{n-1}}, \tag{17.1}$$

then $g(-w) = -g(w)$ and

$$g\left(\frac{z+w}{\|z+w\|}\right) = \frac{1}{\|z+w\|}\ \left[\|z\|\ g\left(\frac{z}{\|z\|}\right) + \|w\|\ g\left(\frac{w}{\|w\|}\right)\right] \tag{17.2}$$

for $z, w \in T_PM$ with $z, w, z+w \neq 0$.

(ii) Show that if g satisfies $g(-w) = -g(w)$ and

$$g\left(\frac{z+w}{\|z+w\|}\right) = \frac{1}{\|z+w\|}\ \left[\|z\|\ g\left(\frac{z}{\|z\|}\right) + \|w\|\ g\left(\frac{w}{\|w\|}\right)\right]$$

for $z, w \in T_PM$ with $z, w, z+w \neq 0$, then there exists a linear function $G : T_PM \to X$ for which

$$G\Big|_{\mathbb{S}_P^{n-1}} = g. \tag{17.3}$$

(iii) Show that if g satisfies $g(-w) = -g(w)$ and

$$g\left(\frac{z+w}{\|z+w\|}\right) = \frac{1}{\|z+w\|}\ \left[\|z\|\ g\left(\frac{z}{\|z\|}\right) + \|w\|\ g\left(\frac{w}{\|w\|}\right)\right]$$

for $z, w \in T_PM$ with $z, w, z+w \neq 0$, then the linear function $G : T_PM \to X$ obtained in part **(ii)** is uniquely determined by these conditions.

Exercise 17.5. Assuming **DVF3** applies to the directional derivative of a vector field, show that given a unit vector $u \in \mathbb{S}_P^{n-1} \subset T_PM$ at a point P, there holds

(a) $\nabla_u(aw_1+bw_2) = a\nabla_u w_1 + b\nabla_u w_2$ for $a, b \in \mathbb{R}$ and $w_j \in \mathscr{X}(M)$, $j = 1, 2$.

(b) $\nabla_u(fw) = D_u f\ w + f\ D_u w$ for $f \in cC^\infty(M)$ and $w \in \mathscr{X}(M)$.

Notes on the heuristic conditions DVF1, DVF2, and DVF3:

1. We saw in the previous chapter how the directional derivative $D_u f = \nabla_u f$ with $u \in \mathbb{S}_P^{n-1} \subset T_P M$ and $f \in cC^\infty(M)$ determines the differential map $df_P : T_P M \to T_P M$. The construction summarized by point differential property **DVF1** for the differential map $dw_P : T_P M \to T_P M$ associated with the (intrinsic) derivative of a vector field is more complicated. This construction should be given careful attention.

   Exercise 17.6. In my introduction to this section I wrote "...if we take $u \in \mathscr{X}(M)$ a unit field...". Unit vector fields like this do not exist on all manifolds. Show that if there is a global chart $(U, \mathbf{p}) \in \mathcal{A}_*^\infty$ for the manifold M, then there exists a unit field on M.

   Remember that for a unit vector $u \in T_P M$, there is a pretty nice way to understand the intrinsic derivative of a function $f \in cC^\infty(M)$ as the rate of change of f in the direction of $u = [\alpha]$, namely

$$D_u f = \lim_{t \searrow t_0} \frac{f \circ \alpha(t) - f(P)}{\text{length}_M \left[\alpha\Big|_{[t_0,t]} \right]}. \tag{17.4}$$

   I do not know a similar way to understand the intrinsic derivative of a vector field on M. One thing that cannot work (as far as I know) is a naive difference quotient like (17.4). The problem is that the vectors $w_\alpha \in T_\alpha M$ and $w_P \in T_P M$ are not in the same vector spaces. The space M is not flat in general and things may have changed from $\alpha(t)$ to P. In short, the value $w_\alpha - w_P$ you would want in the numerator of a difference quotient is not wel-defined. We use very strongly in (17.4) the fact that the values of f fall in a single well-defined vector space, namely $\mathbb{R}$. Note carefully the codomain of a vector field

$$w : M \to \bigcup_{Q \in M} T_Q M,$$

   which is not a linear space. Sometimes it is convenient to "encode" the restriction $w(P) = w_P \in \mathcal{L}_P M = T_P M$ by taking a cross-product with M

$$M \times \bigcup_{Q \in M} T_Q M$$

and specifying a particular subset of that cross-product in which the "base point" of each vector is referenced or indexed, namely

$$\left\{(P,[\alpha]) \in M \times \bigcup_{Q\in M} T_Q M : [\alpha] \in T_P M\right\}.$$

This subset is usually denoted by TM and is called the **tangent bundle**. This set can be made into a manifold and one can define a map $W : M \to TM$ by

$$W(P) = (P, w_P)$$

using a vector field w. One can get further into bundling if one wishes, but the manifold TM is still not a linear space, and there is no reasonable way to put a linear structure on it in general. Presumably as a result one can still find no reasonable difference quotient definition for the intrinsic derivative of a vector field on a (Riemannian) manifold. At least I have not found such a definition.

But I do know the angles between vectors (at the same point in M) and lengths of curves in M can be measured, and an intrinsic derivative should have something to do how these quantities are changing together. It is quite reasonable to suspect, I think, that a single vector $D_u w$ should contain the information about how w changes in the unit vector direction $u = u_P$. These are perhaps two good starting points:

(a) There should be a derivative $D_u w = \nabla_u w$ of a vector field $w \in \mathscr{X}(M)$ in the direction $u = u_P \in \mathbb{S}^{n-1}_P \subset T_P M$.

(b) That derivative $\nabla_u w(P)$ should be a vector in $T_P M$.

The directional derivative of a traditional vector field $\mathbf{w}$ on $\mathbb{R}^n$ is given by a matrix of partial derivatives,

$$D_{\mathbf{u}}\mathbf{w}(\mathbf{x}) = \left(\frac{\partial w_i}{\partial x_j}(\mathbf{x})\right)$$

so this somehow corresponds to a linear function (at each point) $L : T_{\mathbf{x}}\mathbb{R}^n \to T_{\mathbf{x}}\mathbb{R}^n$. Thus, we might expect something similar for the intrinsic derivative of a vector field on a manifold. On the other hand, the situation is somewhat more complicated on a non-flat/non $\mathbb{R}^n$ space, so we should think carefully about what to expect.

Exercise 17.7. Consider the following property of an assumed intrinsic derivative $\nabla_u w$ of a vector field $w \in \mathscr{X}(M)$ in the direction $u = u_P \in T_P M$.

$$\nabla_{-u} w(P) = -\nabla_u w(P) \qquad \text{for} \qquad u \in \mathbb{S}_P^{n-1},\ w \in \mathscr{X}(M).$$

Do you consider this property natural, necessary, or both? Explain why.

Exercise 17.8. Formulate condition (17.2) as it would apply to the directional derivative $g = \nabla w$ leading to the assumed (point) differential map of property **DVF1**. Do you consider this property natural, necessary, or both? Explain why.

2. Notice the connection property **DVF3** relies crucially on the construction of the global differential map $dw : \mathscr{X}(M) \to \mathscr{X}(M)$ from **DVF2**.

3. As with the determination of the intrinsic derivative $D_u f = \nabla_u f$ of a function in $cC^\infty(M)$, one should expect the metric tensor $\mu : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathbb{R}$ to play a role in the determination of the intrinsic derivative $\nabla_u w$ of a vector field $w \in \mathscr{X}(M)$. Recall, however, that it was possible to define the differential map $df_P : T_P M \to \mathbb{R}$ for a real valued function without using the metric tensor.[1] Thus the formulation of property **DVF3** in particular does not actually depend on the metric tensor.

4. One way to understand that the metric tensor is not used "enough" (since it is not used at all) in the formulation of the properties **DVF1**, **DVF2**, and **DVF3** is that these properties are not adequate to determine the value of the intrinsic directional derivative $\nabla_u w$ of a vector field. To make this assertion a little more precise, it is possible to find many different functions

$$L : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$$

which are

[1]Remember that it is fine to define $D_u f(P)$ as $df_P(u)$ for $u \in \mathbb{S}_P^{n-1} \subset T_P M$, but the definition of $\mathbb{S}_P^{n-1} = \{z \in T_P M : \mu_P(z,z) = 1\}$ depends on the metric tensor. Similarly, you can have a differential map, but no way to connect it to the geometry of (functions on) M.

(a) bilinear when $\mathscr{X}(M)$ is considered as a linear space over $\mathbb{R}$,

(b) $cC^\infty(M)$ (modular) linear in the second argument

$$L(w, fz_1 + gz_2, w) = fL(z_1, w) + gL(z_2, w)$$
$$(f, g \in cC^\infty(M),\ z_1, z_2 \in \mathscr{X}(M))$$

when $\mathscr{X}(M)$ is considered as a module over $cC^\infty(M)$, and

(c) Leibnizian

$$L(fw, z) = df(z)\, w + fL(w, z) \qquad (f \in cC^\infty(M),\ z, w \in \mathscr{X}(M))$$

in the first argument.

In fact, there is a different such function L with $L(w, z) = dw(z)$ and $dw(u) = \nabla_u w$ for every different metric tensor on M. (And there are even others on certain manifolds that do not even come from Riemannian metric tensors at all, so these presumably have nothing to do with geometry.)

A function $L : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ on a topological manifold with an atlas $\mathcal{A}_*^\infty$ satisfying **(a)**, **(b)** and **(c)** is called a "connection." One might mistakenly take a connection and define a differential map $dw : \mathscr{X}(M) \to \mathscr{X}(M)$ for filament fields by

$$dw(z) = L(w, z). \tag{17.5}$$

If you're just interested in topology, maybe there is not much harm done by such a definition. It doesn't have anything to do with geometry however and specifically with the notion of an intrinsic derivative of a vector field. More precisely, the real problem with such a definition arises if your topological manifold happens to be a Riemannian manifold with a metric tensor $\mu : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathbb{R}$. Then $dw(z) = L(w, z)$ and $dw_u(P) = L(w, u)_P$ for $u \in \mathbb{S}_P^{n-1}$ in particular (based on some arbitrary connection) doesn't have anything to do with an intrinsic derivative of the vector field $w \in \mathscr{X}(M)$. Of course, this problem or concerns about "what is geometry?" do not stop a lot of people from making the atrocious definition (17.5) using some arbitrary connection. And such people often go further. Specifically one might be tempted to define a directional derivative of a filament field w in the direction z to be

$$\nabla_z w = L(w, z). \tag{17.6}$$

Again, this may be some kind of amusing algebraic exercise, but it's not geometry and it's certainly not geometry with calculus. There's nothing wrong with amusing algebraic exercises per se, but the real problem arises when there is actually a Riemannian metric around with which to measure angles and distance and do geometry. In that case, if you take an arbitrary connection $L : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ and write down the value given in (17.6), then we know first of all that the value has some complicated scaling business going on that differentiates it somewhat from any kind of honest directional derivative. But even if we restrict to a unit vector $u \in \mathbb{S}^{n-1}_P$, then $L(w,u)_P$ is not the directional derivative of w in the direction u. There is a real meaning of $\nabla_u w$ as a derivative on a Riemannian manifold, so we shouldn't waste that on a connection. We should figure out what the value is.

It turns out there are two more properties. One of them is certainly pretty natural, and it depends pretty strongly on the metric. It can be stated like this:

DVF4 metric compatibility Given any two vector fields $w_1, w_2 \in \mathscr{X}(M)$ and a (locally) unit[2] u in $\mathscr{X}(M)$ there holds

$$D_u\mu(w_1,w_2) = \mu(\nabla_u w_1, w_2) + \mu(w_1, \nabla_u w_2). \tag{17.7}$$

This says differentiation satisfies a product rule (or is Leibnizian if you like) with respect to the metric. That is, it says something about the way angles between vector fields change.

This is a key property, and it is certainly something that holds in Euclidean space. As a consequence it will also hold for submanifolds of Euclidean space. Incidentally, it is property **DVF4** which suggests inner products of vector fields $\mu(w_1,w_2)$ give interesting real valued functions in $cC^\infty(M)$ to differentiate.

Notice that by the module linearity in the argument z of

1. the differential $df(z)$ for real value functions f,

[2]Note that there may be no vector field $u \in \mathscr{X}(M)$ with $\mu_Q(u_Q,u_Q) = 1$ for all $Q \in M$. It is always possible to take any vector field $z \in \mathscr{X}(M)$ with $z_P \neq \mathbf{0}$ and find an open set $\mathbf{p}(U) \subset M$ with $P \in \mathbf{p}(U)$ and another vector field $u \in \mathscr{X}(M)$ with $u = z/\mu(z,z)^{1/2}$ on $\mathbf{p}(U)$ so that u is locally a unit field.

2. the metric tensor $\mu(z, w)$ for $\mu : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathbb{R}$, and
3. the differential $dw(z)$ for vector fields from property **DVF2**

it follows from (17.7) that for any $z, w_1, w_2 \in \mathscr{X}(M)$ there should hold

$$d\mu(w_1, w_2)(z) = \mu(dw_1(z), w_2) + \mu(w_1, dw_2(z))$$

in terms of the intrinsic differential map and of course

$$d\mu(w_1, w_2)(z) = \mu(L(w_1, z), w_2) + \mu(w_1, L(w_2, z))$$

in terms of the intrinsic connection L.

The final property is the one I find the most difficult to motivate, but I think I have a calculation that does motivate it pretty well (in some sense). The final property is also independent of the metric tensor, but it has something to do with charts and their interaction with the intrinsic derivative of vector fields.

DVF5 symmetry Given any $P \in M$ and a chart $(U, \mathbf{p}) \in \mathcal{A}_*^\infty$ there holds

$$[d(d\mathbf{p}(\mathbf{e}_i))(d\mathbf{p}(\mathbf{e}_j))]_Q = [d(d\mathbf{p}(\mathbf{e}_j))(d\mathbf{p}(\mathbf{e}_i))]_Q \tag{17.8}$$

for $i, j = 1, 2, \ldots, n$ and Q in some open set $\mathbf{p}(V)$ with $\xi(P) \in V \subset\subset U$. This says, for one thing, that the vector fields induced by the coordinate fields in a chart are somehow special. In terms of the notation $v_j = d\mathbf{p}(\mathbf{e}_j)$ for $j = 1, 2, \ldots, n$ for the coordinate induced fields introduced above, the relation (17.8) reads

$$dv_i(v_j) = dv_j(v_i) \tag{17.9}$$

for each chart $(U, d\mathbf{p}) \in \mathcal{A}_*^\infty$.

It should be noted that by $v_j = d\mathbf{p}(\mathbf{e}_j)$ for $j = 1, 2, \ldots, n$ in (17.8) we mean any field w in $\mathscr{X}(M)$ for which

$$w\Big|_{\mathbf{p}(V)} = d\mathbf{p}(\mathbf{e}_j). \tag{17.10}$$

Technically, $d\mathbf{p}(\mathbf{e}_j)$ is properly only an element of $\mathscr{X}(\mathbf{p}(U))$ defined on the (full dimension) open submanifold $\mathbf{p}(U) \subset M$. But given any $V \subset\subset U$, it can be shown that $d\mathbf{p}(\mathbf{e}_j)$ has an extension to a vector field $w \in \mathscr{X}(M)$ satisfying (17.10).

Exercise 17.9. Express (17.8) in terms of the intrinsic connection $L : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ defined in property **DVF3**.

A connection satisfying the symmetry property **DVF5** is also said to be **torsion free**. One can reasonably object to the notation used in (17.8) though I think it is basically correct. One has there of course the differential map associated with a particular vector field $d(d\mathbf{p}(\mathbf{e}_i)) = dv_i : T_PM \to T_PM$ which (especially the first expression used in (17.8)) is rather ugly. I have attempted to restrict the use of the symbol D_u and/or ∇_u for an intrinsic derivative to situations when u is a unit length vector to better capture and illustrate the geometry in the manifold. It is common to allow the "direction" of differentiation appearing in these symbols to be any tangent vector. From this point of view the value $dw(z)$ of the differential map may be written as $\nabla_z w$. Again, I have tried to avoid this in order to emphasize the geometry. If this is allowed, the symmetry conditions (17.8) and (17.9) can take the considerably prettier (and familiar) form

$$\nabla_{v_i} v_j = \nabla_{v_j} v_i, \qquad i, j = 1, 2, \ldots, n.$$

A good deal more should be said in terms of the motivation and/or necessity of properties **DVF1-DVF5**. One interesting direction, especially with respect to the properties **DVF1-DVF3** is to consider each of the properties for the derivatives of the traditional vector fields $d\xi(v)$ induced locally using the translator differential of a coordinate function $\xi : \mathbf{p}(U) \to \mathbb{R}^n$.

Exercise 17.10. Consider generalizing Exercise 17.4 for application to the (directional) derivative of a traditional vector field $\mathbf{w} : U \to \mathbb{R}^n$ induced in coordinates by a vector field $w \in \mathscr{X}(M)$ on a manifold M:

(a) Given $P = \mathbf{p}(\mathbf{x})$ with $\mathbf{x} \in U$, consider the set

$$\Sigma_{\mathbf{x}} = \{d\xi_P(u) : u = u_P \in \mathbb{S}^{n-1}_P\} \subset T_{\mathbf{x}}\mathbb{R}^n = \mathbb{R}^n.$$

$\mathbb{S}^{n-1}_P$ is a sphere. What kind of set is $\Sigma_{\mathbf{x}}$?

(b) Compose and complete a version of Exercise 17.4 replacing M with $\mathbb{R}^n$, T_PM with $T_{\mathbf{x}}\mathbb{R}^n$ and $\mathbb{S}^{n-1}_P$ with $\Sigma_{\mathbf{x}}$.

(c) What are the conditions required for the extension of $g : \Sigma_{\mathbf{x}} \to X$ to a linear function $G : T_{\mathbf{x}}\mathbb{R}^n \to T_{\mathbf{x}}\mathbb{R}^n$? Are these natural and/or necessary conditions for the function $g : \Sigma_{\mathbf{x}} \to T_{\mathbf{x}}\mathbb{R}^n$ given by $g(\mathbf{u}) = D_{\mathbf{u}}\mathbf{w}$?

Another approach might be through local embedding.

Exercise 17.11. Read the initial history section of the paper THE IMBEDDING PROBLEM FOR RIEMANNIAN MANIFOLDS (1955) by John Nash and the introduction to the preprint *Counterexamples for local isometric embedding* (2002) by Nicolai Nadirashvili and Yu Yuan.

(a) What interesting things do you find there?

(b) Is the question of (isometrically) embedding some neighborhood $\mathbf{p}(U)$ containing a point P in a two-dimensional Riemannian manifold as a submanifold of $\mathbb{R}^3$ settled?

Rather than pursue these directions further at the moment, let me proceed under the assumption that at least properties **DVF1-DVF3** hold for the (directional) derivative of a vector field $w \in \mathscr{X}(M)$. There is also a section below specifically intended to motivate property **DVF5**.

17.2.2 First reduction

Say we have a filament/vector $u \in \mathbb{S}_P^{n-1} \subset T_PM$ at a point P in a Riemannian manifold M, and we also have a vector field $w \in \mathscr{X}(M)$. Our objective is to determine an expression for the directional derivative $\nabla_u w(P) = (\nabla_u w)_P \in T_PM$.

With respect to a local chart (function) $\mathbf{p} : U \to M$ with $\xi(P) \in U$, we can write

$$w = \sum_{j=1}^{n} b_j v_j$$

where $b_1, b_2, \dots, b_n \in C^\infty(U)$ are real valued functions. Using the properties **DVF1-DVF3** and leaving off the evaluation so that $\nabla_u w = \nabla_u w(P)$,

$$\begin{aligned}
\nabla_u w &= dw(u) \\
&= \nabla_u \left(\sum_{j=1}^{n} b_j v_j \right) \\
&= \sum_{j=1}^{n} [D_u b_j \ v_j + b_j \ \nabla_u v_j] \\
&= \sum_{j=1}^{n} [D_u b_j(P) \ (v_j)_P + b_j(P) \ (\nabla_u v_j)_P].
\end{aligned}$$

Recall that we already know the values of the directional derivatives $D_u b_j(P)$ given by

$$D_u b_j(P) = \frac{d}{dt}(b_j \circ \mathbf{p})(\mathbf{x} + td\xi_P(u))\Big|_{t=0} = \langle D(b_j \circ \mathbf{p})(\mathbf{x}), d\xi_P(u)\rangle_{T_{\mathbf{x}}U}$$

where $\mathbf{x} = \xi(P)$. That is,

$$D_u b_j(P) = \sum_{i=1}^{n} a_i \; \frac{\partial(b_j \circ \mathbf{p})}{\partial x_i}(\mathbf{x})$$

where

$$u = u_P = \sum_{i=1}^{n} a_i v_i$$

for some $a_1, a_2, \ldots, a_n \in \mathbb{R}$ since

$$d\xi_P(u) = \sum_{i=1}^{n} a_i d\xi_P(v_i) = \sum_{i=1}^{n} a_i \; \mathbf{e}_i = (a_1, a_2, \ldots, a_n)$$

and

$$D(b_j \circ \mathbf{p})(\mathbf{x}) = \left(\frac{\partial(b_j \circ \mathbf{p})}{\partial x_1}(\mathbf{x}), \frac{\partial(b_j \circ \mathbf{p})}{\partial x_2}(\mathbf{x}), \ldots \frac{\partial(b_j \circ \mathbf{p})}{\partial x_n}(\mathbf{x}) \right).$$

We conclude that it is enough to determine the values of the directional derivatives $\nabla_u v_j(P) = (\nabla_u v_j)_P$ for $j = 1, 2, \ldots, n$. Put another way, if we find the values of the special directional derivative vector fields $\nabla_u v_j$, we can find a formula in terms of a chart function for $\nabla_u w$.

Exercise 17.12. Assuming $u \in \mathscr{X}(M)$ is a globally defined unit field,[3] what qualification is necessary if one wishes to write something like $\nabla_u v_j \in \mathscr{X}(M)$? (Hint: Should one expect $\nabla_u v_j$ is defined on all of M?)

It may be noted that for this first reduction, we have used only the special case of **DVF3** described in Exercise 17.5, namely

(DVF3*) directional differentiation of vector fields is linear/Leibnizian
Given $u \in \mathbb{S}^{n-1}_P \subset T_P M$ fixed, the function $\nabla_u : \mathscr{X}(M) \to \mathscr{X}(M)$ satisfies

[3] Global unit fields do not exist on some manifolds, but they can exist on some manifolds.

(a) ∇_u is linear when $\mathscr{X}(M)$ is considered a linear space over $\mathbb{R}$:

$$\nabla_u(aw_1+bw_2) = a\nabla_u w_1+b\nabla_u w_2 \qquad (a,b \in \mathbb{R},\ w_j \in \mathscr{X}(M),\ j=1,2).$$

(b) ∇_u is Leibnizian when $\mathscr{X}(M)$ is considered a module over $cC^\infty(M)$:

$$\nabla_u(fw) = D_u f\ w + f\ \nabla_u w \qquad (f \in cC^\infty(M),\ w \in \mathscr{X}(M)).$$

17.2.3 Second reduction

Again we start with a filament/vector $u \in \mathbb{S}_P^{n-1} \subset T_P M$ at a point P in a Riemannian manifold M. Our objective now is to start with a local chart (function) $\mathbf{p} : U \to M$ with $\xi(P) \in U$ and determine a formula for

$$\nabla_u v_j \qquad \text{for} \qquad j = 1, 2, \ldots, n$$

where $v_j \in \mathscr{X}(\mathbf{p}(U))$ given by $(v_j)_Q = d\mathbf{p}_{\xi(Q)}(\mathbf{e}_j)$ for $Q \in \mathbf{p}(U)$ is a vector field on the open submanifold $\mathbf{p}(U)$ of M.

As above we can write

$$u = u_P = \sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} a_i (v_i)_P$$

where $a_1, a_2, \ldots, a_n$ are real numbers. real valued functions.

Again, we suppress the evaluation at P so $\nabla_u v_j = (\nabla_{u_P} v_j)_P$ and now we use **DVF1** to write

$$\begin{aligned} \nabla_u v_j &= dv_j(u) \\ &= dv_j\left(\sum_{i=1}^{n} a_i v_i\right) \\ &= \sum_{i=1}^{n} dv_j(v_i). \end{aligned}$$

The conclusion here is that if we can determine the differential values $dv_j(v_i)$ or equivalently the directional derivatives

$$\|v_i\| \nabla_{u_i} v_j \qquad \text{for} \qquad i, j = 1, 2, \ldots, n,$$

then we will be able to determine an expression for the directional derivative $\nabla_u w(P) = (\nabla_u w)_P \in T_P M$.

Putting the reductions considered above together, using the properties **DVF1-DVF3** freely, and leaving off the evaluation so that $\nabla_u w = \nabla_u w(P)$,

$$\begin{aligned}
\nabla_u w &= dw(u) \\
&= dw\left(\sum a_i v_i\right) \\
&= \sum_{i=1}^{n} a_i dw(v_i) \\
&= \sum_{i=1}^{n} a_i L(w, v_i) \\
&= \sum_{i=1}^{n} a_i L\left(\sum_{j=1}^{n} b_j v_j, v_i\right) \\
&= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} [db_j(v_i)\ v_j + b_j\ L(v_j, v_i)] \\
&= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} [db_j(v_i)\ v_j + b_j\ dv_j(v_i)].
\end{aligned}$$

Introducing the basis $\{u_1, u_2, \ldots, u_n\}$ consisting of unit vectors

$$u_i = \frac{v_i}{\|v_i\|_{T_P M}} \qquad \text{for} \qquad i = 1, 2, \ldots, n,$$

we can express this result in terms of directional derivatives while respecting the convention that such directional derivatives are only taken in unit vector directions:

$$\begin{aligned}
\nabla_u w &= \sum_{i=1}^{n} a_i \|v_i\| \sum_{j=1}^{n} [D_{u_i} b_j\ v_j + b_j\ \nabla_{u_i} v_j] \\
&= \sum_{i=1}^{n} a_i \|v_i\|_{T_P M} \sum_{j=1}^{n} [D_{u_i} b_j(P)\ (v_j)_P + b_j(P)\ \nabla_{u_i} v_j(P)] \\
&= \sum_{i=1}^{n} a_i \|v_i\|_{T_P M} \sum_{j=1}^{n} [D_{u_i} b_j(P)\ (v_j)_P + b_j(P)\ (\nabla_{u_i} v_j)_P].
\end{aligned}$$

Again, we conclude that it is enough to determine the values of the directional derivatives $\nabla_{u_i} v_j(P) = (\nabla_{u_i} v_j)_P$ and/or the differential values $dv_j(v_i)$ for

$i, j = 1, 2, \ldots, n$ in particular. Put another way, if we find the values of the special directional derivative vector fields $\nabla_{u_i} v_j$, we can find a formula in terms of a chart function for $\nabla_u w$.

Exercise 17.13. What qualification is necessary if one wishes to write something like $\nabla_{u_i} v_j \in \mathscr{X}(M)$? (Hint: Should one expect $\nabla_{u_i} v_j$ is defined on all of M?)

17.2.4 Levi-Civita calculation

The determination $dv_j(v_i)$ depends on **DVF4** and **DVF5**: Consider $i, j, k \in \{1, 2, \ldots, n\}$.

$$d\mu(v_j, v_k)(v_i) = \mu(dv_j(v_i), v_k) + \mu(v_j, dv_k(v_i)) \tag{17.11}$$
$$d\mu(v_k, v_i)(v_j) = \mu(dv_k(v_j), v_i) + \mu(v_k, dv_i(v_j)) \tag{17.12}$$
$$d\mu(v_i, v_j)(v_k) = \mu(dv_i(v_k), v_j) + \mu(v_i, dv_j(v_k)). \tag{17.13}$$

It was mentioned in the introduction of the (axiomatic) Leibnizian/metric compatibility property of the differential of a vector field **DVF4** that a real valued function given by the inner product of two vector fields can be an interesting function to differentiate. On the left of each identity (17.11-17.13) the differential of a real valued function with values given by the inner product of two special (coordinate induced) vector fields on $\mathbf{p}(U) \subset M$ appears. These real valued functions are essentially the metric coefficients g_{jk}, g_{ki}, and g_{ij}. Thus, (17.11-17.13) may also be written as

$$\mu(dv_j(v_i), v_k) + \mu(v_j, dv_k(v_i)) = d(g_{jk} \circ \xi)(v_i) \tag{17.14}$$
$$\mu(dv_k(v_j), v_i) + \mu(v_k, dv_i(v_j)) = d(g_{ki} \circ \xi)(v_j) \tag{17.15}$$
$$\mu(dv_i(v_k), v_j) + \mu(v_i, dv_j(v_k)) = d(g_{ij} \circ \xi)(v_k). \tag{17.16}$$

Exercise 17.14. Rewrite the three identities (17.14-17.16) in terms of directional derivatives.

Adding the first two identities (17.14) and (17.15), subtracting the third (17.16), and noting the symmetry condition **DVF5** $dv_j(v_i) = dv_i(v_j)$ for coordinate induced fields we find

$$2\mu(dv_j(v_i), v_k) = d(g_{jk} \circ \xi)(v_i) + d(g_{ki} \circ \xi)(v_j) - d(g_{ij} \circ \xi)(v_k).$$

Thus,

$$\mu(dv_j(v_i), v_k) = \frac{1}{2}[d(g_{jk} \circ \xi)(v_i) + d(g_{ki} \circ \xi)(v_j) - d(g_{ij} \circ \xi)(v_k)] \qquad (17.17)$$

is determined for each $k = 1, 2, \ldots, n$ by known quantities (or at least directional derivatives/differentials of real valued functions which we know how to intrinsically differentiate). This is the main calculation which lies at the heart of a famous theorem of Levi-Civita (1917). Notice that the key idea was permuting the indices in the Leibniz/metric compatibility formula and then using the symmetry/torsion free property. It should be at least intuitively clear that if we know the inner products

$$\mu(dv_j(v_i), v_k) = \mu_P(dv_j(v_i), v_k)$$

for $k = 1, 2, \ldots, n$, then we should know the vector $dv_j(v_i)$. To make this explicit, let us express $dv_j(v_i)$ in terms of the basis $\{v_1, v_2, \ldots, v_n\}$:

$$dv_j(v_i) = \|v_i\| \; \nabla_{u_i} v_j = \sum_{\ell=1}^{n} \Gamma_{ij}^{\ell} \; v_\ell. \qquad (17.18)$$

The coefficients Γ_{ij}^{ℓ} in the linear combination of the vectors in the basis $\{v_1, v_2, \ldots, v_n\}$ of T_PM for $dv_j(v_i)$ are called the **Christoffel symbols**.

The notation in which the Christoffel symbols Γ_{ij}^{k} appear here is a generalization of the notation used for submanifolds: In that case, say we have a regularly parameterized n-dimensional submanifold $M = \mathcal{S}$ of $\mathbb{R}^N$ given (locally) by a parameterization

$$X : U \to \mathcal{S} \subset \mathbb{R}^N.$$

In this case, there is a basis

$$\left\{ \frac{\partial X}{\partial x_1}, \frac{\partial X}{\partial x_2}, \ldots, \frac{\partial X}{\partial x_n} \right\} = \left\{ \frac{\partial X}{\partial x_1}(\mathbf{x}), \frac{\partial X}{\partial x_2}(\mathbf{x}), \ldots, \frac{\partial X}{\partial x_n}(\mathbf{x}) \right\}$$

of traditional vectors for $T_P\mathcal{S}$ considered as a linear/vector subspace $\mathbb{R}^N$ where $X(\mathbf{x}) = P \in \mathcal{S}$. It will be noted that this basis of traditional vectors corresponds (via differential translation) to the filament basis $\{v_1, v_2, \ldots, v_n\}$ for T_PM. Furthermore, for each $j = 1, 2, \ldots, n$ there is a (local) vector field

$$\frac{\partial X}{\partial x_j}$$

tangent to $M = \mathcal{S}$, and the partial derivative

$$\frac{\partial}{\partial x_i}\left(\frac{\partial X}{\partial x_j}\right) = \frac{\partial^2 X}{\partial x_j \partial x_i}$$

is a natural vector to consider (in $T_P\mathbb{R}^N = \mathbb{R}^N$). Typically,

$$\frac{\partial^2 X}{\partial x_j \partial x_i} \notin T_P\mathcal{S}$$

but there is a decomposition

$$\frac{\partial^2 X}{\partial x_j \partial x_i} = \left(\frac{\partial^2 X}{\partial x_j \partial x_i}\right)^T + \left(\frac{\partial^2 X}{\partial x_j \partial x_i}\right)^\perp$$

into tangent and normal components with

$$\left(\frac{\partial^2 X}{\partial x_j \partial x_i}\right)^T \in T_P\mathcal{S}$$

and

$$\left\langle \left(\frac{\partial^2 X}{\partial x_j \partial x_i}\right)^\perp, \frac{\partial X}{\partial x_k}\right\rangle_{\mathbb{R}^N} = 0, \qquad k = 1, 2, \ldots, n.$$

In this case, we write

$$\left(\frac{\partial^2 X}{\partial x_j \partial x_i}\right)^T = \sum_{j=1}^n \Gamma_{ij}^k \, \frac{\partial X}{\partial x_k}$$

where the coefficients Γ_{ij}^k are the (traditional) Christoffel symbols. To the extent that $dv_j(v_i) = \|v_i\| \, \nabla_{u_i} v_j$ corresponds to

$$\left(\frac{\partial^2 X}{\partial x_j \partial x_i}\right)^T,$$

the simple fact that partial derivatives commute

$$\frac{\partial^2 X}{\partial x_j \partial x_i} = \frac{\partial^2 X}{\partial x_i \partial x_j}$$

gives some motivation for the symmetry property **DVF5** according to which $dv_j(v_i) = dv_i(v_j)$ or

$$\|v_i\| \; \nabla_{u_i} v_j = \|v_j\| \; \nabla_{u_j} v_i,$$

though we will attempt to offer additional motivation from another more intrinsic perspective below.

Returning to the determination of the Christoffel symbols in the case of a Riemannian manifold, if we substitute (17.18) in (17.17) we find

$$\sum_{\ell=1}^{n} \Gamma_{ij}^{\ell} \mu(v_\ell, v_k) = \frac{1}{2}[d(g_{jk} \circ \xi)(v_i) + d(g_{ki} \circ \xi)(v_j) - d(g_{ij} \circ \xi)(v_k)]. \quad (17.19)$$

Note that in the formulation above the metric coefficients have domain the chart U so that $g_{ij} \in C^\infty(U)$, but the Christoffel symbols have $\Gamma_{ij}^k \in cC^\infty(\mathbf{p}(U))$. This distinction of domain is often the victim of, as Spivak puts it, casual confusion and relations like (17.17) are expressed as

$$\mu(dv_j(v_i), v_k) = \frac{1}{2}[dg_{jk}(v_i) + dg_{ki}(v_j) - dg_{ij}(v_k)].$$

It's difficult to argue with the elegance, but I'm usually inclined to try to avoid the confusion.

The left side of (17.19) is the k-th entry in the product

$$(g_{k\ell} \circ \xi) \begin{pmatrix} \Gamma_{ij}^1 \\ \Gamma_{ij}^2 \\ \vdots \\ \Gamma_{ij}^n \end{pmatrix}.$$

Thus, denoting the inverse of the matrix $(g_{ij}) = (g_{ij} \circ \xi)$ of metric coefficients by

$$(g^{ij}) = (g_{ij})^{-1},$$

we conclude

$$\Gamma_{ij}^k = \Gamma_{ij}^k \circ \mathbf{p} = \frac{1}{2} \sum_{\ell=1}^{n} g^{k\ell}[dg_{j\ell}(v_i) + dg_{\ell i}(v_j) - dg_{ij}(v_\ell)] \quad (17.20)$$

$$= \frac{1}{2} \sum_{\ell=1}^{n} g^{k\ell} \left[\frac{\partial g_{j\ell}}{\partial x_i} + \frac{\partial g_{\ell i}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_\ell} \right]. \quad (17.21)$$

Computationally, probably the most useful formula for the Christoffel symbols is given by (17.21) and indeed, when casual confusion happens to be avoided, the Γ_{ij}^k are often considered as functions with domain U rather than $\mathbf{p}(U)$ as we have considered them here.

Notice the special differential values $dv_i(v_j)$ and the special directional derivatives

$$\nabla_{u_i} v_j = \frac{1}{\|v_i\|_{T_P M}}\ dv_j(v_i) = \frac{1}{(g_{ii} \circ \xi)^{1/2}} \sum_{k=1}^n \Gamma_{ij}^k\ v_k$$

in particular, depend very strongly and in a very nonobvious way on the Riemannian metric (coefficients) through (17.20). This is, as far as I know, the way the formula for the intrinsic (directional) derivative of a vector field on a Riemannian manifold is obtained.

To review, the special directional derivatives

$$\nabla_{u_i} v_j = \frac{1}{(g_{ii} \circ \xi)^{1/2}} \sum_{k=1}^n \Gamma_{ij}^k\ v_k$$

are expressed in terms of the differential values

$$dv_j(v_i) = \sum_{k=1}^n \Gamma_{ij}^k\ v_k.$$

The coefficients determining these special vector fields are given by (17.20) in terms of the metric coefficients (and thier derivatives).

If you have a general vector field $w \in \mathscr{X}(M)$, and you want to differentiate it in the direction $u \in \mathbb{S}_P^{n-1}$ at a point $P \in M$, then find a chart function $\mathbf{p} : U \to M$ with $\xi(P) = \mathbf{x} \in U$, and expand w and u in terms of the chart induced basis

$$\{v_1, v_2, \ldots, v_n\} = \{d\mathbf{p}(\mathbf{e}_1), d\mathbf{p}(\mathbf{e}_2), \ldots, d\mathbf{p}(\mathbf{e}_n)\}$$

with

$$u = \sum_{i=1}^n a_i v_i \qquad \text{and} \qquad w = \sum_{j=1}^n b_j v_j.$$

Use the linear and Leibnizian properties to express

$$\begin{aligned}
\nabla_u w &= \sum_{i=1}^n a_i \sum_{j=1}^n [D_{v_i} b_j \ v_j + b_j \ dv_j(v_i)] \\
&= \sum_{i=1}^n a_i \sum_{j=1}^n [D_{v_i} b_j(P) \ (v_j)_P + b_j(P) \ (dv_j(v_i))_P] \\
&= \sum_{j=1}^n D_u b_j(P) \ (v_j)_P + \sum_{j=1}^n b_j(P) \ \nabla_u v_j(P) \\
&= \sum_{j=1}^n D_u b_j(P) \ (v_j)_P + \sum_{i,j=1}^n a_i \ b_j(P) \ \|u_i\| \ \nabla_{u_i} v_j(P) \\
&= \sum_{j=1}^n D_u b_j(P) \ (v_j)_P + \sum_{i,j,k=1}^n a_i \ b_j(P) \ \Gamma_{ij}^k \ (v_k)_P.
\end{aligned}$$

17.2.5 Alternative forms/discussion/review

Given the formula

$$\nabla_u w = \sum_{j=1}^n D_u b_j(P) \ (v_j)_P + \sum_{i,j,k=1}^n a_i \ b_j(P) \ \Gamma_{ij}^k \ (v_k)_P \tag{17.22}$$

it is common to change indices in the first summation to express the result in terms of the basis $\{v_1, v_2, \ldots, v_n\}$ for $T_P M$:

$$\nabla_u w = \sum_{k=1}^n D_u b_k(P) \ (v_k)_P + \sum_{i,j,k=1}^n a_i \ b_j(P) \Gamma_{ij}^k \ (v_k)_P.$$

The coefficients/real numbers a_i, $i = 1, 2, \ldots, n$ are often commonly interpreted in a different way. Specifically, we know $u = [\alpha]$ has a generating curve/path $\alpha : I \to M$, and $\xi \circ \alpha : I \to U$ is a traditional[4] embedded path in $\mathfrak{I}^\infty(U)$. In this way the isolated vector $u = u_P \in T_P M$ is taken as the

[4]What we mean by "traditional" here is that $\xi \circ \alpha$ has a traditional derivative $(\xi \circ \alpha)' \in \mathbb{R}^n$ in contrast to the filament $u = [\alpha] \in T_P M$ for which $\alpha \in cC^\infty(M)$ and $\alpha : I \to M$ may admit no notion of a derivative α', though many authors simply denote the filament $u = [\alpha]$ in this case by α'.

value at P of a vector field $u \in \mathscr{X}(A)$ or $u \in \mathscr{X}(\mathbf{p}(A_0))$ with respect to the one-dimensional submanifolds

$$A = \{\alpha(t) : t \in I\} \qquad \text{or} \qquad A_0 = \{\xi \circ \alpha(t) : t \in I,\ \alpha(t) \in \mathbf{p}(U)\}.$$

An emphasis may then be put on the parameter t so that for example the coefficients $a_1, a_2, \ldots, a_n$ can be expressed differently: The path $\xi \circ \alpha$ has coordinate functions $\xi^1 \circ \alpha, \xi^2 \circ \alpha, \ldots, \xi^n \circ \alpha$ and

$$\xi \circ \alpha(t) = \sum_{i=1}^{n} \xi^i \circ \alpha \ \mathbf{e}_i.$$

Thus, the traditional tangent vector to A_0 is

$$(\xi \circ \alpha)'(t) = \sum_{i=1}^{n} (\xi^i \circ \alpha)'(t) \ \mathbf{e}_i$$

with

$$\sum_{i=1}^{n} (\xi^i \circ \alpha)'(t_0) \ \mathbf{e}_i = d\xi_P(u).$$

This means

$$u = \sum_{i=1}^{n} a_i \ v_i = \sum_{i=1}^{n} (\xi^i \circ \alpha)'(t_0) \ v_i$$

and

$$a_i = (\xi^i \circ \alpha)'(t_0) \qquad (i = 1, 2, \ldots, n).$$

This alternative expression is often written simply as

$$a_i = \frac{d(\xi^i \circ \alpha)}{dt}$$

or even

$$a_i = \frac{d\xi^i}{dt}$$

so that (17.22) is written something like

$$\nabla_u w = \sum_{j=1}^{n} D_u b_j(P) \ (v_j)_P + \sum_{i,j,k=1}^{n} \frac{d(\xi^i \circ \alpha)}{dt} \ b_j(P) \ \Gamma_{ij}^k \ (v_k)_P.$$

One can then further interpret the directional derivative $\nabla_u w(P)$ as a derivative "along the path α" and write

$$\nabla_u w = \frac{Dw}{dt}$$

where the complicated operator

$$\frac{D}{dt} : \mathscr{X}(M) \to \mathscr{X}(A)$$

depends on the path A and is called the **covariant derivative along** α. Notice that this allows extension (and we have incorporated) some kind of derivative of the vector field $w \in \mathscr{X}(M)$ at each point $\alpha(t)$ along the path instead of simply at $P = \alpha(t_0)$ though this will only be a proper directional derivative at points $Q = \alpha(t)$ where the filament $[\alpha] \in T_Q M$ has unit length.[5] Also, the covariant derivative of w along α depends on the entire path $A \subset M$, but in reality the value in principle depends only on the filament $[\alpha]$ at each point $Q = \alpha(t)$. Like other intrinsic derivatives, the covariant derivative along a path applies naturally to real valued functions $f \in cC^\infty(M)$ as well as vector fields with

$$\frac{Df}{dt} = \frac{d(f \circ \alpha)}{dt}$$

so that at P

$$D_u b_j(P) = (b_j \circ \alpha)'(t_0) = \frac{Db_j}{dt} \qquad (j = 1, 2, \ldots, n)$$

and (17.22) is often written as

$$\frac{Dw}{dt} = \sum_{j=1}^{n} \frac{Db_j}{dt}\; v_j + \sum_{i,j,k=1}^{n} \frac{D\xi^i}{dt}\; b_j\; \Gamma_{ij}^k\; v_k.$$

Compare the formulas on the bottom of page 52 and in the middle of page 56 of Chapter 2 of [2].

Returning to (17.22)

$$\nabla_u w = \sum_{j=1}^{n} D_u b_j(P)\; (v_j)_P + \sum_{i,j,k=1}^{n} a_i\; b_j(P)\; \Gamma_{ij}^k\; (v_k)_P$$

[5] Of course a representative path α for the filament may be chosen (by reparameterization for example) to have constant unit length $\mu_\alpha([\alpha], [\alpha]) = 1$, and then $Dw/dt = \nabla_{[\alpha]} w$ at each $Q = \alpha(s)$.

consider a special case (or two). If $M = U \subset \mathbb{R}^n$, then the coordinate basis fields v_j for $j = 1, 2, \ldots, n$ are given by $v_j = [\gamma_j]$ with $\gamma_j(t) = P + t\mathbf{e}_j = \mathbf{x} + t\mathbf{e}_j$ for $j = 1, 2, \ldots, n$, and the (local/global) metric coefficients are

$$g_{ij} = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

are constants for $i, j = 1, 2, \ldots, n$. In particular, all the differential values

$$d\mu(v_j, v_k)(v_i) = d(g_{jk} \circ \xi)(v_i)$$

appearing in (17.19)

$$\sum_{\ell=1}^{n} \Gamma_{ij}^{\ell} \mu(v_\ell, v_k) = \frac{1}{2}[d(g_{jk} \circ \xi)(v_i) + d(g_{ki} \circ \xi)(v_j) - d(g_{ij} \circ \xi)(v_k)].$$

are zero. This implies the important numbers

$$\urcorner_{ij}^{k} = \frac{1}{2}[d(g_{jk} \circ \xi)(v_i) + dg_{ki}(v_j) - dg_{ij}(v_k)] = 0 \qquad (i, j, k = 1, 2, \ldots, n)$$

vanish as well so that (17.20) becomes

$$\Gamma_{ij}^{k} = \sum_{\ell=1}^{n} g^{k\ell} \, \urcorner_{ij}^{\ell} = 0$$

for $i, j, k = 1, 2, \ldots, n$ or

$$\begin{pmatrix} \Gamma_{ij}^1 \\ \Gamma_{ij}^2 \\ \vdots \\ \Gamma_{ij}^n \end{pmatrix} = (g^{km}) \begin{pmatrix} \urcorner_{ij}^1 \\ \urcorner_{ij}^2 \\ \vdots \\ \urcorner_{ij}^n \end{pmatrix} = \mathbf{0} \tag{17.23}$$

and formula (17.22) becomes simply

$$\nabla_u w = \sum_{j=1}^{n} D_u b_j(P) \, (v_j)_P. \tag{17.24}$$

In this case, w and u correspond by differential translation $d(\mathrm{id}_U) : \mathscr{X}(U) \to \mathbb{R}^n$ to a traditional vector field $\mathbf{w} = (b_1, b_2, \ldots, b_n)$ and a traditional unit vector $\mathbf{u} = (a_1, a_2, \ldots, a_n) \in T_P\mathbb{R}^n$. The number $D_u b_j(P)$ is $D_{\mathbf{u}} b_j(P) = \langle Db_j(P), \mathbf{u}\rangle_{\mathbb{R}^n}$. The main point is that the formula for the intrinsic derivative of the vector field w in (17.24) says that one should simply (intrinsically) differentiate the coefficients of w with respect to the basis $\{v_1, v_2, \ldots, v_n\}$. In particular, if u is taken to be a standard basis vector as well so that $\mathbf{u} = \mathbf{e}_k$ for some $k \in \{1, 2, \ldots, n\}$, then

$$\nabla_u w = \sum_{j=1}^{n} \frac{\partial b_j}{\partial x_k}(P)\ (v_j)_P$$

corresponding to

$$D_{\mathbf{e}_k}\mathbf{w} = \sum_{j=1}^{n} \frac{\partial b_j}{\partial x_k}(P)\ \mathbf{e}_j.$$

If we take a chart, even for this trivial manifold $M = U \subset \mathbb{R}^n$, then we get a first hint as to how the metric tensor/distance on M might come into play in taking a derivative of a vector field on M and indeed must come into play. Say we consider $M \subset \mathbb{R}^2$ again with the standard/trivial metric tensor, but let us assume M is a region in the plane admitting a chart function given by polar coordinates $\mathbf{p} : U \to M$ with $U \subset (0, \infty) \times (\theta_0, \theta_0 + 2\pi)$ where $\theta_0 \in \mathbb{R}$ is fixed and

$$\mathbf{p}(r, \theta) = (r\cos\theta, r\sin\theta).$$

Here again the coordinate filaments $v_j = [\alpha_j] = d\mathbf{p}(\mathbf{e}_j)$ for $j = 1, 2$ with $\alpha_1(t) = \mathbf{x} + t(\mathbf{x}/|\mathbf{x}|)$ and $\alpha_2(t) = |\mathbf{x}|(\cos(\theta + t), \sin(\theta + t))$ where $\mathbf{x} = |\mathbf{x}|(\cos\theta, \sin\theta)$ correspond to the traditional vector fields $\mathbf{v} = \mathbf{x}/|\mathbf{x}|$ and $\mathbf{w} = \mathbf{x}^{\perp} = |\mathbf{x}|(-\sin\theta, \cos\theta)$. This situation is discussed in more detail with illustrations in the next section. The metric coefficients in this case, however are not (all) constant with $g_{ij} : U \to \mathbb{R}$ by

$$\begin{aligned} g_{11}(r, \theta) &= |\mathbf{v}|^2 = 1 \\ g_{12}(r, \theta) &= \mathbf{v} \cdot \mathbf{w} = (\cos t_0, \sin t_0) \cdot (-r\sin t_0, r\cos t_0) = 0 \\ g_{22}(r, \theta) &= |\mathbf{w}|^2 = r^2. \end{aligned}$$

In particular,

$$d(g_{22} \circ \xi)(v_1) = D_{\mathbf{v}}(g_{22} \circ \xi) = D_{\mathbf{e}_1} g_{22} = \frac{\partial g_{22}}{\partial r} = 2r.$$

The other differential values $d(g_{ij} \circ \xi)(v_k)$ are zero. Computing

$$\daleth_{ij}^k \circ \mathbf{p} = \frac{1}{2}\left[\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ki}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k}\right]$$

for $i, j, k = 1, 2, \ldots, n$ we find

$$\begin{aligned}\daleth_{11}^1 &= \frac{1}{2}\left[\frac{\partial g_{11}}{\partial r} + \frac{\partial g_{11}}{\partial r} - \frac{\partial g_{11}}{\partial r}\right] \\ &= 0\end{aligned}$$

because $g_{11} \equiv 1$.

$$\begin{aligned}\daleth_{11}^2 &= \frac{1}{2}\left[\frac{\partial g_{12}}{\partial r} + \frac{\partial g_{21}}{\partial r} - \frac{\partial g_{11}}{\partial \theta}\right] \\ &= 0\end{aligned}$$

because $g_{12} = g_{21} \equiv 0$ and $g_{11} \equiv 1$.

$$\begin{aligned}\daleth_{12}^1 &= \frac{1}{2}\left[\frac{\partial g_{21}}{\partial r} + \frac{\partial g_{11}}{\partial \theta} - \frac{\partial g_{12}}{\partial r}\right] \\ &= 0\end{aligned}$$

because $g_{12} = g_{21} \equiv 0$ and $g_{11} \equiv 1$.

$$\begin{aligned}\daleth_{12}^2 &= \frac{1}{2}\left[\frac{\partial g_{22}}{\partial r} + \frac{\partial g_{21}}{\partial \theta} - \frac{\partial g_{12}}{\partial \theta}\right] \\ &= r\end{aligned}$$

because $g_{22}(r, \theta) = r^2$, and $g_{12} = g_{21} \equiv 0$.

$$\begin{aligned}\daleth_{21}^1 &= \frac{1}{2}\left[\frac{\partial g_{11}}{\partial \theta} + \frac{\partial g_{11}}{\partial r} - \frac{\partial g_{21}}{\partial r}\right] \\ &= 0\end{aligned}$$

because $g_{21} \equiv 0$ and $g_{11} \equiv 1$.

$$\begin{aligned}\daleth_{21}^2 &= \frac{1}{2}\left[\frac{\partial g_{12}}{\partial \theta} + \frac{\partial g_{22}}{\partial r} - \frac{\partial g_{21}}{\partial \theta}\right] \\ &= r\end{aligned}$$

because $g_{22}(r, \theta) = r^2$, and $g_{12} = g_{21} \equiv 0$.

Exercise 17.15. Give an alternative justification that $(\urcorner^1_{21}, \urcorner^2_{21}) = (0, r)$ by showing the symmetry relation

$$\urcorner^k_{ji} = \urcorner^k_{ij} \qquad \text{for} \qquad i, j, k = 1, 2, \ldots, n$$

in general.

$$\begin{aligned} \urcorner^1_{22} &= \frac{1}{2}\left[\frac{\partial g_{21}}{\partial \theta} + \frac{\partial g_{12}}{\partial \theta} - \frac{\partial g_{22}}{\partial r}\right] \\ &= -r \end{aligned}$$

because $g_{21} = g_{21} \equiv 0$ and $g_{22}(r, \theta) = r^2$.

$$\begin{aligned} \urcorner^2_{22} &= \frac{1}{2}\left[\frac{\partial g_{22}}{\partial \theta} + \frac{\partial g_{22}}{\partial \theta} - \frac{\partial g_{22}}{\partial \theta}\right] \\ &= 0 \end{aligned}$$

because g_{ij} is independent of θ for $i, j = 1, 2$.

Exercise 17.16. Compute the Christoffel symbols for polar coordinates and show the values were correctly computed in the video

https://www.youtube.com/watch?v=4UVJTYL4pQk

Hint: Assemble the McAwful symbols $\urcorner^k_{ij}$ computed for you above into three or four vectors and then apply the formula appearing (17.23).

17.2.6 Symmetry for (certain) flat manifolds

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