Some care of a different flavor is required with respect to differentiation. First of all, differentiability at a point x in the open interval (a, b) as we know is defined by

$$h'(x) = \lim_{v \to 0} \frac{h(x+v) - h(x)}{v}. \tag{1.19}$$

Specifically, the function $h : (a, b) \to \mathbb{R}$ is said to be **differentiable at** $x \in (a, b)$ if the limit in (1.19) exists in the sense that there is some number $L \in \mathbb{R}$ for which given any $\epsilon > 0$, there exists some $\delta > 0$ for which

$$\left| \frac{h(x+v) - h(x)}{v} - L \right| < \epsilon \qquad \text{whenever} \qquad |v| < \delta. \tag{1.20}$$

If this condition holds, we write $h'(x) = L$. If $h : (a, b) \to \mathbb{R}$ is differentiable at every point $x \in (a, b)$, then we say h is **differentiable on the entire interval** (a, b). In this case, a function $h' : (a, b) \to \mathbb{R}$ is defined, and if this function is continuous, we write $h \in C^1(a, b)$. This is the basic condition defining what it means for a function to be **continuously differentiable** though the terminology is perhaps slightly opaque. Note carefully, however, that the definition just given for $C^1(a, b)$ applies for a, b **extended real numbers** with $a < b$. In particular, the values $a = -\infty$ and $b = +\infty$ are included/allowed.

Exercise 1.22. Define the vector space of functions $C^0(a, b)$ for $a \in [-\infty, \infty)$ and $b \in (-\infty, \infty]$ with $a < b$, and show $C^1(a, b) \subset C^0(a, b)$.

An extension of the definition of differentiability at a point associated with (1.19) and (1.20) to the closed interval $[a, b]$ is readily obtained using the same approach used to define continuity above:

The function $h : [a, b] \to \mathbb{R}$ is said to be **differentiable at** $x \in [a, b]$ if the limit in (1.19) exists in the sense that there is some number $L \in \mathbb{R}$ for which given any $\epsilon > 0$, there exists some $\delta > 0$ for which

$$\left| \frac{h(x+v) - h(x)}{v} - L \right| < \epsilon \qquad \text{whenever} \qquad |v| < \delta \quad \textbf{and } x + v \in [a, b].$$

If this condition holds, we again write $h'(x) = L$, though often it may be convenient to denote the value of the derivative at an endpoint by

$$h'(a^+) = \lim_{v \searrow 0} \frac{h(a+v) - h(a)}{v} \qquad \text{or} \qquad h'(b^-) = \lim_{v \nearrow 0} \frac{h(b+v) - h(b)}{v}.$$

If $h : (a, b) \to \mathbb{R}$ is differentiable at every point $x \in [a, b]$, we say h is **differentiable on the entire interval** $[a, b]$. In this case, a function $h' : [a, b] \to \mathbb{R}$ is defined, and if $h' \in C^0[a, b]$ we write $h \in C^1[a, b]$ and say h is **continuously differentiable**.

Exercise 1.23. Show $C^1[a, b]$ is a vector space of functions, and the following statements are equivalent:

(i) $h \in C^1[a, b]$,

(ii) There exists some open interval $I \supset [a, b]$ and a function $g \in C^1(I)$ with

$$g\Big|_{(a,b)} = h.$$

(iii) There exists a function $g \in C^1(\mathbb{R})$ with

$$g\Big|_{(a,b)} = h.$$

The difference quotient formulation of the condition **(i)** in Exercise 1.23 does not carry over to higher dimensions. The alternative formulation **(ii)** allows the consideration of functions adequate for many purposes.[6]

Given an open set $U \subset \mathbb{R}^n$, a point $P \in U$, and $j \in \{1, 2, \ldots, n\}$, the **$j$-th partial derivative of $h : U \to \mathbb{R}$ at P** is defined by

$$\frac{\partial h}{\partial x_j}(P) = \lim_{v \to 0} \frac{h(P + v\mathbf{e}_j) - h(P)}{v}$$

when this limit exists. The value may also be denoted $D_j h(P)$, $h_{x_j}(P)$, or $D^{\mathbf{e}_j} h(P)$. If the j-th partial derivative of h exists at every point $P \in U$, then $D_j h : U \to \mathbb{R}$ is a well-defined function. If $D_j h : U \to \mathbb{R}$ is a well-defined function for each $j = 1, 2, \ldots, n$, then we say h is **partially differentiable** in all of U. Note: This is not the same as **differentiability** or the condition that h is **differentiable** even at a single point $P \in U$ which we define below. Partial differentiability is not the same (and does not imply in general) differentiability.

If h is partially differentiable and $D_j h \in C^0(U)$ for each $j = 1, 2, \ldots, n$, then we say h is **continuously partially differentiable** on all of U and write $h \in C^1(U)$.

[6] We are going to give a definition of $C^1(\overline{U})$ which is somewhat more restrictive than the one usually given.

Exercise 1.24. Show $C^1(U) \subset C^0(U)$.

If $U \subset \mathbb{R}^n$ is open and bounded and $h \in C^1(U)$, we say $h \in C^1(\overline{U})$ if the following condition(s) hold: There exists an open set $V \supset \overline{U}$ and a function $g \in C^1(V)$ for which

$$g\big|_U = h.$$

Exercise 1.25. Let $h \in C^1(\overline{U})$.

(a) If there exists an open set $V \supset \overline{U}$ and functions $g, \tilde{g} \in C^1(V)$ for which

$$g\big|_U = \tilde{g}\big|_U = h,$$

then

$$g\big|_{\overline{U}} = \tilde{g}\big|_{\overline{U}}.$$

(b) Show

$$C^1(\overline{U}) = \left\{ g\big|_U : g \in C^0(\overline{U}) \text{ and } g\big|_U \in C^1(\overline{U}) \right\}.$$

(c) In view of Exercise 1.24 and parts **(a)** and **(b)** above, it makes sense to say $C^1(\overline{U}) \subset C^0(\overline{U})$ and in particular, $h \in C^0(\overline{U})$. (Explain.)

In view of the discussion above, we state what hopefully are considered natural geometry problems from calculus:

Newton's first differentiation problem (Problem 6): Given $h \in C^1[a, b]$, find the derivative $h' \in C^0[a, b]$.

Newton's general differentiation problem (Problem 6): Given an open and bounded set $U \subset \mathbb{R}^n$ and a function $h \in C^1(\overline{U})$, find the partial derivatives $D_j h \in C^0(\overline{U})$ for $j = 1, 2, \ldots, n$.

The following important definition was probably not known to Newton. It was certainly known to Fréchet (1878-1973).

Definition 1. Let U be an open set in $\mathbb{R}^n$, and let $h : U \to \mathbb{R}$ be a real valued function with domain U.

(i) The function $h : U \to \mathbb{R}$ is **differentiable at** $\mathbf{x} \in U$ if there exists a linear function $L : \mathbb{R}^n \to \mathbb{R}$ for which

$$\lim_{|\mathbf{v}|\to 0} \frac{h(\mathbf{x}+\mathbf{v}) - h(\mathbf{x}) - L(\mathbf{v})}{|\mathbf{v}|} = 0.$$

The linear function $L : \mathbb{R}^n \to \mathbb{R}$ is called the **differential** of h at $\mathbf{x}$ and is denoted by $dh_{\mathbf{x}} : \mathbb{R}^n \to \mathbb{R}$.

(ii) The function h is said to be **differentiable on all of** U if h is differentiable at each $\mathbf{x} \in U$.

(iii) We will say the function h is **differentiable on all of** $\overline{U}$ if there is some open set $V \subset \mathbb{R}^n$ with $V \supset \overline{U}$ and h is differentiable on all of V.

Exercise 1.26. Let U be a bounded open subset of $\mathbb{R}^n$, and let $h : U \to \mathbb{R}$ be a real valued function defined on U.

(a) Show that if h is differentiable at $\mathbf{x} \in U$, then h is continuous at $\mathbf{x} \in U$. Consequently, if h is differentiable on all of U, then $h \in C^0(U)$.

(b) Show that if h is differentiable at $\mathbf{x} \in U$, then h is partially differentiable at $\mathbf{x} \in U$.

(c) Show that if $h \in C^1(U)$, then h is differentiable on all of U.

(d) Show that if $h \in C^1(\overline{U})$ then h is differentiable on all of $\overline{U}$.

Newton's general differentiation problem (Problem 6a): Given an open set $U \subset \mathbb{R}^n$ and a function $h \in C^1(U)$, find the differential $dh_{\mathbf{x}} : \mathbb{R}^n \to \mathbb{R}$ for each $\mathbf{x} \in U$.

Exercise 1.27. (challenge(s)) Let U be a open subset of $\mathbb{R}^n$ with $h \in C^1(U)$.

(a) Consider the function $g : U \times \mathbb{R}^n \to \mathbb{R}$ by

$$g(\mathbf{x}, \mathbf{v}) = dh_{\mathbf{x}}(\mathbf{v}).$$

Characterize the regularity of g?

(b) Consider the function $Dh : U \to \mathbb{R}^n$ by

$$Dh(\mathbf{x}) = \left(\frac{\partial}{\partial x_1}(\mathbf{x}), \frac{\partial}{\partial x_2}(\mathbf{x}), \ldots, \frac{\partial}{\partial x_n}(\mathbf{x}) \right).$$

Characterize the regularity of Dh.

(c) Consider the function $\ell : U \to \beth(\mathbb{R}^2)$ by

$$\ell(\mathbf{x}) = dh_{\mathbf{x}}$$

where $\beth(\mathbb{R}^2)$ denotes the vector space of real valued linear functions $L : \mathbb{R}^2 \to \mathbb{R}$. Characterize the regularity of ℓ.

(d) If U is also bounded and $h \in C^1(\overline{U})$ under what conditions may it be said that the functions g, Dh, and ℓ considered above extend to $\overline{U}$, and what can be said about the regularity of each function in that case? Hint: Consider the case in which $U \subset \mathbb{R}^2$ and $\Gamma = \partial U$ is a cyclic path.

Once the delicate issue of determining how to say a function $h \in C^1(U)$ satisfies $h \in C^1(\overline{U})$ when $U \subset \mathbb{R}^n$ is an open set is addressed on one way or another, then the question of higher derivatives is more or less straightforward. For an open set $U \subset \mathbb{R}^n$ and for an integer $k \geq 2$, we say $h \in C^k(U)$ if all (partial) derivatives of order k or lower are in $C^0(U)$. All these partial derivatives are perhaps easiest to express in **multiindex notation**. A multiindex β is just an element of $\mathbb{N}_0^n$ where $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ denotes the natural numbers with zero. Thus, a multiindex $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ has $\beta_j \in \mathbb{N}_0$ for $j = 1, 2, \ldots, n$, and we write

$$D^\beta h = \frac{\partial h^{|\beta|}}{\partial^{\beta_1} x_1 \partial^{\beta_2} x_2 \cdots \partial^{\beta_n} x_n}$$

where $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_n$. Thus, for $k = 1, 2, \ldots$,

$$C^k(U) = \{h \in C^0(U) : D^\beta h \in C^0(U),\ |\beta| \leq k\}.$$

The vector space of functions $C^k(\overline{U})$ is defined inductively with

$$C^2(\overline{U}) = \{h \in C^1(\overline{U}) : D^\beta h \in C^1(\overline{U}),\ |\beta| = 1\}$$

with whatever definition is given to $C^1(\overline{U})$, and

$$C^k(\overline{U}) = \{h \in C^{k-1}(\overline{U}) : D^\beta h \in C^1(\overline{U}),\ |\beta| = k-1\}$$

for $k = 2, 3, 4, \ldots$ in general.

We will mostly deal with derivatives of orders two and lower, so we will often use the more traditional notation. There is also a few other vector spaces that might be worth mentioning at this point. Given U an open set in $\mathbb{R}^n$,

$$C^\infty(U) = \bigcap_{k=0}^{\infty} C^k(U),$$

$$C^\infty(\overline{U}) = \bigcap_{k=0}^{\infty} C^k(\overline{U}),$$

and $C^\omega(U)$ denotes the collection of **real analytic functions**, that is functions $h \in C^\infty(U)$ with the following property:

For each $\mathbf{p} \in U$, there exists some $r > 0$ for which the series

$$\sum_{j=0}^{\infty} \sum_{|\beta|=j} \frac{D^\beta(\mathbf{p})}{\beta!} (\mathbf{x} - \mathbf{p})^\beta$$

where $\beta! = \beta_1!\beta_2! \cdots \beta_n!$ and

$$(\mathbf{x} - \mathbf{p})^\beta = (x_1 - p_1)^{\beta_1}(x_2 - p_2)^{\beta_2} \cdots (x_n - p_n)^{\beta_n}$$

converges for $|\mathbf{x} - \mathbf{p}| < r$ and

$$f(\mathbf{x}) = \sum_{j=0}^{\infty} \sum_{|\beta|=j} \frac{D^\beta(\mathbf{p})}{\beta!} (\mathbf{x} - \mathbf{p})^\beta \qquad \text{for} \qquad |\mathbf{x} - \mathbf{p}| < r.$$

1.4.2 A new problem: curvature

Euclid's curvature problem (Problem 7): Given a circle of radius $r > 0$, what is the curvature of the circle?

This problem[7] is stated to suggest the formulation of the definition of curvature as much as the calculation of the particular number $k = 1/r$. What is curvature? In this case, a first approximation of the answer might be something like "Curvature is a number associated with a circle or a straight line which is zero for a straight line, positive for a circle, and decreasing with the radius for circles and tending to $+\infty$ as the radius decreases to zero." There are many choices of course, but the particular choice $k = 1/r$ with the limiting value

$$\lim_{r \nearrow \infty} \frac{1}{r} = 0$$

[7] Perhaps first answered by Nicolas Oresme (c.1320–1382).