Isometric embedding of Riemannian surfaces: Some comments

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I am going to attempt here to construct some kind of framework in which to prove Janet's 1926 theorem (also perhaps better known as the Janet-Cartan theorem from 1927 in the case $n = 2$):

Theorem 1 [Janet, 1926] Given a disk

$$B_R(\mathbf{0}) = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < R^2\}$$

of some radius $R > 0$ and a positive definite matrix assignment $(g_{ij}) : B_R(\mathbf{0}) \to GL_2(\mathbb{R})$ where

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}$$

and each of the three functions $g_{ij} : B_R(\mathbf{0}) \to \mathbb{R}$ is real analytic

$$g_{ij}(\mathbf{x}) = \sum_{|\beta| \geq 0} \frac{D^\beta g_{ij}(\mathbf{0})}{\beta!} \mathbf{x}^\beta$$

with the series convergent[1] on $\overline{B_R(\mathbf{0})}$, that is $g_{ij} \in C^\omega(\overline{B_R(\mathbf{0})})$ for $i, j = 1, 2$, consider for each r with $0 < r < R$, the Riemannian manifold $M = B_r(\mathbf{0})$ with Riemannian metric tensor $\mu : M \to \mathcal{T}^2(M)$ satisfying $\mu_{\mathbf{x}} : T_{\mathbf{x}}M \times T_{\mathbf{x}}M \to \mathbb{R}$ by

$$\mu_{\mathbf{x}}(v, w) = \langle (g_{ij})\mathbf{v}, \mathbf{w} \rangle_{\mathbb{R}^2} \tag{1}$$

for $\mathbf{x} \in B_r(\mathbf{0})$, and $v = d\mathbf{p}_{\mathbf{x}}(\mathbf{v}), w = d\mathbf{p}_{\mathbf{x}}(\mathbf{w}) \in T_{\mathbf{x}}M$ where $\mathbf{p} : B_r(\mathbf{0}) \to M$ is considered a global chart function and $d\mathbf{p}_{\mathbf{x}} : T_{\mathbf{x}}\mathbb{R}^2 \to T_{\mathbf{x}}M$. Under these assumptions, there is some $r > 0$ for which M is **realizable** as a surface $\mathcal{S}$ in $\mathbb{R}^3$ with respect to the Riemannian metric induced from $\mathbb{R}^3$. More precisely, there exists a function $X : B_r(\mathbf{0}) \to \mathbb{R}^3$ for which the following hold:

1. $X \in C^\omega(\overline{B_r(\mathbf{0})} \to \mathbb{R}^3)$,

2. $X : B_r(\mathbf{0}) \to X(B_r(\mathbf{0})) = \mathcal{S}$ is a bijection,

3. $\mathcal{S}$ is a regular parameterized surface with

$$\left\{ \frac{\partial X}{\partial x_1}, \frac{\partial X}{\partial x_2} \right\}$$

linearly independent for each $\mathbf{x} \in \overline{B_r(\mathbf{0})}$, and

[1] See the follow-up section at the end for some clarification on this.

4. The metric tensor $\nu : \mathcal{S} \to \mathcal{T}^2(\mathcal{S})$ satisfying $\nu_X : T_X\mathcal{S} \times T_X\mathcal{S} \to \mathbb{R}$ by

$$\nu_X(v, w) = \langle \mathbf{v}, \mathbf{w} \rangle_{\mathbb{R}^3}$$

for $X \in \mathcal{S}$ and

$$\begin{aligned} \mathbf{v} &= \alpha'(\alpha^{-1}(X)), \qquad & v &= [\alpha] \\ \mathbf{w} &= \alpha'(\alpha^{-1}(X)), \qquad & w &= [\alpha], \end{aligned}$$

also satisfies

$$\mu_{\mathbf{x}}(v, w) = \nu_X(dX_{\mathbf{x}}(v), dX_{\mathbf{x}}(w)) \tag{2}$$

where

$$\begin{aligned} dX_{\mathbf{x}}(v) &= [X \circ \alpha], \qquad & v &= [\alpha] \\ dX_{\mathbf{x}}(w) &= [X \circ \alpha], \qquad & w &= [\alpha]. \end{aligned}$$

Condition 4, and (2) in particular, is the main condition which makes $X : M \to \mathcal{S}$ an isometry of Riemannian manifolds as well as an embedding of the Riemannian manifold M as a submanifold (specifically a surface) in $\mathbb{R}^3$.

In the discussion that follows I hope to also mention something about a topic that I consider potentially somewhat more interesting than the basic embedding problem for Riemannian manifolds which I will call *flexibility*. A fair amount of attention has been given to the **rigidity** of certain embeddings, but I don't know that much has been written about the question(s) I'm going to suggest. Perhaps there are things known, and I just do not know them.

Finally, it should be noted that the real analyticity is apparently required as Nadarishvili and Yuan (2002) claim a counterexample for a C^∞ metric assignment on the disk. My impression is that few people have gone through these counterexamples (starting with a $C^{2,1}$ example of Pogorelov in 1971) very carefully.

1 Preliminary normalizations

Dilating $M_R = B_R(\mathbf{0})$ by a factor $\mu_{\mathbf{0}}(d\mathbf{p_0}(\mathbf{e}_1), d\mathbf{p_0}(\mathbf{e}_1))^{-1/2}$ we may assume $\|d\mathbf{p_0}(\mathbf{e}_1)\|_M = 1$.

Exercise 1 Explain the details of this normalization.

Consequently, we look for a function $X \in C^\omega(B_r(\mathbf{0}) \to \mathbb{R}^3)$ with

$$\left| \frac{\partial X}{\partial x_1}(\mathbf{0}) \right| = 1.$$

Let us denote the three real analytic coordinate functions of X by $(u, v, w) = (u_1, u_2, u_3)$. It is also required that

$$\left\{ \frac{\partial X}{\partial x_1}, \frac{\partial X}{\partial x_2} \right\} \subset \mathbb{R}^3$$

is a linearly independent set. By a rotation of $\mathbb{R}^3$, we may assume

$$\frac{\partial X}{\partial x_1}(\mathbf{0}) = \mathbf{e}_1 \in \mathbb{R}^3. \tag{3}$$

After this, we still have a one parameter family of rotations of $\mathbb{R}^3$ leaving $\mathbf{e}_1$ fixed which may be used for the purpose of normalization. In particular, we can assume the nonzero vector

$$\frac{\partial X}{\partial x_2}(\mathbf{0}) = \left(\frac{\partial u}{\partial x_2}(\mathbf{0}), \frac{\partial v}{\partial x_2}(\mathbf{0}), \frac{\partial w}{\partial x_2}(\mathbf{0}) \right)$$

satisfies

$$\frac{\partial w}{\partial x_2}(\mathbf{0}) = \frac{\partial u_3}{\partial x_2}(\mathbf{0}) = 0 \tag{4}$$

and

$$\frac{\partial v}{\partial x_2}(\mathbf{0}) = \frac{\partial u_2}{\partial x_2}(\mathbf{0}) > 0. \tag{5}$$

Exercise 2 Find the explicit rotations of $\mathbb{R}^3$ involved in the normalizations giving (3), (4) and (5) and explain their use in detail.

With these normalizations the condition for the linear independence of

$$\left\{\frac{\partial X}{\partial x_1}(\mathbf{0}),\ \frac{\partial X}{\partial x_2}(\mathbf{0})\right\} \subset \mathbb{R}^3$$

at $\mathbf{x} = \mathbf{0} \in \mathbb{R}^2$ becomes

$$\frac{\partial u}{\partial x_1}(\mathbf{0})\ \frac{\partial v}{\partial x_2}(\mathbf{0}) - \frac{\partial v}{\partial x_1}(\mathbf{0})\ \frac{\partial u}{\partial x_2}(\mathbf{0}) = \frac{\partial v}{\partial x_2}(\mathbf{0}) = \frac{\partial u_2}{\partial x_2}(\mathbf{0}) > 0.$$

2 First fundamental form; metric relations

The surface $\mathcal{S}$ inherits a metric tensor I with $I_X : T_X\mathcal{S} \times T_X\mathcal{S} \to \mathbb{R}$ given on traditional vectors[2] in $T_X\mathcal{S} \subset \mathbb{R}^3$ by

$$I_X(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}, \mathbf{w}\rangle_{\mathbb{R}^3}.$$

It is easy to check that the value of I_X is obtained in $\mathbf{x} = (x_1, x_2) \in U \subset \mathbb{R}^2$ coordinates in general associated with a embedding $X : U \to \mathcal{S}$ with $X \in C^1(U \to \mathcal{S})$ by

$$I_X(\mathbf{v}, \mathbf{w}) = E\ \alpha_1'\ \beta_1' + F(\alpha_1'\ \beta_2' + \alpha_2'\ \beta_1') + G\ \alpha_2'\ \beta_2' = \left\langle \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \alpha_1' \\ \alpha_2' \end{pmatrix}, \begin{pmatrix} \beta_1' \\ \beta_2' \end{pmatrix} \right\rangle_{\mathbb{R}^2} \tag{6}$$

where

$$\begin{aligned} \alpha' &= \alpha'(\alpha^{-1}(X)) = d(X^{-1})_X(\mathbf{v}), \\ \beta' &= \beta'(\alpha^{-1}(X)) = d(X^{-1})_X(\mathbf{w}), \end{aligned} \tag{7}$$

and

$$\begin{aligned} E &= \left|\frac{\partial X}{\partial x_1}\right|^2, \\ F &= \left\langle \frac{\partial X}{\partial x_1}, \frac{\partial X}{\partial x_2} \right\rangle_{\mathbb{R}^3} \\ G &= \left|\frac{\partial X}{\partial x_2}\right|^2. \end{aligned}$$

In fact, if $\alpha(t) = \mathbf{x} + td(X^{-1})_X(\mathbf{v})$ and $\beta(t) = \mathbf{x} + td(X^{-1})_X(\mathbf{w})$, then

$$\begin{aligned} \langle (X\circ\alpha)', (X\circ\beta)'\rangle_{\mathbb{R}^3} &= \left\langle \alpha_1'\frac{\partial X}{\partial x_1}(\alpha) + \alpha_2'\frac{\partial X}{\partial x_2}(\alpha), \beta_1'\frac{\partial X}{\partial x_1}(\beta) + \beta_2'\frac{\partial X}{\partial x_2}(\beta) \right\rangle_{\mathbb{R}^3} \\ &= \alpha_1'\beta_1' \left\langle \frac{\partial X}{\partial x_1}(\alpha), \frac{\partial X}{\partial x_1}(\beta) \right\rangle_{\mathbb{R}^3} + \alpha_1'\beta_2' \left\langle \frac{\partial X}{\partial x_1}(\alpha), \frac{\partial X}{\partial x_2}(\beta) \right\rangle_{\mathbb{R}^3} \\ &\quad + \alpha_2'\beta_1' \left\langle \frac{\partial X}{\partial x_2}(\alpha), \frac{\partial X}{\partial x_1}(\beta) \right\rangle_{\mathbb{R}^3} + \alpha_2'\beta_2' \left\langle \frac{\partial X}{\partial x_2}(\alpha), \frac{\partial X}{\partial x_2}(\beta) \right\rangle_{\mathbb{R}^3} \end{aligned}$$

[2]Note: This "first fundamental form" metric tensor for surfaces is nominally different from the Riemannian metric tensors considered on filaments. This is possible because each filament $v \in \mathcal{L}_X\mathcal{S}$ on an embedded surface considered as a Riemannian manifold corresponds to traditional vector $\mathbf{v} \in T_X\mathcal{S} \subset \mathbb{R}^3$.

so that evaluating at $t=0$ where $\alpha(0)=\beta(0)=X$ we obtain (6).

Exercise 3 Show that a regular parameterization $X \in C^\infty(U \to \mathbb{R}^3)$ of a surface may be used/interpreted as a global chart function $\mathbf{q} : U \to X(U)$ making the image surface $X(U)$ and Riemannian manifold with Riemannian metric $\nu : X(U) \to \mathcal{T}^2(X(U))$ satisfying

$$\nu_X(v,w) = I_X(\mathbf{v},\mathbf{w})$$

where $v = d\mathbf{q} \circ d(X^{-1})(\mathbf{v})$ and $v = d\mathbf{q} \circ d(X^{-1})(\mathbf{w})$.

Note the differential $d(X^{-1})_X : T_X\mathcal{S} \to T_{\mathbf{x}}\mathbb{R}^2$ appearing in Exercise 3 and in (7) of the map $X \in C^1(U \to \mathcal{S})$ is a slightly different kind of differential than we have seen/considered before. We have traditional differentials $d\psi_{\mathbf{x}} : T_{\mathbf{x}}\mathbb{R}^n \to T_{\psi(\mathbf{x})}\mathbb{R}^n$ on open subsets of Euclidean space mapping traditional vectors $\mathbf{v} \in T_{\mathbf{x}}\mathbb{R}^n$ to traditional vectors $d\psi_{\mathbf{x}}(\mathbf{v}) \in T_{\psi(\mathbf{x})}\mathbb{R}^m$. We have also introduced *translator* differentials $d\mathbf{p}_{\mathbf{x}} : T_{\mathbf{x}}\mathbb{R}^n \to T_PM$ taking traditional vectors $\mathbf{v} \in T_{\mathbf{x}}\mathbb{R}^n$ to filaments $v = d\mathbf{p}_{\mathbf{x}}(\mathbf{v}) \in T_PM$ where M is a manifold and $P = \mathbf{p}(\mathbf{x})$. Here $dX : T_{\mathbf{x}}\mathbb{R}^2 \to T_X\mathcal{S}$ and $d(X^{-1})_X : T_X\mathcal{S} \to T_{\mathbf{x}}\mathbb{R}^2$ are special to a surface $\mathcal{S}$ regularly parameterized by a function $X \in C^1(U \to \mathcal{S})$. These differentials also map traditional vectors, but only involve vectors in the tangent space of the surface at a particular point $X = X(\mathbf{x})$. There is a simple relation between traditional differentials and these **surface differentials**. Specifically, if $\bar{X} : U \times \mathbb{R} \to \mathbb{R}^3$ is any C^1 extension of X such that

$$\bar{X}\Big|_V \in C^1(V \to \mathbb{R}^3)$$

is a diffeomorphism of open sets V and $\bar{X}(V)$ in $\mathbb{R}^3$, then

$$d(\bar{X}^{-1})_X(\mathbf{v}) = d(X^{-1})_X(\mathbf{v}) \qquad \text{for} \qquad \mathbf{v} \in T_X\mathcal{S}.$$

As a result of the discussion above the metric relation (2) asserted in the statement of Jenet's theorem above reduces to three first order partial differential equations:

$$\begin{cases} g_{11} &= \left(\dfrac{\partial u}{\partial x_1}\right)^2 + \left(\dfrac{\partial v}{\partial x_1}\right)^2 + \left(\dfrac{\partial w}{\partial x_1}\right)^2 \\[2ex] g_{12} &= \dfrac{\partial u}{\partial x_1}\,\dfrac{\partial u}{\partial x_2} + \dfrac{\partial v}{\partial x_1}\,\dfrac{\partial v}{\partial x_2} + \dfrac{\partial w}{\partial x_1}\,\dfrac{\partial w}{\partial x_2} \\[2ex] g_{22} &= \left(\dfrac{\partial u}{\partial x_2}\right)^2 + \left(\dfrac{\partial v}{\partial x_2}\right)^2 + \left(\dfrac{\partial w}{\partial x_2}\right)^2. \end{cases}$$

It will be noted that these equations amount to $E = g_{11}$, $F = g_{12}$ and $G = g_{22}$. Thus, if we can find some $r > 0$ so that these equations hold for (real analytic functions) u, v, w on $B_r(\mathbf{0}) \subset \mathbb{R}^2$, then we have proved Janet's theorem.

3 Series expansions

Of course, we also have the normalization/point conditions

$$u(\mathbf{0}) = 0,\ v(\mathbf{0}) = 0,\ w(\mathbf{0}) = 0,$$

$$\frac{\partial u}{\partial x_1}(\mathbf{0}) = 1,\ \frac{\partial v}{\partial x_1}(\mathbf{0}) = 0 = \frac{\partial w}{\partial x_1}(\mathbf{0}),$$

and

$$\frac{\partial v}{\partial x_2}(\mathbf{0}) > 0 = \frac{\partial w}{\partial x_1}(\mathbf{0}).$$

These imply the (multivariable) series expansions

$$u = x_1 + \sum_{|\beta|\geq 2} \frac{D^\beta u(\mathbf{0})}{\beta!}\, \mathbf{x}^\beta,$$

$$v = \frac{\partial v}{\partial x_2}(\mathbf{0})\, x_2 + \sum_{|\beta|\geq 2} \frac{D^\beta v(\mathbf{0})}{\beta!}\, \mathbf{x}^\beta, \qquad \text{and}$$

$$w = \sum_{|\beta|\geq 2} \frac{D^\beta w(\mathbf{0})}{\beta!}\, \mathbf{x}^\beta.$$

3.1 First order derivatives/coefficients

If we evaluate the PDEs at $\mathbf{x} = \mathbf{0}$ we see the relations

$$\begin{cases} 1 &= \left(\dfrac{\partial u}{\partial x_1}(\mathbf{0})\right)^2 \\ g_{12}(\mathbf{0}) &= \dfrac{\partial u}{\partial x_1}(\mathbf{0})\, \dfrac{\partial u}{\partial x_2}(\mathbf{0}) \\ g_{22}(\mathbf{0}) &= \left(\dfrac{\partial u}{\partial x_2}(\mathbf{0})\right)^2 + \left(\dfrac{\partial v}{\partial x_2}(\mathbf{0})\right)^2. \end{cases}$$

The first equation corresponds to a normalization of which we have already taken account so that the second equation becomes

$$\frac{\partial u}{\partial x_2}(\mathbf{0}) = g_{12}(\mathbf{0}).$$

Taking into account the normalization

$$\frac{\partial v}{\partial x_2}(\mathbf{0}) > 0,$$

The third equation implies

$$\frac{\partial v}{\partial x_2}(\mathbf{0}) = \sqrt{g_{22}(\mathbf{0}) - [g_{12}(\mathbf{0})]^2} = \sqrt{g_{11}(\mathbf{0})g_{22}(\mathbf{0}) - [g_{12}(\mathbf{0})]^2} > 0.$$

Thus, all the values $u(\mathbf{0}) = 0$, $v(\mathbf{0}) = 0$, and $w(\mathbf{0}) = 0$, which are the same as the zero order coefficients in the series, are determined by normalization, and all the first order derivatives

$$\frac{\partial u}{\partial x_1}(\mathbf{0}) = 1, \quad \frac{\partial u}{\partial x_2}(\mathbf{0}) = g_{12}(\mathbf{0})$$

$$\frac{\partial v}{\partial x_1}(\mathbf{0}) = 0, \qquad \frac{\partial v}{\partial x_2}(\mathbf{0}) = \sqrt{g_{22}(\mathbf{0}) - [g_{12}(\mathbf{0})]^2},$$

and

$$\frac{\partial w}{\partial x_1}(\mathbf{0}) = 0, \qquad \frac{\partial w}{\partial x_2}(\mathbf{0}) = 0,$$

which are the same as the first order coefficients in the series are either determined by normalization or directly by the PDEs evaluated at $\mathbf{x} = \mathbf{0}$.

3.2 Second order derivatives/coefficients

I'm going to change notation at this point in order to simplify the forms of the expressions and better keep track of what is going on. The changes I am going to make have already been reference above. First, I will index the unknown functions writing $X = (u, v, w) = (u_1, u_2, u_3)$. Second I will use multi-index notation for partial derivatives so that

$$D^{\mathbf{e}_1} = \frac{\partial}{\partial x_1} \qquad \text{and} \qquad D^{\mathbf{e}_2} = \frac{\partial}{\partial x_2}.$$

With these changes the system of PDEs can be written as

$$\sum_{j=1}^{3} (D^{\mathbf{e}_1} u_j)^2 = g_{11}$$

$$\sum_{j=1}^{3} (D^{\mathbf{e}_1} u_j)(D^{\mathbf{e}_2} u_j) = g_{12}$$

$$\sum_{j=1}^{3} (D^{\mathbf{e}_2} u_j)^2 = g_{22}.$$

We note that there are three second order partial derivatives of each of the functions u_j for $j = 1, 2, 3$, making nine second order coefficients in the three series for $u = u_1$, $v = u_2$ and $w = u_3$. The second order partial differential operators are

$$D^{2\mathbf{e}_1} = \frac{\partial^2}{\partial x_1^2}, \qquad D^{\mathbf{e}_1+\mathbf{e}_2} = \frac{\partial^2}{\partial x_1 \partial x_2}, \qquad \text{and} \qquad D^{2\mathbf{e}_2} = \frac{\partial^2}{\partial x_2^2}.$$

Applying $D^{\mathbf{e}_1}$ to the three equations, we obtain three quasilinear second order PDEs

$$\sum_{j=1}^{3} (D^{\mathbf{e}_1} u_j) D^{2\mathbf{e}_1} u_j = \frac{1}{2} D^{\mathbf{e}_1} g_{11}$$

$$\sum_{j=1}^{3} [(D^{\mathbf{e}_2} u_j) D^{2\mathbf{e}_1} u_j + (D^{\mathbf{e}_1} u_j) D^{\mathbf{e}_1+\mathbf{e}_2} u_j] = D^{\mathbf{e}_1} g_{12}$$

$$\sum_{j=1}^{3} (D^{\mathbf{e}_2} u_j) D^{\mathbf{e}_1+\mathbf{e}_2} u_j = \frac{1}{2} D^{\mathbf{e}_1} g_{22}.$$

Similarly, differentiating with respect to x_2 we get three quasilinear second order PDEs:

$$\sum_{j=1}^{3} (D^{\mathbf{e}_1} u_j) D^{\mathbf{e}_1+\mathbf{e}_2} u_j = \frac{1}{2} D^{\mathbf{e}_2} g_{11}$$

$$\sum_{j=1}^{3} [(D^{\mathbf{e}_2} u_j) D^{\mathbf{e}_1+\mathbf{e}_2} u_j + (D^{\mathbf{e}_1} u_j) D^{2\mathbf{e}_2} u_j] = D^{\mathbf{e}_2} g_{12}$$

$$\sum_{j=1}^{3} (D^{\mathbf{e}_2} u_j) D^{2\mathbf{e}_2} u_j = \frac{1}{2} D^{\mathbf{e}_2} g_{22}.$$

Evaluating these six equations at $\mathbf{x} = 0$, we obtain six linear equations for the nine unknown values $D^{\mathbf{e}_i+\mathbf{e}_j} u_k(\mathbf{0})$, $i, j = 1, 2$, $k = 1, 2, 3$. Substituting the known first order coefficient values, this system of six linear equations becomes

$$D^{2\mathbf{e}_1} u_1(\mathbf{0}) = \frac{1}{2} D^{\mathbf{e}_1} g_{11}(\mathbf{0})$$

$$g_{12}(\mathbf{0}) D^{2\mathbf{e}_1} u_1(\mathbf{0}) + \sqrt{g_{22}(\mathbf{0}) - [g_{12}(\mathbf{0})]^2}\, D^{2\mathbf{e}_1} u_2(\mathbf{0}) + D^{\mathbf{e}_1+\mathbf{e}_2} u_1(\mathbf{0}) = D^{\mathbf{e}_1} g_{12}(\mathbf{0})$$

$$g_{12}(\mathbf{0}) D^{\mathbf{e}_1+\mathbf{e}_2} u_1(\mathbf{0}) + \sqrt{g_{22}(\mathbf{0}) - [g_{12}(\mathbf{0})]^2}\, D^{\mathbf{e}_1+\mathbf{e}_2} u_2(\mathbf{0}) = \frac{1}{2} D^{\mathbf{e}_1} g_{22}(\mathbf{0})$$

$$D^{\mathbf{e}_1+\mathbf{e}_2} u_1(\mathbf{0}) = \frac{1}{2} D^{\mathbf{e}_2} g_{11}(\mathbf{0})$$

$$g_{12}(\mathbf{0}) D^{\mathbf{e}_1+\mathbf{e}_2} u_1(\mathbf{0}) + \sqrt{g_{22}(\mathbf{0}) - [g_{12}(\mathbf{0})]^2}\, D^{\mathbf{e}_1+\mathbf{e}_2} u_2(\mathbf{0}) + D^{2\mathbf{e}_2} u_1(\mathbf{0}) = D^{\mathbf{e}_2} g_{12}(\mathbf{0})$$

$$g_{12}(\mathbf{0}) D^{2\mathbf{e}_2} u_1(\mathbf{0}) + \sqrt{g_{22}(\mathbf{0}) - [g_{12}(\mathbf{0})]^2}\, D^{2\mathbf{e}_2} u_2(\mathbf{0}) = \frac{1}{2} D^{\mathbf{e}_2} g_{22}(\mathbf{0}).$$

As I look at this system the first obvious (striking) observations are that the values of the two derivatives

$$D^{2\mathbf{e}_1} u_1(\mathbf{0}) \qquad \text{and} \qquad D^{\mathbf{e}_1+\mathbf{e}_2} u_1(\mathbf{0})$$

are completely determined by the first and fourth equations respectively and all three values

$$D^{2\mathbf{e}_1} u_3(\mathbf{0}) = D^{2\mathbf{e}_1} w(\mathbf{0}), \qquad D^{\mathbf{e}_1+\mathbf{e}_2} u_3(\mathbf{0}) = D^{\mathbf{e}_1+\mathbf{e}_2} w(\mathbf{0}), \qquad \text{and} \qquad D^{2\mathbf{e}_2} u_3(\mathbf{0}) = D^{2\mathbf{e}_2} w(\mathbf{0})$$

are entirely absent from the system of equations. Rewriting the system in terms of the four remaining unknown (and appearing) second derivative values, we have

$$\sqrt{g_{22}(\mathbf{0}) - [g_{12}(\mathbf{0})]^2}\, D^{2\mathbf{e}_1} u_2(\mathbf{0}) = D^{\mathbf{e}_1} g_{12}(\mathbf{0}) - \frac{1}{2} g_{12}(\mathbf{0}) D^{\mathbf{e}_1} g_{11}(\mathbf{0}) - \frac{1}{2} D^{\mathbf{e}_2} g_{11}(\mathbf{0})$$

$$\sqrt{g_{22}(\mathbf{0}) - [g_{12}(\mathbf{0})]^2}\, D^{\mathbf{e}_1+\mathbf{e}_2} u_2(\mathbf{0}) = \frac{1}{2} D^{\mathbf{e}_1} g_{22}(\mathbf{0}) - \frac{1}{2} g_{12}(\mathbf{0}) D^{\mathbf{e}_2} g_{11}(\mathbf{0})$$

$$\sqrt{g_{22}(\mathbf{0}) - [g_{12}(\mathbf{0})]^2}\, D^{\mathbf{e}_1+\mathbf{e}_2} u_2(\mathbf{0}) + D^{2\mathbf{e}_2} u_1(\mathbf{0}) = D^{\mathbf{e}_2} g_{12}(\mathbf{0}) - \frac{1}{2} g_{12}(\mathbf{0}) D^{\mathbf{e}_2} g_{11}(\mathbf{0})$$

$$g_{12}(\mathbf{0}) D^{2\mathbf{e}_2} u_1(\mathbf{0}) + \sqrt{g_{22}(\mathbf{0}) - [g_{12}(\mathbf{0})]^2}\, D^{2\mathbf{e}_2} u_2(\mathbf{0}) = \frac{1}{2} D^{\mathbf{e}_2} g_{22}(\mathbf{0}).$$

These remaining four equations determine the four unknowns

$$D^{2\mathbf{e}_1} u_2(\mathbf{0}), \qquad D^{\mathbf{e}_1+\mathbf{e}_2} u_2(\mathbf{0}), \qquad D^{2\mathbf{e}_2} u_1(\mathbf{0}), \qquad \text{and} \qquad D^{2\mathbf{e}_2} u_2(\mathbf{0})$$

uniquely. Thus, all second order coefficients for $u_1 = u$ and $u_2 = v$ are determined uniquely, while all three second order coefficients for $u_3 = w$ are seemingly left entirely unprescribed.

3.3 Flexibility (questions)

It is not surprising that some of the second order coefficients are left unprescribed or, as one might say, that there is some flexibility in choosing these coefficients. Presumably, one can continue to obtain conditions on the coefficients of all orders leading to convergent power series for u, v, and w giving Janet's embedding. I assume this will require a more organized treatment of taking higher order derivatives of the system of PDE's. Hopefully, the basic procedure of differentiating the system of PDEs and evaluating at $\mathbf{x} = \mathbf{0} \in \mathbb{R}^2$ to determine some of the coefficients is illustrated by the discussion above.

Were we to have one solution giving an embedding however, one heuristically expects there should be many other such solutions corresponding to "flexing" the surface that has been obtained. If we assume for a moment that a solution could be obtained by taking

$$D^{2\mathbf{e}_1} w(\mathbf{0}) = D^{\mathbf{e}_1+\mathbf{e}_2} w(\mathbf{0}) = D^{2\mathbf{e}_2} w(\mathbf{0}) = 0, \tag{8}$$

then perhaps we could say that a solution/embedding obtained for any other choice of these three constants corresponds to a "flexing" of the original solution surface. Thus, we might imagine there is at least a three-parameter family of flexings. It is likely that more dimensions are picked up at each level of determining the coefficients leading to an infinite dimensional family of flexings.

Exercise 4 If it is assumed that there exists a solution surface satisfying (8), can you prove that there is a solution/flexing corresponding to each nonzero choice for these three derivative values? Say you have convergent series solutions for u, v, and w that solve the entire problem and the coefficients for w satisfy (8). How would the coefficients need to be modified to give a solution with $D^{2\mathbf{e}_1} w(\mathbf{0}) = 2$ corresponding to adding x_1^2 to the value of w?

Exercise 5 Do you expect some surfaces to be more "flexible" than others? For example, should a convex cap be less flexible than a negatively curved saddle? Can you quantify this?

Exercise 6 Can you imagine a quantitative measure of comparison among flexings of a given surface, so that one surface might be considered more "relaxed" than another, and perhaps one could look for an optimally relaxed embedding of a Riemannian manifold?

Exercise 7 Can you connect the discussion/construction above with the 1926 paper of Janet or with the 1927 paper of Cartan?

4 Follow-up

In the statement of Janet's theorem my wording suggests that $g \in C^\omega(\overline{B_R(\mathbf{0})})$ amounts to the same thing as the convergence of the multivariable power series to the value of the function

$$g(\mathbf{x}) = \sum_{\beta \in \mathbb{N}^2} \frac{D^\beta g(\mathbf{0})}{\beta!} \mathbf{x}^\beta$$

on $\overline{B_R(\mathbf{0})}$. This is not quite correct.

Exercise 8 Consider the function $f : \mathbb{R} \to (0, \infty)$ by

$$f(x) = \frac{4}{4 + x^2}.$$

Show that this function satisfies $f \in C^\omega(\mathbb{R})$ but the radius of convergence at $x = 0$ is finite. Hint: To understand how the radius of convergence is determined at each point, consider $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \frac{4}{4 + z^2}.$$

If we take the matrix assignment

$$(g_{ij}) = \begin{pmatrix} \dfrac{16}{(4+|\mathbf{x}|^2)^2} & 0 \\ 0 & \dfrac{16}{(4+|\mathbf{x}|^2)^2} \end{pmatrix}$$

for all $\mathbf{x} = (x, y) = (x_1, x_2) \in \mathbb{R}^2$, then we know a solution for the embedding problem described above, namely $X = (u, v, w)$ with

$$u(x,y) = \frac{4x}{4+x^2+y^2}, \qquad v(x,y) = \frac{4y}{4+x^2+y^2}, \qquad \text{and} \qquad w(x,y) = \frac{2(x^2+y^2)}{4+x^2+y^2} \tag{9}$$

corresponding to a kind of stereographic projection.

Exercise 9 Check that (9) gives a solution to Janet's embedding problem resulting in a sphere

$$\mathcal{S} = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + (x_3 - 1)^2 = 1,\ x_3 < 2\}$$

punctured at the north pole. What are the series coefficients (and the radius of convergence) in this case? In particular, do the conditions (8) hold?

There are a couple other cases where an explicit solution of Janet's theorem is known. One is for the flat tensor $(g_{ij}) = (\delta_{ij})$ on the plane.

Exercise 10 Analyze the flexings of $\mathcal{S} = \{(x_1, x_2, 0) : \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2\}$ given by $X(\mathbf{x}) = (x_1, x_2, 0)$ using power series.

Another explicit solution is given (along with explicit flexing) for a deformation of a portion of Enneper's minimial surface to a portion of the catenoid. I won't write this down at the moment, but it involves the Weierstrass representation theorem for minimal surfaces.