Exercise 3.1.12 Find formulas for the Euclidean areas and Riemannian areas of some of the regions mentioned in **(a)**, **(b)**, and/or **(c)** of Exercise 3.1.11 above.

Exercise 3.1.13 If the Riemannian length of the radial segment indicated in Figure 3.2 is $b = \pi/4$, what is the Euclidean length a of the corresponding segment $\{t(\cos\theta, \sin\theta) \in B_1(\mathbf{0}) : 0 < t < a\}$?

Exercise 3.1.14 For some fixed a_0 and θ_0 in Problem 3.1.3, plot the value of $\text{length}_B[\alpha]$ as a function of x. Do you notice anything interesting?

3.2 Example $\mathcal{C}$

As a **point set** $\mathcal{B}$ is identical to $B_1(\mathbf{0}) \subset \mathbb{R}^2$. I've suggested that in order to think about and understand the identity of $\mathcal{B}$ as a Riemannian manifold, it is useful to "leave all the structure of $B_1(\mathbf{0}) \subset \mathbb{R}^2$ behind." After reflection on this suggestion, it strikes me that $\mathcal{B}$ is such a nice point set that it may be quite difficult to ignore the structure from $B_1(\mathbf{0})$ when contemplating $\mathcal{B}$. In anticipation of this problem, I've devised an alternative example designed to illustrate (to some extent) just how bad a Riemannian manifold's point set[3] can be. The second example is called $\mathcal{C}$, and $\mathcal{C}$ is a subset of $\mathbb{R}^3$. In order to describe $\mathcal{C}$, I'm going to use two familiar subsets of $\mathbb{R}$ and two familiar "quantities" associated with points in $\mathbb{R}^2$. The relevant subsets of $\mathbb{R}$ are the rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q} : q \in \mathbb{N} = \{1, 2, 3, \ldots,\} \text{ and } p \in \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\} \right\}$$

and the irrational numbers $\mathbb{R}\backslash\mathbb{Q}$. The quantities associated with a point $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ are the polar radius

$$r = \sqrt{x_1^2 + x_2^2}$$

[3]Keep in mind that the point set by itself is not the Riemannian manifold in its entirety, but the Riemannian manifold is the point set along with the "structure" I am trying to describe and hopefully you are trying to understand. In principle, however, the point set has an obvious importance in the definition, and I view assertions like "... in the context of the preceeding definitions, one cannot distinguish between two homeomorphic manifolds nor between two diffeomorphic differentiable manifolds" which appears on page 3 in [1] as somewhat counterproductive, at least for those who are trying to understand those definitions. Hopefully, many differences between $\mathcal{B}$ and $\mathcal{C}$ are fairly obvious, though it is equally clear one does not discern those differences by looking at the charts alone.

and the argument $\theta \in [0, 2\pi)$ for which $\mathbf{x} = r(\cos\theta, \sin\theta)$. There is one more (hopefully familiar) concept I need, namely that of a **characteristic function**: Given any set S and any subset $A \subset S$, the **characteristic function with support on** A is the function $\chi_A : S \to \mathbb{R}$ by

$$\chi_A(p) = \begin{cases} 1, & p \in A \\ 0, & p \in S \backslash A. \end{cases}$$

In terms of these quantities, I'm going to define a function $h : \mathbb{R}^2 \to \mathbb{R}$ by

$$h(x_1, x_2) = \begin{cases} \chi_{\mathbb{R}\backslash\mathbb{Q}}(r) - \chi_{\mathbb{Q}}(r), & \theta/\pi \in \mathbb{Q} \\ \\ \chi_{\mathbb{Q}}(r) - \chi_{\mathbb{R}\backslash\mathbb{Q}}(r), & \theta/\pi \in \mathbb{R}\backslash\mathbb{Q}. \end{cases} \tag{3.6}$$

Then

$$\mathcal{C} = \left\{ \left(x_1, x_2, h(x_1, x_2)\sqrt{x_1^2 + x_2^2} \right) : \mathbf{x} = (x_1, x_2) \in B_1(\mathbf{0}) \right\}. \tag{3.7}$$

The important points are the following:

1. $\mathcal{C}$ is in one-to-one correspondence with $B_1(\mathbf{0})$.

2. Riemannian lengths ($\text{length}_{\mathcal{C}}$), Riemannian angles ($\theta_{\mathcal{C}}$), and Riemannian areas ($\text{area}_{\mathcal{C}}$) can be computed using the same formulas used to find $\text{length}_{\mathcal{B}}$, $\theta_{\mathcal{B}}$, and $\text{area}_{\mathcal{B}}$.

3. $\mathcal{C}$ is an example of a Riemannian manifold just as much as $\mathcal{B}$.

Here are some exercises to walk you through some of the details of $\mathcal{C}$. The first question you might ask is: Is $\mathcal{C}$ well-defined?

Exercise 3.2.1 The quantity $r = \sqrt{x_1^2 + x_2^2}$, i.e., the polar radius, clearly corresponds to a well-defined function $r : \mathbb{R}^2 \to \mathbb{R}$. (I've written down the formula for this function.) The polar angle or argument θ is somewhat more complicated.

(a) Find/write down a formula for the argument $\theta : \mathbb{R}^2 \to \mathbb{R}$.

(b) Make an illustration of the graph

$$\left\{ \left(x_1, x_2, \sqrt{x_1^2 + x_2^2} \right) : \mathbf{x} = (x_1, x_2) \in B_1(\mathbf{0}) \right\}.$$

(c) Make an illustration of the graph

$$\{(x_1, x_2, \theta(x_1, x_2)) : \mathbf{x} = (x_1, x_2) \in B_1(\mathbf{0})\}.$$

Exercise 3.2.2 One consequence of the solution of Exercise 3.2.1 should be that the function h defined in (3.6) and the point set $\mathcal{C}$ defined in (3.7) are well-defined. In particular, there exists a well-defined **chart function**[4] $\mathbf{p} : B_1(\mathbf{0}) \to \mathcal{C}$ given by

$$\mathbf{p}(x_1, x_2) = (x_1, x_2, h(x_1, x_2)\ r(x_1, x_2)).$$

Find/write down a formula for the inverse $\xi : \mathcal{C} \to B_1(\mathbf{0})$ of $\mathbf{p}$. This function should have two coordinate functions $\xi = (\xi^1, \xi^2)$ with $\xi^j : \mathcal{C} \to \mathbb{R}$ for $j = 1, 2$. It's probably easiest to write down formulas in the form $\xi^j = \xi^j(x_1, x_2, x_3)$ for $j = 1, 2$ where $(x_1, x_2, x_3) \in \mathcal{C}$.

Exercise 3.2.3 Draw an illustration of the graph

$$\mathcal{C} = \left\{ \left(x_1, x_2, h(x_1, x_2)\sqrt{x_1^2 + x_2^2} \right) : \mathbf{x} = (x_1, x_2) \in B_1(\mathbf{0}) \right\}.$$

Terminology of charts

I think the idea Riemann (and Gauss) had in mind is best illustrated by imagining you are looking at a paper map. The next chapter, Chapter 4, is intended to help you put your mind in this mode of thinking. The entity corresponding to the paper map in the discussion of this chapter so far is the Euclidean disk $B_1(\mathbf{0})$. Generally, we can refer to the entity (or set) playing this role as the **chart**. So a **chart** for us is an open subset of (coordinatized) Euclidean space $\mathbb{R}^n$, in this case $B_1(\mathbf{0}) \subset \mathbb{R}^2$. As you look at the paper map/chart, you have in mind some kind of identification with some other "world," like the surface of the earth perhaps, and the functions

$$\mathbf{p} : B_1(\mathbf{0}) \to \mathcal{B} \qquad \text{and} \qquad \mathbf{p} : B_1(\mathbf{0}) \to \mathcal{C}$$

play this role. I wish to call functions like these **chart functions**.

[4] See below for further discussion of this terminology.

Definition 2 (chart and chart function[5]) Given a Riemannian manifold M (whatever that is—some point set with a "structure") an open subset U of $\mathbb{R}^n$ in which geometric calculation can be carried out in reference to M, i.e., in which geometry in M can be "done," is called a **chart**. This set should be compared to a paper map which is used to, for example, determine distances in some "world" M in which the geometry may not be seen directly.

Associated with each chart, is a **chart function** $\mathbf{p} : U \to M$. If there is only one chart and only one chart function,[6] then for most practical purposes, the roles played by both the chart function and the point set M are secondary. The only property required of $\mathbf{p} : U \to M$ is that $\mathbf{p}$ is a bijection.

Of course, the matrix assignment (g_{ij}) on U is of central importance. You need to use (g_{ij}) in order to understand or "do" geometry in M. We will discuss that more later.

Also, generally a chart function is not required to be a bijection, but a chart function is always required to be a bijection onto the image

$$\mathbf{p}(U) = \{\mathbf{p}(\mathbf{x}) : \mathbf{x} \in U\} \subset M.$$

That is, $\mathbf{p} : U \to M$ is always required to be an injection. In cases where $\mathbf{p}$ is not surjective, i.e., when $\mathbf{p}(U)$ is a proper subset of M, then there must be other charts and chart functions around in order to navigate to all points in the world M. This almost goes without saying. The crucial consideration of situations in which there is more than one chart may be found in Section 3.4 below. For now, we can make a distinction among these two different kinds of charts and chart functions:

A chart U with a bijective chart function $\mathbf{p} : U \to M$ is called a **global chart/global chart function pair**. A chart U with a chart function $\mathbf{p} : U \to M$ for which $\mathbf{p}(U) \neq M$ is called a **local chart/local chart function pair**.

[5]More properly, this should probably be called an "informal definition" or a "preliminary definition," because it starts with the prerequisite assumption of a Riemannian manifold which we are in the process of trying to define. On the other hand, if one understands the definition of a Riemannian manifold, then this is a perfectly fine definition. It is perhaps just a little premature in a technical sense.

[6]Note carefully that this is the case in both example $\mathcal{B}$ and example $\mathcal{C}$ where $U = B_1(\mathbf{0})$ and the single chart function $\mathbf{p} : B_1(\mathbf{0}) \to M$ is a bijection for $M = \mathcal{B}$ or $\mathcal{C}$.

The set $\mathcal{B}$ or the set $\mathcal{C}$, which we wish to think of as the manifold M, is (when you are looking at the map/chart $B_1(\mathbf{0})$) **not seen directly**. The set M which "is" the manifold may be radically different as a point set from the way it is represented on the chart as illustrated by the Riemannian manifold $\mathcal{C}$. In order to understand the manifold, you have primary recourse to the chart itself, and both the manifold and the chart function play a secondary role with respect to the Riemannian geometry, though this state of affairs may be somewhat dependent on what you find when you actually "go out into the world" and consider the manifold as a point set. It may be that you find M resembles the information contained in the chart so closely, it is difficult to tell M from U. This is the case with the Riemannian manifold $\mathcal{B}$ and the chart $B_1(\mathbf{0})$. It may also be the case that the point set M is easy to tell apart from the chart U as in the case of $\mathcal{C}$ and $B_1(\mathbf{0})$ or in the case of a paper map of Atlanta and a drive from Skiles to Stone Mountain.

In summary, Riemannian geometry is not a "visible" geometry. Though the angles are still "seen" in example $\mathcal{B}$ (and example $\mathcal{C}$) above, the notions of "length" (Riemannian length) and "area" (Riemannian area) are not directly seen, and this is the point. Very specifically, the observed Euclidean distance from the point $P_0 = \mathbf{p}(\mathbf{0}) \in \mathcal{B}$ corresponding to the origin $\mathbf{0} = (0,0) \in B_1(\mathbf{0})$ to the point $\mathbf{x} \in B_1(\mathbf{0})$ is $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2}$, and this is in general different from

$$\text{length}_{\mathcal{B}}[\alpha] = \int_0^{|\mathbf{x}|} \frac{4}{4+t^2}\, dt$$

which is the Riemannian distance[7] from $P_0 \in \mathcal{B}$ to $P = \mathbf{p}(\mathbf{x}) \in \mathcal{B}$. This is a length which is not "seen" and reminds me of something Spengler wrote about classical (western) mathematics:

[7]As Weierstrass might point out, it is also far from clear that this value is actually the Riemannian length of the path of **shortest** Riemannian length from P_0 to P.

> Every product of the waking consciousness of the Classical world, then, is elevated to the rank of actuality by way of sculptural definition. That which cannot be drawn is not "number."
>
> —Oswald Spengler

In contrast what one has in Riemannian geometry requires a strikingly different perspective:

> Numbers are the images of the perfectly desensualized understanding, of pure thought, and contain their abstract validity within themselves.
>
> —Oswald Spengler

We have already given three examples, though you may not have noticed the third: The Euclidean ball $B_1(\mathbf{0})$ is also a Riemannian manifold.

Exercise 3.2.4 What is the matrix assignment for each point in the Euclidean unit ball $B_1(\mathbf{0})$ when considered as a Riemannian manifold? (If you do not know immediately, guess the simplest thing you can think of.) Go back and apply the definitions of Riemannian length, angle, and area to the Riemannian manifold $B_1(\mathbf{0})$. Explain carefully what you get in each case.

Exercise 3.2.5 Obviously regularity is not of primary interest in my application of the functions r and θ in constructing the example $\mathcal{C}$. However, these are interesting functions in general, and regularity is interetsting in general.

(a) Show the polar radius satisfies $r \in C^0(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \backslash \{\mathbf{0}\})$.

(b) Show $r \notin C^1(\mathbb{R}^2)$.

(c) Show $r \in \mathrm{Lip}(\mathbb{R}^2)$

(d) Show $\theta \in C^\infty(U)$ where

$$U = B_1(\mathbf{0}) \backslash \{(x, 0) : 0 \le x \le 1\}.$$

(e) Show θ has no continuous extension to

$$\overline{U} = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 1\}$$

where U is given in part **(d)** above.

(f) Let $\epsilon > 0$ and

$$V = \{\mathbf{x} = (x_1, x_2) \in B_1(\mathbf{0}) \backslash (\overline{B_\epsilon(\mathbf{0})} \cup \{s(\cos t, \sin t) : s \geq 0,\ |t| \leq \epsilon\}).$$

Draw an illustration of $V \subset \mathbb{R}^2$.

(g) Let V be the open set defined in part **(f)** above. Show $\theta \in C^\infty(\overline{V})$ in the sense that for each partial derivative $D^\beta \theta : V \to \mathbb{R}$ there is some continuous function $g \in C^0(\overline{V})$ for which

$$g\big|_V = D^\beta \theta.$$

(h) With V as in part **(f)** above, show $\theta \in C^\infty(\overline{V})$ in the much stronger sense that $\theta \in C^\infty(V)$ and there is some open set $W \subset \mathbb{R}^2$ with $\overline{V} \subset W$ and a function $g \in C^\infty(W)$ such that

$$g\big|_V = \theta.$$

(i) Challenge: With V as in parts **(f)**, **(g)**, and **(h)** above, show there exists a function $g \in C^\infty(\mathbb{R}^2)$ for which

$$g\big|_V = \theta.$$

Here is another exercise which is a follow-up to the consideration of general bilinear forms as considered in Exercises 3.1.8 and 3.1.9. This is also a little outside the most direct narrative leading to the definition of a Riemannian manifold, but it may be interesting, and it will undoubtedly come up in the discussion at some point.

Exercise 3.2.6 Given a general bilinear form $B : V \times V \to \mathbb{R}$ as in Exercise 3.1.8, the **associated quadratic form** is the function $Q : V \to \mathbb{R}$ by $Q(v) = B(v, v)$.

(a) Show B is symmetric if and only if the **polarization identity**

$$B(v, w) = \frac{1}{2}[Q(v + w) - Q(v) - Q(w)]$$

holds.

(b) Show that if the matrix of a symmetric bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is invertible and $B(\mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{w} \in \mathbb{R}^n$, then $\mathbf{v} = \mathbf{0}$.